MAT 535: HOMEWORK 10

Due THU April 21

Problems marked by asterisk (*) are optional and will not be graded. Problems marked by ($\star$) are for extra credit.

1. Determine the Galois group of $x^5 - 2$ over $\mathbb{Q}$ and all the subfields of the splitting field of this polynomial.

2. Let F be a field, $n > 0$ an integer relatively prime with the characteristic of F, and assume that F contains a primitive n-th root of unity. Prove that if K/F is a Galois extension with the Galois group being cyclic of order n, then there is $\alpha \in K$ such that $K = F(\alpha)$ and α is a root of a polynomial $x^n - a$ for some $a \in F$.

(*Hint:* Apply Hilbert's Theorem 90 to ζ_n^{-1}).

3. D&F, Exercise 26 on p. 584.

4. Consider a polynomial $f(x) = x^p - x - a \in \mathbb{F}_p[x]$, where $a \neq 0$.

(a) Prove that $\alpha \mapsto \alpha + 1$ is an automorphism and using it show explicitly that the Galois group of $f(x)$ is cyclic of order p.

(b) Let $K/\mathbb{F}_p$ be a Galois extension with the Galois group being cyclic of order p, then $K = \mathbb{F}_p(\alpha)$, where α is a root of $f(x)$ for some $a \in \mathbb{F}_p$.

(*Hint:* For part (b) use Problem 3).

5. D&F, Exercise 11* on p. 589, exercises 2*, 5* and 9 on pp. 595–596 and exercise 8 on p. 603.

6. Determine all the subfields and corresponding minimal polynomials for $\mathbb{Q}[\zeta_7]$.

Extra Credit

$\star$ **7.** Let p be a prime number and let n be relatively prime to p. Prove that if n-th cyclotomic polynomial $\Phi_n(x)$ has a root in $\mathbb{F}_p$, then n divides $p - 1$.

(*Hint:* Let $\Phi_n(\alpha) = 0$ in $\mathbb{F}_p$ and let m be the order of an element α is the group $\mathbb{F}_p^*$. Prove that $m = n$.)

$\star$ **8.** Prove that there are infinitely many primes $p \equiv 1 \bmod n$.

(*Hint*: Suppose that there are finitely many such primes $p_1, \dots, p_k$. Put $m = np_1 \cdots p_k$, consider $\Phi_m[x]$ and let an integer $a > 0$ be such that $\Phi_m[am] \geq 2$. Consider the prime divisor p of $\Phi_m[am]$ and use Problem 7).