MAT 545 FALL 2025
HOMEWORK 3 SOLUTIONS

1. (Leray's covering) Let $\underline{U}=\{U_\alpha\}_{\alpha\in A}$ be an open cover of a complex manifold X such that $H^{0,1}_{\bar\partial}(U_\alpha)=0$ for every $\alpha\in A$, and let $\mathcal{O}$ be the structure sheaf of X.

(a) Let ρ_α be a partition of unity subordinated to $\underline{U}$ and let $\sigma=\{\sigma_{\alpha\beta}\}\in Z^1(\underline{U},\mathcal{O})$ be a 1-cocycle. Show that

$$\varphi_\alpha=\sum_\beta \rho_\beta\sigma_{\alpha\beta}\in C^\infty(U_\alpha)$$

satisfies $\delta\varphi=\sigma$ and that $\theta_\alpha=\bar\partial\varphi_\alpha$ are restrictions to U_α of a $\bar\partial$-closed $(0,1)$-form θ on X.

(b) Prove that the mapping $\sigma\mapsto\alpha(\sigma)=\theta$ defines a linear map $\alpha: H^1(\underline{U},\mathcal{O})\to H^{0,1}_{\bar\partial}(X)$ from Čech cohomology to Dolbeault cohomology.

c) Prove that the map α establishes the isomorphism

$$H^1(\underline{U},\mathcal{O})\simeq H^{0,1}_{\bar\partial}(X).$$

2. Let X be a complex manifold and let $L\to X$ be a holomorphic line bundle. Prove that L is a trivial bundle if and only if it has a nowhere vanishing global holomorphic section.

Problems **1-2** are standard; solution of Problem **5** is a careful computation using defining equations of corresponding hypersurfaces.

3. Let X be a complex manifold, $\dim X=n$, and let $\mathcal{T}_X$ be the holomorphic tangent bundle over X. The *first Chern class* $c_1(X)$ of the complex manifold X is the first Chern class of line bundle $\Lambda^n\mathcal{T}_X$. Let Ω_X be the holomorphic cotangent bundle and $\mathcal{K}_X=\Lambda^n\Omega_X$ be the so-called canonical line bundle over X. Prove that $c_1(X)=-c_1(\mathcal{K}_X)$.

Solution. Since Ω_X is the dual vector bundle to $\mathcal{T}_X$, line bundle $\mathcal{K}_X$ is dual to $\Lambda^n\mathcal{T}_X$. Hermitian metric in the dual line bundle L^* is the inverse of the metric in the line bundle L, so the curvature form of the Chern connection in L^* is negative of the curvature form of the Chern connection in L.

4. Let X be a complex manifold and let V be a smooth divisor on X (a complex submanifold of codimension 1). The *normal bundle* $\mathcal{N}_V$ to V is the holomorphic line bundle

$$\mathcal{N}_V=\mathcal{T}_X|_V/\mathcal{T}_V,$$

the quotient of $\mathcal{T}_X|_V$, restriction to V of the holomorphic tangent bundle $\mathcal{T}_X$ of X, over the the holomorphic tangent bundle $\mathcal{T}_V$ of V. Prove the following adjunction formulas

(a) $N_V \simeq [-V]|_V$

(b) $\mathcal{K}_V \simeq (\mathcal{K}_X \otimes [V])|_V$.

Solution. There is a **typo in part (a)**, it should read $N_V \simeq [V]|_V$! For solution, see Griffiths-Harris, pp. 146-147, or Huybrechts, Proposition 2.2.17 and Proposition 2.4.7.

5. Prove that the following complex submanifolds in $\mathbb{P}^n$ have vanishing first Chern class by showing that they admit a nowhere vanishing top degree holomorphic differential form (so they are Calabi-Yau manifolds).

(a) X is a Fermat cubic in $\mathbb{P}^2$, defined by the equation

$$F(z_0, z_1, z_2) = z_0^3 + z_1^3 + z_2^3 = 0$$

(actually, X is an elliptic curve). Let ω be a $(1,0)$-form on X, which in the coordinate chart $X \cap U_0$ is defined by

$$\omega_0 = \frac{du_1}{u_2^2}, \quad \text{where} \quad u_1 = \frac{z_1}{z_0},\ u_2 = \frac{z_2}{z_0}.$$

Show that ω is a nowhere vanishing holomorphic $(1,0)$-form on X.

Hint: Prove that on $X \cap U_0$ one has $du_1/(\partial f/\partial u_2) = -du_2/(\partial f/\partial u_1)$.

(b) X is Fermat quartic in $\mathbb{P}^3$, defined by the equation

$$F(z_0, z_1, z_2, z_3) = z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$$

(actually, X is a K3 surface). Let ω be a $(2,0)$-form on X, which in the coordinate chart $X \cap U_0$ is defined by

$$\omega_0 = \frac{du_1 \wedge du_2}{u_3^3}, \quad \text{where} \quad u_1 = \frac{z_1}{z_0},\ u_2 = \frac{z_2}{z_0},\ u_3 = \frac{z_3}{z_0}.$$

Show that ω is a nowhere vanishing holomorphic $(2,0)$-form on X.

(c) X is a Fermat quintic in $\mathbb{P}^4$, defined by the equation

$$F(z_0, z_1, z_2, z_3, z_4) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$$

(actually, X is a Calabi-Yau threefold). Let ω be a $(3,0)$-form on X, which in the coordinate chart $X \cap U_0$ is defined by

$$\omega_0 = \frac{du_1 \wedge du_2 \wedge du_3}{u_4^4}, \quad \text{where} \quad u_1 = \frac{z_1}{z_0},\ u_2 = \frac{z_2}{z_0},\ u_3 = \frac{z_3}{z_0},\ u_4 = \frac{z_4}{z_0}.$$

Show that ω is a nowhere vanishing holomorphic $(3,0)$-form on X.