MAT 535: HOMEWORK 4

DUE THU Feb 25

Problems marked by asterisk (*) are optional.

1. (a) Find 2×2 matrix a satisfying

$$a^2 = (a^*)^2 = 0 \quad \text{and} \quad aa^* + a^*a = I,$$

where a^* is a Hermitian conjugate of a and I is the identity matrix.

(b) Find $2^n \times 2^n$ matrices $a_1, \dots, a_n$ satisfying

$$[a_i, a_j]_+ = [a_i^*, a_j^*]_+ = 0 \quad \text{and} \quad [a_i, a_j^*]_+ = \delta_{ij} I$$

for all $i, j = 1, \dots, n$, where a_i^* is a Hermitian conjugate of a_i and I is the identity matrix.

2. (a) Prove that vectors $v_1, \dots, v_k \in V$ in a F-vector space V are linear independent if and only if

$$v_1 \wedge \cdots \wedge v_k \neq 0.$$

(b) Let W_1 and W_2 be a subspaces of V with bases $u_1, \dots, u_k$ and $v_1, c \dots, v_k$. Prove that

$$x_1 \wedge \cdots \wedge x_k = c y_1 \wedge \cdots \wedge y_k$$

with some non-zero $c \in F$ if and only if $W_1 = W_2$.

3. (Cartan's lemma) Suppose that $v_1, \dots, v_k \in V$ are linear independent in a F-vector space V and $u_1, \dots, u_k \in V$ are such that

$$u_1 \wedge v_1 + \cdots + u_k \wedge v_k = 0.$$

Prove that there is a symmetric $k \times k$ matrix A such that

$$u_i = \sum_{j=1}^{k} a_{ij} v_j, \quad i = 1, \dots, k.$$

***4.** Prove Lemma 1 in the notes.

***5.** Prove Lemma 2 in the notes.

***6.** Prove that Koszul dual of $A = \mathrm{Sym}^\bullet(V)$ is $A^! = \Lambda^\bullet V^*$ (see the notes).