MAT 535: HOMEWORK 2

DUE THU Feb 11

Problems marked by asterisk (*) are optional.

***1.** By an orthogonal transformation transform the quadratic form $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n$ to the canonical form.

2. Let $f(x_1, \dots, x_n) = X^tAX$ be a positive-definite quadratic form with symmetric matrix $A = \{a_{ij}\}_{i,j=1}^n$, and let $D_f = \det A$ be the discriminant of f.

(a) Prove that the discriminant of the form

$$f(x_1, \dots, x_n) + (b_1x_1 + \cdots + b_nx_n)^2$$

is greater than D_f.

(b) Put $\varphi(x_2, \dots, x_n) = f(0, x_2, \dots, x_n)$. Prove that

$$D_f \leq a_{11}D_\varphi.$$

***3.** Let $f(x_1, \dots, x_n) = X^tAX$ and $g(x_1, \dots, x_n) = X^tBX$ be positive-definite quadratic forms with the matrices $A = \{a_{ij}\}_{i,j=1}^n$ and $B = \{b_{ij}\}_{i,j=1}^n$. Prove that the quadratic form (f, g) with the matrix $C = \{a_{ij}b_{ij}\}_{i,j=1}^n$ (no summation) is also positive-definite.

4. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be skew-symmetric. Prove that its *Cayley transform* $O = (I - A)(I + A)^{-1}$ is well-defined and is orthogonal.

5. Consider the complex vector space $V = \mathrm{Mat}_{n\times n}(\mathbb{C})$ and let Tr be the matrix trace. Prove that $(A, B) = \mathrm{Tr}\, AB^*$ defines a Hermitian inner product in V.

***6.** Let A be a normal operator such that $A^2 = A$. Prove that A is self-adjoint.

7. Let V be a real vector space. A *complex structure* on V is a linear operator $J : V \to V$ such that $J^2 = -I$.

(a) Prove that a complex structure J on V turns V into a complex vector space V_J, where $iv \overset{\text{def}}{=} Jv$ for all $v \in V$.

(b) Let ω be a non-degenerate alternating form on V which is *compatible with the complex structure* J, that is,

$$\omega(Ju, Jv) = \omega(u, v) \quad \text{for all} \quad u, v \in V,$$

and also suppose that $\omega(v, Jv) > 0$ for all non-zero $v \in V$. Prove that

$$\langle u, v\rangle \overset{\text{def}}{=} \omega(u, Jv) - i\omega(u, v)$$

determines a Hermitian inner product on V_J.

(c) Let $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V. Prove that the operator J extends to $V^{\mathbb{C}}$, has eigenvalues $\pm i$ and

$$V_J \cong V^{1,0},$$

where $V^{1,0} \subset V^{\mathbb{C}}$ is the eigenspace of J with eigenvalue i.