MAT 534: HOMEWORK 2

DUE THU SEP 12

Problems marked by asterisk (*) are optional.

1. Show that if $G/Z(G)$ is cyclic, then G is Abelian.

2. Prove the Third Isomorphism Theorem: if $H, K \trianglelefteq G$ with $K \subseteq H$, then one has a group isomorphism
$$G/H \cong \overline{G}/\overline{H},$$
where $\overline{G} = G/K$, $\overline{H} = H/K$.

3. Let $H \leq A_4$ be the subgroup generated by elements $x = (12)(34)$, $y = (13)(24)$. Describe the structure of H (i.e., is it isomorphic to a cyclic group? a product of cyclic groups? how large is it). Prove that $H \trianglelefteq A_4$.

4. Show that the groups S_3, S_4 are solvable.

5. Let $\mathrm{Aut}(G)$ be the group of all automorphisms of G, i.e., all isomorphisms $\varphi : G \to G$. Prove that $\mathrm{Aut}(\mathbb{Z}_n) = \mathbb{Z}_n^*$, the group of invertible elements in $\mathbb{Z}_n$ with respect to the multiplication in $\mathbb{Z}_n$.

6. Prove that $\mathrm{Aut}(\mathbb{Z}_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and use it to describe all semidirect products $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$. One of these semidirect products is the dihedral group — which one?

7. Let G be a group. For any $g \in G$, let $\varphi_g : G \to G$ be the conjugation by g, $\varphi_g(x) = gxg^{-1}$ for all $x \in G$.

(a) Prove that each φ_g is an automorphism of G. (Automorphisms of this form are called inner automorphisms).

(b) Prove that $\varphi_g \varphi_h = \varphi_{gh}$. Deduce from it that inner automorphisms form a group, isomorphic to $G/Z(G)$:
$$\mathrm{Inn}(G) \cong G/Z(G),$$
where $Z(G)$ is the center of G.

(c) Prove that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ by showing that for any (not necessarily inner) automorphism σ, we have $\sigma \circ \varphi_g \circ \sigma^{-1} = \varphi_{\sigma(g)}$.

8. Let H be any group. Show that there is a group G such that $H \trianglelefteq G$ and that for every $\sigma \in \mathrm{Aut}(H)$ there is $g \in G$ such that $ghg^{-1} = \sigma(h)$ for all $h \in H$ (i.e., every automorphism of H is obtained as an inner automorphism of G restricted to H). (*Hint*: Use a semi-direct product construction).

***9.** From Dummit and Foote: exercises 19 on p. 96, 7 on p. 101 and 12 on p. 111.