MAT 536 SPRING 2021 HOMEWORK 11

More challenging problems are marked by *.

1. Give an example of an equicontinuous, but not normal, family of functions $f : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} = \{|z| < 1\}$.

2. Give an example of a normal, but not sequentially closed, family of functions, contained in $\mathcal{O}(D)$, for some domain D.

3. Show that the family of holomorphic functions $f : \mathbb{H} \to \mathbb{H}$ is normal.

4. Prove that the family of holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk $\mathbb{D}$ satisfying $|a_n| \le n$ for each n is normal.

5. Let $f(z)$ and $g(z)$ be entire functions satisfying
$$e^{f(z)} + e^{g(z)} = 1$$
for all z. Prove that they are constants.

6*. Let $f(z)$ and $g(z)$ be entire functions satisfying $f(0) = g(0)$ and let $P(z)$ and $Q(z)$ be polynomials such that for all z
$$e^{f(z)} + P(z) = e^{g(z)} + Q(z).$$
Prove that $f(z) = g(z)$ so that $P(z) = Q(z)$.
(*Hint:* Write $P - Q = e^g(1 - e^{f-g})$ and use Picard theorem to show that the function $1 - e^{f(z)-g(z)}$ either is identically zero or has infinitely many zeros).

7*. Prove that
$$\sum_{n=1}^{\infty} \frac{n}{(n^2 - 3)\sqrt{4n^2 - 3}} = \int_0^{\sqrt{3}/2} \frac{x \cot \pi x}{(3 - x^2)\sqrt{3 - 4x^2}} dx + \frac{1}{6} \cot[\pi(2 - \sqrt{3})].$$

(*Hint:* Evaluate the integral $\int_{C_n} \dfrac{z \cot \pi z}{(z^2 - 3)\sqrt{4z^2 - 3}} dz$, where C_n is the contour below).