MAT 536 SPRING 2021 HOMEWORK 7

More challenging problems are marked by *.

1. Let f be holomorphic on a disk $|z| < R$ and suppose that $|f(z)| < M$ for all $|z| < R$. Prove that if $f(z_0) = 0$, then

$$|f(z)| \leq M \frac{R|z - z_0|}{|R^2 - z\bar{z}_0|} \quad (|z| < R) \quad \text{and} \quad |f'(z_0)| \leq \frac{MR}{R^2 - |z_0|^2}.$$

2. Let f be holomorphic on the unit disk and suppose that $|f(z)| < 1$ for all $|z| < 1$. Prove that if $f(z_k) = 0$, $k = 1, \ldots, n$, then

$$|f(z)| \leq \frac{|z - z_1| \cdots |z - z_n|}{|1 - z\bar{z}_1| \cdots |1 - z\bar{z}_n|}, \quad |z| < 1.$$

3. Let f be holomorphic on the strip $|\operatorname{Re} z| < \pi/4$ and suppose that $|f(z)| < 1$ and $f(0) = 0$. Prove that in the strip

$$|f(z)| \leq |\tan z|.$$

4. Let f be holomorphic on the right half-plane $\operatorname{Re} z > 0$ and suppose that $|f(z)| < 1$. Prove that if $f(z_k) = 0$, $k = 1, \ldots, n$, then

$$|f(z)| \leq \frac{|z - z_1| \cdots |z - z_n|}{|z + \bar{z}_1| \cdots |z + \bar{z}_n|}.$$

5. Is there a holomorphic function f which maps the unit disk onto itself and satisfies $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$?

6. Let $w = f(z)$ be a holomorphic map of the unit disk into the right half-plane $\operatorname{Re} w \geq 0$. Prove that if $f(0) = 1$, then $\operatorname{Re} f(z) > 0$ and

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|} \quad \text{for all} \quad |z| < 1.$$

(*Hint*: Prove that if $\operatorname{Re} f(z) > 0$ for all $|z| < 1$, the function $1/f(z)$ also maps the unit disk into the right half-plane.)

7* (a) Let f be holomorphic on the disk $|z| < R$ and let $M(r) = \max_{|z|=r} |f(z)|$, $0 \leq r < R$. Prove that $M(r)$ is continuous non-decreasing function, which is strictly increasing if f is not a constant.

(b) Let f be a polynomial of degree n. Prove that

$$\frac{M(r_1)}{r_1^n} \geq \frac{M(r_2)}{r_2^n}$$

for $0 < r_1 < r_2$, and the equality for one pair of $r_1 < r_2$ holds only for $f(z) = az^n$.

8*. Let f be a conformal automorphism of the square $-3 \leq x, y \leq 3$, $z = x + iy$, and suppose that $f(0) = 2 + 2i$. Prove that

$$\frac{1}{9\sqrt{2}} < |f'(0)| < \frac{10}{9\sqrt{2}}.$$

(Stony Brook comprehensive exam, January 1998).