MAT 538 RIEMANN SURFACES
HOMEWORK 1

1. Let X be a topological manifold, $\mathscr{F}$ be a presheaf of abelian groups over X and $\tilde{\mathscr{F}}$ is the corresponding étalé space with the projection $p : \tilde{\mathscr{F}} \to X$. Prove the following statements.

(a) For open $U \subseteq X$ let $\bar{\mathscr{F}}(U)$ be the space of all sections of $\tilde{\mathscr{F}}$ over U — continuous maps

$$f : U \to \tilde{\mathscr{F}}$$

such that $p \circ f = \mathrm{id}_U$. Prove that $\bar{\mathscr{F}}$ with natural inclusion maps is a sheaf.

(b) There is a natural isomorphism of the stalks

$$\mathscr{F}_x \xrightarrow{\sim} \tilde{\mathscr{F}}_x \quad \text{for all} \quad x \in X.$$

(c) If $\mathscr{F}$ is a sheaf, then the natural mapping

$$\mathscr{F}(U) \ni \sigma \to \tilde{\sigma} = \{\sigma_x\}_{x \in U} \in \tilde{\mathscr{F}}(U)$$

defines an isomorphism of sheaves

$$\mathscr{F} \simeq \tilde{\mathscr{F}}.$$

2. Let X be a complex manifold. Prove that

$$\mathscr{F}(U) = \mathcal{O}(U)/\exp \mathcal{O}(U),$$

where $\exp f = e^{2\pi\sqrt{-1}f}$ is a presheaf but not a sheaf.

3. (Leray) Let X be a topological space with a sheaf $\mathscr{F}$ of abelian groups and open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ such that $H^i(U_\alpha, \mathscr{F}) = 0$ for all $\alpha \in A$. Prove that

$$H^i(X, \mathscr{F}) \simeq H^i(\mathfrak{U}, \mathscr{F}).$$

4. Let

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$$

be an exact sequence of sheaves over the topological space X. Describe the connecting homomorphism (Bockstein homomorphism)

$$\delta^* : H^i(X, \mathscr{G}) \to H^{i+1}(X, \mathscr{E})$$

in the corresponding long exact sequence of the sheaf cohomology groups.

5. Let $X = \mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\,y_i$ and with the Hermitian metric

$$h = \sum_{i=1}^{n} dz_i \otimes d\bar{z}_i.$$

The corresponding Riemannian metric

$$g := \operatorname{Re} h = \sum_{i=1}^{n} (dx_i^2 + dy_i^2)$$

and the associated $(1,1)$-form

$$\omega := -\operatorname{Im} h = \sum_{i=1}^{n} dx_i \wedge dy_i$$

determine the volume form on $\mathbb{C}^n$,

$$\mathrm{dVol} = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Let $*$ be the Hodge star operator on differential forms on $\mathbb{C}^n$ associated with the Riemannian metric g, and let A, B and M be pairwise disjoint ordered subsets of $\{1, 2, \ldots, n\}$ of cardinalities a, b and m. For $A = \{i_1, \ldots, i_a\}, B = \{j_1, \ldots, j_b\}$ and $M = \{\mu_1, \ldots, \mu_m\}$ put

$$dz_A = dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_a}, \quad d\bar{z}_B = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \cdots \wedge d\bar{z}_{j_b}$$

and

$$\omega_M = dz_{\mu_1} \wedge d\bar{z}_{\mu_1} \wedge \cdots \wedge dz_{\mu_m} \wedge d\bar{z}_{\mu_m}.$$

Prove the following result (A. Weil).

$$*(dz_A \wedge d\bar{z}_B \wedge \omega_M) = \gamma(a, b, m) dz_A \wedge d\bar{z}_B \wedge \omega_{M'},$$

where $M' = \{1, 2, \ldots, n\} \setminus (A \cup B \cup M)$ and

$$\gamma(a, b, m) = i^{b-a}(-2i)^{p-n}(-1)^{\frac{p(p+1)}{2}}, \quad p = a + b + 2m.$$

6. Prove that Hodge theorem $\mathbb{I} = P + \Delta G$ implies the following decompositions into the direct orthogonal sum:

$$\mathcal{A}^p(X) = \mathscr{H}^p(X) \oplus d\mathcal{A}^{p-1}(X) \oplus d^* \mathcal{A}^{p+1}(X)$$

for smooth compact manifolds, and

$$\mathcal{A}^{p,q}(X) = \mathscr{H}^{p,q}(X) \oplus \bar{\partial}\mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^* \mathcal{A}^{p,q+1}(X)$$

for complex compact manifolds.

7. Let $L \to X$ be a holomorphic line bundle over a complex manifold X. Prove the 'twisted' Dollbeault isomorphism

$$H^q(X, \mathcal{O}^p(L)) = H^{p,q}_{\bar{\partial}}(X, L),$$

where $\mathcal{O}^p(L)$ is the sheaf of germs of holomorphic L-valued p-forms on X ('twisted by L differential forms'), and $H^{p,q}_{\bar{\partial}}(X, L)$ is the Dollbeault cohomology group of L-valued (p, q)-forms on X.