

MAT 536 SPRING 2021 HOMEWORK 9

More challenging problems are marked by *.

1. Evaluate

$$(a) \operatorname{Res}_{z=\infty} z^n e^{a/z}, \quad (b) \operatorname{Res}_{z=\infty} \frac{e^{z^2}}{z^{2n} + 1}.$$

2. Let $f(z)$ be an entire function. Prove that

$$\operatorname{Res}_{z=\infty} \left\{ f(z) \log \frac{z-b}{z-a} \right\} = \int_a^b f(z) dz,$$

where for $\log \frac{z-b}{z-a}$ we take any branch that is regular at $z = \infty$.

3. Evaluate the following integral

$$\int_{|z|=2} \frac{dz}{z^3(z^{10} - 2)}.$$

4. Evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{\cosh x} dx, \quad a \in \mathbb{R}.$$

5. Problem 4(h) on p. 161 in Ahlfors.

6. Problem 4(i) on p. 161 in Ahlfors.

7. Prove that

$$\int_0^{\infty} \frac{x^{2m} dx}{1 + x^{2n}} = \frac{\pi}{2n} \frac{1}{\sin \frac{2m+1}{2n} \pi},$$

where m and n are non-negative integers, $m < n$.

8*. Let $F(z)$ be entire function satisfying $|f(x + iy)| \leq C e^{a|y|}$ for $C > 0$ and some $-\pi < a < \pi$. Prove that

$$\frac{F(z)}{\sin \pi z} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N (-1)^n \frac{F(n)}{z - n}.$$

9*. Prove that for real a ,

$$\int_0^{\infty} \frac{\sin ax}{e^{2\pi x} - 1} dx = \frac{1}{4} \frac{e^a + 1}{e^a - 1} - \frac{1}{2a}. \quad (\text{Legendre.})$$

(Hint: Integrate $e^{\pm aiz}/(e^{2\pi z} - 1)$ over a contour formed by the rectangle with vertices 0 , R , $R + i$, i (the rectangle being indented at 0 and i) and let $R \rightarrow \infty$).