

MAT 545 FALL 2025
HOMEWORK 3

1. (Leray's covering) Let $\underline{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a complex manifold X such that $H_{\bar{\partial}}^{0,1}(U_\alpha) = 0$ for every $\alpha \in A$, and let $\mathcal{O}$ be the structure sheaf of X .
- (a) Let ρ_α be a partition of unity subordinated to $\underline{U}$ and let $\sigma = \{\sigma_{\alpha\beta}\} \in Z^1(\underline{U}, \mathcal{O})$ be a 1-cocycle. Show that

$$\varphi_\alpha = \sum_{\beta} \rho_\beta \sigma_{\alpha\beta} \in C^\infty(U_\alpha)$$

satisfies $\delta\varphi = \sigma$ and that $\theta_\alpha = \bar{\partial}\varphi_\alpha$ are restrictions to U_α of a $\bar{\partial}$ -closed $(0,1)$ -form θ on X .

- (b) Prove that the mapping $\sigma \mapsto \alpha(\sigma) = \theta$ defines a linear map $\alpha : H^1(\underline{U}, \mathcal{O}) \rightarrow H_{\bar{\partial}}^{0,1}(X)$ from Čech cohomology to Dolbeault cohomology.
- (c) Prove that the map α establishes the isomorphism

$$H^1(\underline{U}, \mathcal{O}) \simeq H_{\bar{\partial}}^{0,1}(X).$$

2. Let X be a complex manifold and let $L \rightarrow X$ be a holomorphic line bundle. Prove that L is a trivial bundle if and only if it has a nowhere vanishing global holomorphic section.
3. Let X be a complex manifold, $\dim X = n$, and let $\mathcal{T}_X$ be the holomorphic tangent bundle over X . The *first Chern class* $c_1(X)$ of the complex manifold X is the first Chern class of line bundle $\Lambda^n \mathcal{T}_X$. Let Ω_X be the holomorphic cotangent bundle and $\mathcal{K}_X = \Lambda^n \Omega_X$ be the so-called canonical line bundle over X . Prove that $c_1(X) = -c_1(\mathcal{K}_X)$.
4. Let X be a complex manifold and let V be a smooth divisor on X (a complex submanifold of codimension 1). The *normal bundle* $\mathcal{N}_V$ to V is the holomorphic line bundle

$$\mathcal{N}_V = \mathcal{T}_X|_V / \mathcal{T}_V,$$

the quotient of $\mathcal{T}_X|_V$, restriction to V of the holomorphic tangent bundle $\mathcal{T}_X$ of X , over the holomorphic tangent bundle $\mathcal{T}_V$ of V . Prove the following adjunction formulas

- (a) $\mathcal{N}_V \simeq [-V]|_V$
(b) $\mathcal{K}_V \simeq (\mathcal{K}_X \otimes [V])|_V$.

5. Prove that the following complex submanifolds in $\mathbb{P}^n$ have vanishing first Chern class by showing that they admit a nowhere vanishing

top degree holomorphic differential form (so they are Calabi-Yau manifolds).

- (a) X is a Fermat cubic in $\mathbb{P}^2$, defined by the equation

$$F(z_0, z_1, z_2) = z_0^3 + z_1^3 + z_2^3 = 0$$

(actually, X is an elliptic curve). Let ω be a $(1, 0)$ -form on X , which in the coordinate chart $X \cap U_0$ is defined by

$$\omega_0 = \frac{du_1}{u_2^2}, \quad \text{where} \quad u_1 = \frac{z_1}{z_0}, \quad u_2 = \frac{z_2}{z_0}.$$

Show that ω is a nowhere vanishing holomorphic $(1, 0)$ -form on X .

Hint: Prove that on $X \cap U_0$ one has $du_1/(\partial f/\partial u_2) = -du_2/(\partial f/\partial u_1)$.

- (b) X is Fermat quartic in $\mathbb{P}^3$, defined by the equation

$$F(z_0, z_1, z_2, z_3) = z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$$

(actually, X is a K3 surface). Let ω be a $(2, 0)$ -form on X , which in the coordinate chart $X \cap U_0$ is defined by

$$\omega_0 = \frac{du_1 \wedge du_2}{u_3^3}, \quad \text{where} \quad u_1 = \frac{z_1}{z_0}, \quad u_2 = \frac{z_2}{z_0}, \quad u_3 = \frac{z_3}{z_0}.$$

Show that ω is a nowhere vanishing holomorphic $(2, 0)$ -form on X .

- (c) X is a Fermat quintic in $\mathbb{P}^4$, defined by the equation

$$F(z_0, z_1, z_2, z_3, z_4) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$$

(actually, X is a Calabi-Yau threefold). Let ω be a $(3, 0)$ -form on X , which in the coordinate chart $X \cap U_0$ is defined by

$$\omega_0 = \frac{du_1 \wedge du_2 \wedge du_3}{u_4^4}, \quad \text{where} \quad u_1 = \frac{z_1}{z_0}, \quad u_2 = \frac{z_2}{z_0}, \quad u_3 = \frac{z_3}{z_0}, \quad u_4 = \frac{z_4}{z_0}.$$

Show that ω is a nowhere vanishing holomorphic $(3, 0)$ -form on X .