MAT 536 SPRING 2021 HOMEWORK 12

More challenging problems are marked by *.

1. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, C be an arc of the unit circle $S^1 = \partial\mathbb{D}$, and $D \subseteq \mathbb{D}$ be a domain such that $C \subseteq \partial D$. Suppose a holomorphic function $f \in \mathcal{O}(D)$ has the property $f(D) \subseteq \mathbb{H} = \{w : \operatorname{Im} w > 0\}$. Prove that if f is continuous on C and $f(C) \subseteq \mathbb{R}$, then f can be analytically continued across C by the relation

$$f(z) = \overline{f(1/\bar{z})}.$$

2. Let f be as in Problem **1**, but suppose that instead of taking real values on C, $f(C) \subseteq S^1$. Show that the analytic continuation of f across C is now given by

$$f(z) = 1/\overline{f(1/\bar{z})}.$$

3. Suppose a holomorphic function $f \in \mathcal{O}(\mathbb{D})$ on the unit disk extends continuously to the boundary $S^1 = \partial\mathbb{D}$. Suppose moreover that $f(S^1) \subseteq S^1$. Prove that f is a rational function.

4. Let $f(z)$ be holomorphic on the strip $0 < \mathrm{Re} z < 1$, continuous on its closure $0 \le \mathrm{Re} z \le 1$ and on the lines $\mathrm{Re} z = 0$ and $\mathrm{Re} z = 1$ takes only real values. Prove that $f(z)$ can be extended to an entire function which is periodic with period 2.

5*. Let $f_n \in \mathcal{O}(D)$ be a sequence of holomorphic functions that converges to $f \in \mathcal{O}(D)$ uniformly on every $K \Subset D$. Suppose there is an open set U such that $f_n(D) \subset U$ for all n. Show that either $f(D) \subset U$ or there is $c \in \partial U$ such that $f(z) = c$ for all $z \in D$.

6*. Suppose that $f(z)$ is a conformal mapping of a rectangle $|\operatorname{Re} z| < h$, $|\operatorname{Im} z| < h'$ onto some disk or half-plane. Prove that f can be extended as a meromorphic functions on the whole complex plane with periods $4h$ and $4ih'$.