MAT 545 FALL 2025
HOMEWORK 2 SOLUTIONS

Problems **1-3** are routine, problems **5** and **6** are multivariable calculus exercises on the use of Stokes' theorem, so here we concentrate on problems **4** and **7**.

1. (i) Let X be a smooth (or complex) manifold with the open covering $\{U_\alpha\}_{\alpha\in A}$ and the assignment for each nonempty ordered intersection $U_\alpha \cap U_\beta$ a smooth (holomorphic) map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ (or $\mathrm{GL}(r, \mathbb{C})$), satisfying

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = I \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset, \quad \text{and} \quad g_{\alpha\alpha} = I,$$

where I is the identity operator on $\mathbb{R}^r$ (or $\mathbb{C}^r$). Let E be smooth (or holomorphic) rank r real (or complex) vector bundle over X, defined by

$$E = \bigsqcup_{\alpha\in A} U_\alpha \times \mathbb{K}^r / \sim \qquad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}),$$

where $(x, u) \in U_\alpha \times \mathbb{K}^r$ is equivalent to $(y, v) \in U_\beta \times \mathbb{K}^r$ iff $x = y$ and $u = g_{\alpha\beta}(x)v$. Prove that transition functions for E are $g_{\alpha\beta}$.

(ii) Let X be connected complex manifold and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules over X. Prove that there is a holomorphic vector bundle over X whose sheaf of holomorphic sections is $\mathcal{E}$.

2. Let X be a complex manifold with the holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$, $\dim_{\mathbb{C}} X = n$. Show that the functions $g_{\alpha\beta} = J(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ \varphi_\beta$, where $J(f)$ is the Jacobian matrix of a holomorphic map f of a domain D in $\mathbb{C}^n$ to $\mathbb{C}^n$, are transition functions of the holomorphic tangent bundle $\mathcal{T}_X$ of X.

3. Show that the complex manifold is canonically oriented.

4. Let $D \subset \mathbb{C}^n$ be a polydisk, and suppose that $\omega \in \mathcal{A}^{1,1}(D)$ satisfies $d\omega = 0$ and is real-valued. Prove that there is a smooth function $f : D \to \mathbb{R}$ with the property that $\omega = i\bar{\partial}\partial f$.

Solution By Poincaré lemma, $\omega = d\theta$, where $\theta \in \mathcal{A}^1(D)$. Writing $\theta = \theta^{1,0} + \theta^{0,1}$, where $\theta^{1,0} \in \mathcal{A}^{1,0}(D)$ and $\theta^{0,1} \in \mathcal{A}^{0,1}(D)$. We have

$$\omega = \partial\theta^{0,1} + \bar{\partial}\theta^{1,0} + \partial\theta^{1,0} + \bar{\partial}\theta^{0,1}.$$

Comparing (p, q) degrees, we get $\partial\theta^{1,0} = 0$ and $\bar{\partial}\theta^{0,1} = 0$. Thus by Dolbeault-Grothendieck lemma $\theta^{0,1} = \bar{\partial} g$ and $\theta^{1,0} = \partial h$, so

$$\omega = \partial\bar{\partial}(g - h).$$

Since ω is real, $g - h = if$, where $f : D \to \mathbb{R}$.

5. Consider the Bochner-Martinelli kernel

$$k_{\mathrm{BM}}(z) = c_n \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{z}_k}{|z|^{2n}} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n,$$

where $c_n = (-1)^{\frac{n(n-1)}{2}}(n-1)!/(2\pi i)^n$, and $\widehat{d\bar{z}_k}$ means that the factor $d\bar{z}_k$ is omitted.

(a) Prove that $dk_{\mathrm{BM}} = \bar{\partial} k_{\mathrm{BM}} = 0$ on $\mathbb{C}^n \setminus \{0\}$.

(b) Let $S^{2n-1} \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n$ be the $(2n-1)$-dimensional unit sphere. Show that

$$\int_{S^{2n-1}} k_{\mathrm{BM}} = 1.$$

(c) Let $\pi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be the map $\pi(z, \zeta) = z - \zeta$, and let

$$K(z, \zeta) = \Pi_z^{0,0} \Pi_\zeta^{n,n-1} \pi^* k_{\mathrm{BM}}$$

(note that projector $\Pi_z^{0,0}$ is actually redundant since k_{BM} has type $(n, n-1)$). Let $D \subset \mathbb{C}^n$ be a bounded domain with a C^1 boundary and let $f \in C^1(\bar{D})$. Prove Bochner-Martinelli formula:

$$f(z) = \int_{\partial D} K(z, \zeta) f(\zeta) + \int_D K(z, \zeta) \wedge \bar{\partial} f(\zeta).$$

