MAT 536 SPRING 2021 HOMEWORK 8

More challenging problems are marked by *.

1. Let $f(z)$ be a meromorphic function in a bounded domain D with finitely many poles, and suppose that it is continuous (outside the poles) up to the boundary ∂D of D. Prove that if $\mathrm{Im} f(z) \neq 0$ for all $z \in \partial D$, then the number of zeros of the function $f(z)$ in D is equal to the number of its poles.
(*Hint*: Use the argument principle and observe that the image of each component of ∂D under the map $w = f(z)$ lies either in a half-plane $\mathrm{Im}\, w > 0$ or in the half-plane $\mathrm{Im}\, w < 0$).

2. Let $f(z)$ be a meromorphic function in a bounded domain D with finitely many poles, and suppose that it is continuous (outside the poles) up to the boundary ∂D of D. Let

$$M = \max_{z \in \partial D} |f(z)|.$$

Prove that for $|a| > M$ the number of zeros of the function $f(z) - a$ in D is equal to the number of its poles.

3. Let $f(z)$ and $g(z)$ be holomorphic functions in a bounded domain D, continuous up to the boundary ∂D of D. Prove that if $\mathrm{Im}\, \dfrac{f(z)}{g(z)} \neq 0$ for all $z \in \partial D$, then $f(z)$ and $g(z)$ have the same number of zeros in D.

4. Problem **1** on p. 154 in Ahlfors.

5. Problem **2** on p. 154 in Ahlfors.

6. Problem **3** on p. 154 in Ahlfors.

7. Prove that for $\lambda > 1$ the equation $ze^{\lambda - z} = 1$ has only one root in the disk $|z| \leq 1$ and it is real.

8*. Prove that the equation $z \sin z = 1$ has only real roots.
(*Hint*: Find the number of roots on the interval $[-(n + \frac{1}{2})\pi, (n + \frac{1}{2})\pi]$ and compare it with the number of roots in the disk $|z| < (n + \frac{1}{2})\pi$).