MAT 536 SPRING 2021 HOMEWORK 1

More challenging problems are marked by *.

1. (a) Let $\{z_n\}$ be a sequence of complex numbers such that

$$|z_m - z_n| \leq \frac{1}{1 + |m-n|}$$

for all m and n. Evaluate $\lim_{n\to\infty} z_n$. What more can you say about this sequence?

(b) Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} z_n = 0$ and let $\{w_n\}$ be a bounded sequence. Show that $\lim_{n\to\infty} w_n z_n = 0$.

(c) Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} z_n = A$. Prove that

$$\lim_{n\to\infty} \frac{z_1 + \cdots + z_n}{n} = A.$$

2. Verify the Cauchy-Riemann equations for the function $f(z) = z^3$ by splitting f into real and imaginary parts.

3. Let $x = r\cos\theta$ and $y = r\sin\theta$. Show that the Cauchy-Riemann equations for $f = u + iv$ in polar coordinates take form

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$

4. a Put

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4}, \quad z = x + iy$$

for $z \neq 0$ and $f(0) = 0$. Show that as $z \to 0$ along any straight line, then $f(z)/z$ tends to 0, but as $z \to 0$ along a parabola $x = y^2$, then $f(z)/z$ tends to $\frac{1}{2}$, thus showing that $f'(0)$ does not exist.

(b) Prove that a function

$$f(z) = \begin{cases} \bar{z}^2/z, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equation but is not differentiable at the origin.

5. Determine the radius of convergence of the following infinite series:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=0}^{\infty} n! z^n.$$

6. Prove that if $|a_n| \leq M$, then the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $\rho \geq 1$.

7. Does there exist a holomorphic function f on $\mathbb{C}$, whose real part is

(a) $u(x,y) = e^x$
(b) $u(x,y) = e^x(x\cos y - y\sin y)$.
If the answer is 'yes', exhibit the function, if the answer is 'no', prove it.

8*. Let $f = u + iv$, where $u(x,y)$ and $v(x,y)$ are real-valued differentiable functions, and put $\nabla u = (u_x, u_y)$, $\nabla v = (v_x, v_y)$. Prove that the conditions

$$(\nabla u, \nabla v) = 0 \quad \text{and} \quad \|\nabla u\| = \|\nabla v\|,$$

where $(\ ,\ )$ is the standard Euclidean inner product on $\mathbb{R}^2$ and $\|\cdot\|$ is the corresponding norm, are equivalent to

$$\frac{\partial f}{\partial z}\frac{\partial f}{\partial \bar{z}} = 0.$$

9*. Prove the fundamental theorem of algebra: every polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

with complex coefficients of degree $n \geq 1$ has a root.
(Hint: first show that $|f(z)|$ attains a minimum on $\mathbb{C}$ and some point z_0; by shifting $g(z) = f(z + z_0)$, one can always assume $z_0 = 0$. Next, $g(z)$ has a form $g(z) = \alpha + \beta z^m \cdots + z^n$ with $\beta \neq 0$. Then assume that $\alpha \neq 0$ and show that there exists z such that

$$|g(\omega z)| < |\alpha|,$$

where $\omega^m = -\alpha/\beta$.)

10*. Show that positive integers cannot be partition nontrivially into a finite set of arithmetic progressions with no common differences.
(Hint: you should make a good use of geometric series. Please do not check online for the solution!)