6. Let

$$K^{p,q}(z, \zeta) = \Pi_z^{p,q} \Pi_\zeta^{n-p,n-q-1} \pi^* k_{\mathrm{BM}}$$

(the projector $\Pi_z^{p,q}$ is actually redundant), and let $\varphi \in \mathcal{A}_c^{p,q}(\mathbb{C}^n)$ be a differential form of type (p, q) with compact support. Prove Koppelman formula:

$$\varphi(z) = \bar{\partial}_z \int_{\mathbb{C}^n} K^{p,q-1}(z, \zeta) \wedge \varphi(\zeta) + \int_{\mathbb{C}^n} K^{p,q}(z, \zeta) \wedge \bar{\partial} \varphi(\zeta).$$

(*Hint:* Use relation $\bar{\partial}_\zeta K^{p,q} = -\bar{\partial}_z K^{p,q-1}$, which is derived from part (a) of problem 5).

7. Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball in $\mathbb{C}^n$ centered at 0, $S^{2n-1} = \partial B$ and let $f \in \mathcal{O}(\bar{B})$. Using the following steps, prove Leray-Fantappiè formula

$$f(z) = \int_{S^{2n-1}} f(\zeta) S(z, \zeta),$$

where

$$S(z, \zeta) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k}{(1 - \bar{\zeta} \cdot z)^n} d\bar{\zeta}_1 \wedge \cdots \widehat{d\bar{\zeta}_k} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

and $\bar{\zeta} \cdot z = \bar{\zeta}_1 z_1 + \cdots + \bar{\zeta}_n z_n$.

(a) Put (as in class)

$$\mathcal{K}(z,w,\zeta) = c_n\sum_{k=1}^{n}(-1)^{k-1}\frac{(w_k-\bar\zeta_k)}{((z-\zeta)\cdot(w-\bar\zeta))^n}(dz_1-d\zeta_1)\wedge\cdots\wedge(dz_n-d\zeta_n)\wedge$$
$$(dw_1-d\bar\zeta_1)\wedge\cdots\wedge(dw_{k-1}-d\bar\zeta_{k-1})\wedge(dw_{k+1}-d\bar\zeta_{k+1})\wedge\cdots\wedge(dw_n-d\bar\zeta_n)$$

and

$$\mathcal{K}_0(z,w,\zeta) = \Pi_z^{0,0}\Pi_w^{0,0}\Pi_\zeta^{n,n-1}\mathcal{K}(z,w,\zeta),$$

so that

$$K(z,\zeta) = \mathcal{K}_0(z,\bar z,\zeta)$$

— a pullback of $\mathcal{K}_0$ under the map $\pi : \mathbb{C}^n\times\mathbb{C}^n \to \mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^n$ given by $(z,\zeta)\mapsto(z,\bar z,\zeta)$. For $f\in\mathcal{O}(\bar B)$ and $z,w\in B$ put

$$g(z,w) = \int_{\partial B}\mathcal{K}_0(z,w,\zeta)f(\zeta).$$

Show that $g\in\mathcal{O}(B\times B)$.

Solution. Just observe that $\mathcal{K}_0(z,w,\zeta)$ is holomorphic in z and w.

(b) Using Bochner-Martinelli formula, for $f\in\mathcal{O}(\bar B)$ and $z\in B$ show that $f(z) = g(z,w)|_{w=\bar z}$.

Solution. Follows from **5** part (c) since $\bar\partial f = 0$.

(c) Prove that $\partial_w g(z,w) = 0$, so $g(z,w)$ does not actually depend on w. Get Leray-Fantappiè formula by putting $w=0$ in the integral formula in part (a).

Solution. It follows from part (a) that for small enough z and w we have the power series expansion

$$g(z,w) = \sum_{I,J}a_{IJ}z^Iw^J \quad\text{and}\quad f(z) = \sum_K b_Kz^K$$

where we are using multi-indiex notation. By part (b) we get

$$\sum_{I,J}a_{IJ}z^I\bar z^J = \sum_K b_Kz^K.$$

Replacing z by tz, $t\in\mathbb{R}$, and differentiating with respect to t at $t=0$, we obtain identities

$$\sum_{|I|+|J|=k}a_{IJ}z^I\bar z^J = \sum_{|K|=k}b_Kz^K,$$

which show that $a_{IJ}=0$ unless $J=0$. Thus $g(z,w)$ does not depend on w for small w and, therefore, for all $w\in B$.

8. Prove that $f\in\mathcal{O}(B)$ admits a power series expansion in B, which converges absolutely and uniformly for z in every ball centered at 0 of radius $r<1$.