MAT 536 SPRING 2021 HOMEWORK 5

More challenging problems are marked by *.

1. Problem **1** on p. 123 in Ahlfors. (*Hint*: In the last integral, make use of the equation $z\bar{z} = \rho^2$ and $|dz| = -i\rho dz/z$.)

2 . Let f be an entire function such that $|f(z)| \geq C|z|^N$ with $C > 0$ and $|z|$ large enough (there is no typo!). Prove that f is a polynomial.

3. Problem **4** on p. 130 in Ahlfors.

4. Let f be an entire function. In each case, prove or give a counterexample.

(a) If $\lim_{z\to\infty} f(z) = 0$, then f is a constant.

(b) If there is a sequence $\{z_n\}$ such that

$$\lim_{n\to\infty} z_n = \infty \quad \text{and} \quad \lim_{n\to\infty} f(z_n) = 0,$$

then f is a constant.

(c) If f has a removable singularity at $z = \infty$, then f is a constant.

5. Let f be a holomorphic function on a punctured open unit disk $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. In each case, prove or give a counterexample.

(a) If f has a removable singularity at $z = 0$, so does e^f.

(b) If has a pole at $z = 0$, so does e^f.

(c) If f has an essential singularity at $z = 0$, so does e^f.

6*. Let f be an entire function. Given $a, b \in \mathbb{C}$, let C be a circle of radius $R > |a|, |b|$. Evaluate the integral

$$\int_C \frac{f(z)}{(z-a)(z-b)} dz$$

and give another proof of the Liouville's theorem. (Cannot use the residue theorem, can only use results covered so far).

7*. Problem **5** on p. 123 in Ahlfors.

8*. Suppose that f is holomorphic on a closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, except for one singular point z_0 on the S^1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series for f on $|z| < 1$. Prove that if f has a pole at z_0, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = z_0.$$

9*. Problem **5** on p. 130 in Ahlfors.