MAT 535: HOMEWORK 1

DUE THU Feb 4

1. (a) Find an orthonormal eigenbasis for the operator $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in the standard basis of $\mathbb{R}^2$.

(b) Prove that the cyclic shift operator T in $\mathbb{R}^n$, defined in the standard basis by $Te_k = e_{k-1}$, $k = 1, \ldots, n$ and $e_0 = e_n$, is normal and find its orthonormal eigenbasis.

2. Let V be a fnite-dimensional $\mathbb{R}$-vector space with Euclidean inner product $(\ ,\ )$. Prove that then, for any symmetric operator A, the bilinear form $B_A(u, v) = (Au, v)$ is symmetric and that conversely, every symmetric bilinear form can be written in this form for some symmetric operator A.

3. Let A be the following tri-diagonal $n \times n$ matrix:

$$A = \begin{pmatrix} b & c & & & \\ a & b & c & & \\ & a & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & a & b \end{pmatrix}.$$

Prove that A determines a normal operator in $\mathbb{R}^n$ if and only if $a^2 = c^2$. When $a = c$ diagonalize A for $n = 2, 3$.

4. Let V be a finite-dimensional $\mathbb{C}$-vector space with Hermitian inner product $\langle\ ,\ \rangle$. Prove that $(\ ,\ ) = \mathrm{Re}\langle\ ,\ \rangle$ is a Euclidean inner product on $V_{\mathbb{R}}$ — the space V considered as an $\mathbb{R}$-vector space, and that the bilinear form $\omega(u, v) = \mathrm{Im}\langle u, v\rangle$ on $V_{\mathbb{R}}$ is alternating.

5. Let F be a field and let $\mathcal{A}$ be an associative F-algebra with 1. Let $S \subseteq A$ be such that $st = ts$ for every $s, t \in S$. Let $\mathcal{B}$ be the smallest F-subalgebra of $\mathcal{A}$ which contains the subset S and 1. Prove that $\mathcal{B}$ is commutative. (*Hint:* Prove first that if $st = ts$ then $s^n t^m = t^m s^n$ for all $m, n \in \mathbb{N}$).

6. Let V be a finite-dimensional $\mathbb{C}$-vector space with Hermitian inner product and let A, B be commuting self-adjoint operators on V. Prove that A and B have a common orthonormal eigenbasis.

7. Let V be a finite-dimensional $\mathbb{C}$-vector space with Hermitian inner product, and let A be an invertible, normal operator on V. Prove that there exists a unique factorization $A = UP = PU$, where U is unitary and P is positive, that is, $(Pv, v) > 0$ for all non-zero $v \in V$. (*Hint:* Relate P and A^*A and prove that $U = AP^{-1}$ is unitary. Note that commutativity of U and P is equivalent to A being a normal operator).