MAT 536 SPRING 2021 HOMEWORK 3

1. Suppose that the function $f(z)$ is holomorphic in a domain D and satisfies $|f(z)^2 - 1| < 1$ for all $z \in D$. Prove that either $\operatorname{Re} f(z) > 0$ or $\operatorname{Re} f(z) < 0$ throughout D. Does one need a condition that $f(z)$ is holomorphic?

2. (a) Prove that the action of the Möbius group on $\mathbb{CP}^1$ is *three-transitive*, i.e. for any two triples $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ of distinct points of $\mathbb{CP}^1$, there is a Möbius transformation T such that $Tz_k = w_k$, $k = 1, 2, 3$.

(b) Prove that for two quadruples $\{z_1, z_2, z_3, z_4\}$ and $\{w_1, w_2, w_3, w_4\}$ of distinct points of $\mathbb{CP}^1$ there is a Möbius transformation T such that $Tz_k = w_k$, $k = 1, 2, 3, 4$, if and only if their cross-ratios are equal.

3. Prove that four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ lie on a circle if and only if

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_1 - z_4|$$

and interpret it geometrically (Ptolemy's theorem).

4. (a) Express cross-rations corresponding to all 24 permutations of four distinct points z_1, z_2, z_3, z_4 in terms of the cross-ratio $\lambda = (z_1, z_2, z_3, z_4)$.

(b) Consider the so-called *anharmonic group* — a subgroup of the Möbius group, generated by the transformations $z \to 1/z$ and $z \to 1 - z$. Show that its action on $1, 0, \infty$ gives an isomorphism with S_3, symmetric group on 3 elements. Explain relation with the previous problem and with the isomorphism $S_4/V \simeq S_3$, where V is Klein four-group.

5. Map the domain between two circles $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ conformally onto upper half-plane.

6. Map conformally the complement of the arc $|z| = 1$, $\operatorname{Im} z \geq 0$ on the outside of the unit disk such that ∞ goes to ∞.

7* Suppose that a Möbius transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of corresponding radii are the same.