MAT 536 SPRING 2021 HOMEWORK 6

More challenging problems are marked by *.

1. Let $z = a$ be an essential singularity for a function f, holomorphic in some punctured neighborhood of a point a. Prove that for every $A \in \mathbb{C}$ there is a sequence $\{z_n\}$ such that $\lim_{n\to\infty} z_n = a$ and $\lim_{n\to\infty} f(z_n) = A$ (the Casorati-Sokhotski-Weierstrass theorem).

2. Let
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
has a radius of convergence $R > 0$. Prove that
$$|a_n| \le \inf_{r<R} \frac{M(r)}{r^n}, \quad \text{where} \quad M(r) = \max_{|z|=r} |f(z)|, \;\; r < R.$$

3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function satisfying the inequality $|f(z)| \le M e^{|z|}$ for all z. Prove that
$$|a_n| \le M \left(\frac{n}{e}\right)^{-n}.$$

4. Let $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$ be the Fibonacci numbers.

(a) Prove that
$$\frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} F_n z^n.$$

(b) Prove that
$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right].$$

5. Let f be holomorphic in some neighborhood of a point z_0. Prove that $\lim_{z\to z_0} \frac{\log|f(z)-f(z_0)|}{\log|z-z_0|}$ exists and is a positive integer.

6. Suppose that $f(z)$ is differentiable in a complex sense at $z = 0$ and satisfies the functional equation $f(z) = f(2z)$ in some neighborhood of 0. Prove that f is a constant.

7. Is there a function f, holomorphic in some neighborhood of 0 and satisfying

(a) $f\left(\frac{1}{n}\right) = \sin \frac{\pi n}{2}$;

(b) $f\left(\frac{1}{n}\right) = \frac{1}{n} \cos \pi n$;

(c) $\left| f\left(\frac{1}{n}\right) \right| < e^{-n}$;

(d) $2^{-n} < \left| f\left(\frac{1}{n}\right) \right| < 2^{1-n}$.

8*. Let f be a periodic holomorphic function in some neighborhood of ∞. Prove that f is a constant.