MAT 535: HOMEWORK 3
DUE THU Feb [redacted] 16

Problems marked by asterisk (*) are optional.

1. Exercises 1,2 and 13 on pp. 454–455 in D&F.

2. Let V be a finite-dimensional vector space over a field F. Define the F-linear map $\mathrm{Tr} : \mathrm{End}_F(V) \to F$, the *trace* map, by

$$\mathrm{End}_F V = V^* \otimes_F V \ni v^* \otimes w \mapsto v^*(w) \in F.$$

Prove that for $A, B \in \mathrm{End}_F V$

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}\, A \, \mathrm{Tr}\, B.$$

3. Let V be a finite-dimensional vector space over a field F and $u_1, \dots, u_p$, $v_1, \dots, v_p \in V$ be such that

$$u_1 \wedge \cdots \wedge u_p = c v_1 \wedge \cdots \wedge v_p \neq 0, \quad c \in F.$$

Prove that $u_1, \dots, u_p$ and $v_1, \dots, v_p$ generate the same subspace in V.

***4.** Let R be a commutative ring with 1 and let M be a free R-module[1].

(a) Let M be finitely generated. Prove the following R-algebra isomorphism

$$T(\mathrm{End}_R(M)) \cong \bigoplus_{k=0}^{\infty} \mathrm{End}_R(T^k(M)).$$

(b) Let $M = M' \oplus M''$ be the direct sum of free R-modules. Prove the following graded R-algebra isomorphism

$$\mathrm{Sym}(M) \cong \mathrm{Sym}(M') \otimes_R \mathrm{Sym}(M'').$$

5. Let V be a finite-dimensional vector space over a field F, $\dim_F V = n$ and let $p_A(t)$ be the characteristic polynomial of $A \in \mathrm{End}_F(V)$. Define $\alpha_k(A) = \mathrm{Tr}(\wedge^k A) \in F$, $k = 0, \dots, n$. Prove that

$$p_A(-t) = \sum_{k=0}^{n} \alpha_k(A) t^{n-k}.$$

***6.** Let V be a finite-dimensional vector space over a field F, $\dim_F V = n$ and let $A \in \mathrm{End}_F(V)$.Using that $\wedge^n A$ acts by multiplication by $\det A$ in $\wedge^n V$, prove the Laplace formula (expression for the determinant in terms of cofactors). Prove Laplace expansion by complementary minors.

7. Let A be skew-symmetric $2n \times 2n$ matrix and let

$$\omega(A) = \tfrac{1}{2} \sum_{i,j=1}^{2n} a_{ij} e_i \wedge e_j,$$

[1] For part (a) it is sufficient to assume that M is finitely generated projective module.

where $e_1, \dots, e_{2n}$ is the standard basis of $\mathbb{R}^{2n}$. Prove that

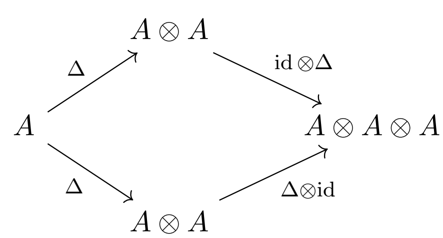
$$\wedge^n \omega(A) = n!\,\mathrm{Pf}(A) e_1 \wedge \cdots \wedge e_{2n},$$

where $\mathrm{Pf}(A)$ is the *Pfaffian* defined in class. Deduce from here that
(a) $\mathrm{Pf}(B^t A B) = \mathrm{Pf}(A) \det B$ for any $2n \times 2n$ matrix B.
(b) $\mathrm{Pf}(A)^2 = \det A$.

***8.** Let R be a commutative ring with 1. Recall that if A is an R-algebra with a multiplication $m : A \otimes_R A \to A$, where $m(a \otimes b) \stackrel{\text{def}}{=} a \cdot b$, then $A \otimes_R A$ is also an R-algebra with the multiplication $m \otimes m$. In other words, $(a \otimes b) \cdot (c \otimes d) \stackrel{\text{def}}{=} (m \otimes m)(a \otimes b \otimes c \otimes d) = ac \otimes bd$ (see Proposition 21 in §10.4 of D&F).

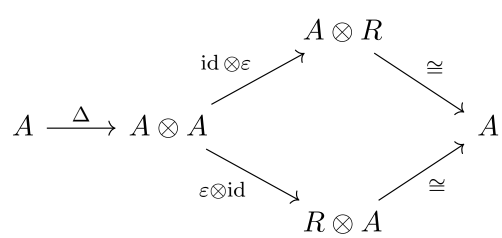
A *Hopf algebra* over R is an R-algebra A with additional operations $\Delta : A \to A \otimes_R A$, called a *comultiplication* or *coproduct*, $\varepsilon : A \to R$, called a *counit* and $S : A \to A$, called an *antipode*, satisfying the following properties.

(i) $\Delta : A \to A \otimes_R A$ is an R-algebra homomorphism satisfying

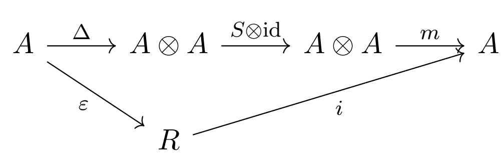


— the *coassociativity*.

(ii) $\varepsilon : A \to R$ is a ring homomorphism satisfying



(iii) $S : A \to A$ is an R-algebra anti-homomorphism ($S(ab) = S(b)S(a)$ for all $a, b \in A$) satisfying



where $i : R \to A$ is a natural inclusion map (maps $1 \in R$ to $\mathbf{1} \in A$). The same property should also hold for $\mathrm{id} \otimes S$.

Prove that the following algebras are the Hopf algebras.

(a) Tensor algebra $T(M)$ of an R-module M, where for $m \in M$ the coproduct, the antipode and counit are given by $\Delta(m) = m \otimes 1 + 1 \otimes m$, $S(m) = -m$, $\varepsilon(m) = 0$, $\varepsilon(1) = 1$. They are extended to $T(M)$ as a homomorphism of R-algebras (for Δ), an R-algebra anti-isomorphism (for S), and a ring homomorphism (for ε).

(b) The group ring $R[G]$ of a group G (see §7.2 in D&F), where for $g \in G$ we have $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$ and $\varepsilon(g) = 1$.

(c) The R-algebra $\mathrm{Fun}_R(G)$ of all maps $f : G \to R$ such that $f(g) = 0$ for all but finitely many $g \in G$ with the pointwise product. Here $\Delta(f)(g_1, g_2) = f(g_1 g_2)$, $S(f)(g) = f(g^{-1})$ and $\varepsilon(f)(g) = f(e)$, where e is the identity in G.