MAT 534: HOMEWORK 9

DUE THU OCT 31

Problems marked by asterisk (*) are optional.

1. Dummit and Foot, problems 1 and 2 on p. 311.

2. Let $f(x) = x^3 - 2x^2 + 3x - 6 \in \mathbb{Q}[x]$.

(a) Prove that $\mathbb{Q}[x]/(f)$ is a direct product of two fields and find these fields.

(b) Find the inverse of $x + 1$ in $\mathbb{Q}[x]/(f)$.

3. Let $\mathbb{F}$ be a field. Define the derivative $D : \mathbb{F}[x] \to \mathbb{F}[x]$ by the usual rule, $D\Big(\sum_{k=0}^{n} a_k x^k\Big) = \sum_{k=1}^{n} k a_k x^{k-1}$.

(a) Prove that $D(fg) = D(f) \cdot g + f \cdot D(g)$.

(b) Prove that a is a multiple root of $f \in \mathbb{F}[x]$ iff a is a root and $D(f)(a) = 0$. In particular, if a multiplicity k of the root a is relatively prime with the characteristic p of the field $\mathbb{F}$, then a is a root of $D(f)$ of multiplicity $k - 1$.

(c) Prove that $f \in \mathbb{C}[x]$ has no multiple roots iff $\gcd(f, D(f)) = 1$.

4. Let $f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$, $g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathbb{F}[x]$ be two monic polynomials of degrees m and n.

(a) Prove that the following conditions are equivalent.

(i) f and g are relatively prime.

(ii) $\deg(\mathrm{lcm}(f, g)) = m + n$.

(iii) The following $m + n$ polynomials $x^i f(x)$, $i = 0, \ldots, n - 1$, and $x^j g(x)$, $j = 0, \ldots, m - 1$, are linear independent over $\mathbb{F}$.

(b) Prove that there is $R \in \mathbb{F}[a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1}]$ such that f and g are relatively prime if and only if $R \neq 0$. (Such polynomial R in variables a_i, b_j is called the *resultant* of two polynomials f and g).

Hint: Collection of k vectors $v_1, \ldots, v_k$ is a k-dimensional vector space is linearly independent if and only if the determinant of the corresponding $k \times k$ matrix is non-zero.

5. Combine two previous problem to prove that if a field $\mathbb{F}$ is algebraically closed, then $f \in \mathbb{F}[x]$ has no multiple roots if and only if the *discriminant* $D = R(f, D(f))$ is non-zero. Compute D for $f(x) = x^2 + px + q$ and $f(x) = x^3 + bx + c$.

6. Let $I \subset \mathbb{C}[x, y]$ be the ideal generated by three monomials x^3, y^3 and xy. Prove that I cannot be generated by two elements in $\mathbb{C}[x, y]$ (not necessarily monomials).

Hint: Show that $I/\mathfrak{m}I$ is a three-dimensional complex vector space, where $\mathfrak{m} \subset \mathbb{C}[x, y]$ is the maximal ideal generated by x and y.

***7.** Prove that for any n there is an ideal in $\mathbb{C}[x, y]$ which cannot be generated by fewer than n elements.