MAT 536 SPRING 2021 HOMEWORK 2

More challenging problems are marked by *.

1. Assume that the function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Let $f = u + iv$ be its decomposition into real and imaginary parts. Show that if $u = v^2$ everywhere, then f is constant.

2. Let $\{y_n\}$ be an increasing sequence of real numbers such that $y_n \to \infty$. Prove that (Stolz theorem)

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

if the limit in the right-hand side exists (or equal $\pm\infty$). Show that Problem 1(c) in the HW 1 (due to Cauchy), immediately follows from Stolz theorem.

3. Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

(a) Show that

$$\overline{\lim}\, a_n b_n \leq \overline{\lim}\, a_n \,\overline{\lim}\, b_n,$$

provided the right-hand side is not of the indeterminate form $0 \times \infty$. Give an example when strict inequality holds.

(b) If $\lim_{n\to\infty} a_n$ exists, show that

$$\overline{\lim}\, a_n b_n = \lim_{n\to\infty} a_n \,\overline{\lim}\, b_n,$$

if the right-hand side is not of the indeterminate form.

4. Let $\{a_n\}$ be a real sequence. Show that

$$\overline{\lim}\, a_n = \sup\{\alpha : \alpha = \lim_{n\to\infty} b_n\},$$

where $\{b_n\}$ is a convergent subsequence of $\{a_n\}$, and

$$\underline{\lim}\, a_n = \inf\{\alpha : \alpha = \lim_{n\to\infty} b_n\},$$

where $\{b_n\}$ is as above.[1]

5. Let $\{a_n\}$ be a positive sequence such that $\lim_{n\to\infty} a_{n+1}/a_n$ exists. Show that $\lim_{n\to\infty} \sqrt[n]{a_n}$ also exists and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

6. Give an example of a power series whose radius of convergence is 1, and such that the corresponding holomorphic function is continuous on the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$.

[1] In this exercise, sequences with limits $\pm\infty$ are considered as convergent.

7. Suppose that the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ both have radius of convergence $R > 0$. Then we have holomorphic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

in the disk $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. Define the sequence $\{c_n\}_{n=0}^{\infty}$ by

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Show that the series $\sum_{n=0} c_n z^n$ converges in $\mathbb{D}_R$ and therefore determines a holomorphic function $h(z)$. Prove that $h(z) = f(z)g(z)$ in $\mathbb{D}_R$. Can $\sum_{n=0}^{\infty} c_n z^n$ have a larger radius of convergence?

8. Define the Bernoulli numbers B_n by the power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Prove that

$$\frac{B_0}{n!0!} + \frac{B_1}{(n-1)!1!} + \cdots + \frac{B_{n-1}}{1!(n-1)!} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

9*. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}.$$

(Hint: The n-th coefficient of this series is not $(-1)^n/n$.)

10*. By definition, a *complex hall* of the set $\{z_1, \ldots, z_k\} \subset \mathbb{C}$ consists of all points $z = \sum_{j=1}^{k} \lambda_j z_j \in \mathbb{C}$, where all $0 \le \lambda_j \le 1$ and $\sum_{j=1}^{k} \lambda_j = 1$. Prove that (Gauss-Lucas theorem) if P is a complex polynomial, then the roots of the derivative P' belong to the convex hull of the roots of P.

(Hint: Use the representation in the proof of Theorem 1 in Ch. 1, §1.3, and obtain a formula for a root of P' which is not a root of P. Do not use online resources!)