MAT 545 FALL 2025
HOMEWORK 1

1. (Hartog's theorem) For $n \geq 2$ prove that holomorphic function f on the domain $\Delta(0,R) \setminus \overline{\Delta(0,r)}$, where $r_1 < R_1, \ldots, r_n < R_n$, extends to a holomorphic function on $\Delta(0,R)$.

Solution: See Griffiths & Harris, p. 7.

2. (Riemann's extension theorem) Let $D \subseteq \mathbb{C}^n$ be an open subset and $f \in \mathcal{O}(D)$. If $g : D \setminus Z(f) \to \mathbb{C}$ is holomorphic and bounded, prove that g extends to a holomorphic function on D.

Solution: see Griffiths & Harris, p. 9.

3. (Zero sets have Lebesgue measure zero)

(a) Prove the Jensen's inequality: if f is holomorphic in an open subset $D \subseteq \mathbb{C}^n$, $\overline{\Delta(0,r)} \subseteq D$ and $f(0) \neq 0$, then $\log|f|$ in integrable on $\overline{\Delta(0,r)}$ and

$$\frac{1}{\mu(\Delta(0,r))} \int\limits_{\overline{\Delta(0,r)}} \log|f| d\mu \geq \log|f(0)|,$$

where $d\mu$ is the Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Solution: see R. Gunning and H. Rossi, "Analytic functions in several complex variables" , pp. 9-10. The key step is to use a monotone convergence theorem for functions $L_n = \max\{-n, \log|f|\}$; one has $L_{n+1} \leq L_n$ so L_n converges to $\log|f|$ monotonically, and each L_n is L^1 function because $-n \leq L_n < |f|$. Since $\log|f| < C$ on $\overline{\Delta(0,r)}$ (is bounded above), $C - \log|f|$ is non-negative, and the non-decreasing sequence of non-negative functions $g_n = -L_n + C$ converges to $C - \log|f|$, which is the standard form of the monotone convergence theorem.

(b) Let $f \in \mathcal{O}(D)$ be not identically zero. Then $\mu(f^{-1}(0)) = 0$.

Solution: see R. Gunning and H. Rossi, p. 10.

4. Problem 1 in `https://www.math.stonybrook.edu/~cschnell/mat545/homework2.pdf`.

Solution: Part (a) is proved by the Leibniz rule. Part (b) is proved by using the implicit function theorem and the fact that the kernel of a ring homomorphism to the principal domain is a prime ideal.

5. Problem 2 in `https://www.math.stonybrook.edu/~cschnell/mat545/homework2.pdf`.

Solution: Follows from Problem 4 (a).

6. Problem 6 in `https://www.math.stonybrook.edu/~cschnell/mat545/homework2.pdf`.

Solution: see Griffiths & Harris, pp. 11-12.