MAT 536 SPRING 2021 HOMEWORK 10

More challenging problems are marked by *.

1. Show that

(a)
$$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$$
on $|z+1| < 1$.

(b)
$$\frac{1}{z^2} = \frac{1}{4} + \frac{1}{4} \sum_{n=1}^{\infty} (-1)^{n+1}(n+1) \left(\frac{z-2}{2} \right)^n$$
on $|z-2| < 2$.

Find the Laurent development of the following functions in the annulus $1 < |z| < 2$:

2. (a)
$$\frac{z^4+1}{(z-1)(z+2)}.$$

(b)
$$\frac{1}{(z^2+1)(z^2-4)}.$$

3. Problem **2** on p. 186 in Ahlfors (only a sketch of the proof is required).

4. Problem **2** on p. 190 in Ahlfors.

5. Problem **4** on p. 190 in Ahlfors.

6. Problem **2** on p. 193 in Ahlfors.

7. Problem **3** on p. 193 in Ahlfors.

8*. Let $f(z)$ be a rational function with poles $a_1, \ldots, a_m \notin \mathbb{Z}$ satisfying $f(z) = O(z^{-2})$ as $|z| \to \infty$. Prove that
$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{k=1}^{m} \mathrm{Res}_{z=a_k} \{ f(z) \cot \pi z \}.$$

9*. Prove that
$$\cosh z - \cos z = z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z^4}{4\pi^4 n^4} \right).$$