MAT 545 FALL 2025
HOMEWORK 1

1. (Hartog's theorem) For $n \geq 2$ prove that holomorphic function f on the domain $\Delta(0,R) \setminus \overline{\Delta(0,r)}$, where $r_1 < R_1, \dots, r_n < R_n$, extends to a holomorphic function on $\Delta(0,R)$.

2. (Riemann's extension theorem) Let $D \subseteq \mathbb{C}^n$ be an open subset and $f \in \mathcal{O}(D)$. If $g : D \setminus Z(f) \to \mathbb{C}$ is holomorphic and bounded, prove that g extends to a holomorphic function on D.

3. (Zero sets have Lebesgue measure zero)

(a) Prove the Jensen's inequality: if f is holomorphic in an open subset $D \subseteq \mathbb{C}^n$, $\overline{\Delta(0,r)} \subseteq D$ and $f(0) \neq 0$, then $\log|f|$ in integrable on $\overline{\Delta(0,r)}$ and

$$\frac{1}{\mu(\Delta(0,r))} \int\limits_{\overline{\Delta(0,r)}} \log|f| d\mu \geq \log|f(0)|,$$

where $d\mu$ is the Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

(*Hint*: use Jensen's inequality from complex analysis class; see Ahlfors book).

(b) Let $f \in \mathcal{O}(D)$ be not identically zero. Then $\mu(f^{-1}(0)) = 0$.

(*Hint*: The set $Z = f^{-1}(0)$ does not have interior points, so $D \setminus Z(f)$ is dense in D. Then prove that there is a sequence $x_k \in D \setminus Z(f)$ and polydisks $\Delta(x_k, r_k)$ such that $D = \bigcup \overline{\Delta(x_k, r_k)}$, and using Jensen's inequality show that f cannot vanish on the set of positive Lebesgue measure).

4. Problem 1 in `https://www.math.stonybrook.edu/~cschnell/mat545/homework2.pdf`.

5. Problem 2 in `https://www.math.stonybrook.edu/~cschnell/mat545/homework2.pdf`.

6. Problem 6 in `https://www.math.stonybrook.edu/~cschnell/mat545/homework2.pdf`.