MAT 534: HOMEWORK 12

DUE TH DEC 5

Problems marked by asterisk (*) are optional.

1. (a) Find an orthonormal eigenbasis for the operator $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in the standard basis of $\mathbb{R}^2$.

(b) Prove that the cyclic shift operator T in $\mathbb{R}^n$, defined in the standard basis by $Te_k = e_{k-1}$, $k = 1, \dots, n$ and $e_0 = e_n$, is normal and find its orthonormal eigenbasis.

2. Let V be a fnite-dimensional $\mathbb{R}$-vector space with Euclidean inner product $(\ ,\ )$. Prove that for any symmetric operator A, the bilinear form $B_A(u, v) = (Au, v)$ is symmetric and that conversely, every symmetric bilinear form can be written in this form for some symmetric operator A.

3. Let A be the following tridiagonal $n \times n$ matrix:

$$A = \begin{pmatrix} b & c & & & \\ a & b & c & & \\ & a & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & a & b \end{pmatrix}.$$

Prove that A determines a normal operator in $\mathbb{R}^n$ if and only if $a^2 = c^2$. When $a = c$ diagonalize A for $n = 2, 3$.

4. Let V be a finite-dimensional $\mathbb{C}$-vector space with Hermitian inner product $\langle\ ,\ \rangle$. Prove that $(\ ,\ ) = \mathrm{Re}\langle\ ,\ \rangle$ is a Euclidean inner product on $V_{\mathbb{R}}$ — the space V considered as an $\mathbb{R}$-vector space, and that the bilinear form $\omega(u, v) = \mathrm{Im}\langle u, v\rangle$ on $V_{\mathbb{R}}$ is alternating.

5. Let V be a finite-dimensional $\mathbb{C}$-vector space with Hermitian inner product and let A, B be commuting self-adjoint operators on V. Prove that A and B have a common orthonormal eigenbasis.

6. Let V be a finite-dimensional $\mathbb{R}$-vector space. A *complex structure* on V is a linear operator $J : V \to V$ such that $J^2 = -I$.

(a) Prove that a complex structure J on V turns V into a complex vector space V_J, where $iv \stackrel{\text{def}}{=} Jv$ for all $v \in V$.

(b) Let ω be a non-degenerate alternating form on V which is *compatible with the complex structure* J, that is,

$$\omega(Ju, Jv) = \omega(u, v) \quad \text{for all} \quad u, v \in V,$$

and also suppose that $\omega(v, Jv) > 0$ for all non-zero $v \in V$. Prove that

$$\langle u, v\rangle \stackrel{\text{def}}{=} \omega(u, Jv) - i\omega(u, v)$$

determines a Hermitian inner product on V_J.

(c) Let $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V. Prove that the operator J extends to $V^{\mathbb{C}}$, has eigenvalues $\pm i$ and

$$V_J \cong V^{1,0},$$

where $V^{1,0} \subset V^{\mathbb{C}}$ is the eigenspace of J with eigenvalue i.

7. Let A be a normal operator such that $A^2 = A$. Prove that A is self-adjoint.

8*. Let V be a finite-dimensional $\mathbb{C}$-vector space. Prove that the

$$(A, B) = \operatorname{Tr} AB^*$$

defines a Hermitian inner product in $\operatorname{End} V$.

9*. Let V be a finite-dimensional $\mathbb{C}$-vector space with Hermitian inner product, and let A be an invertible, normal operator on V. Prove that there exists a unique factorization $A = UP = PU$, where U is unitary and P is positive, that is, $(Pv, v) > 0$ for all non-zero $v \in V$. (*Hint:* Relate P and A^*A and prove that $U = AP^{-1}$ is unitary. Note that commutativity of U and P is equivalent to A being a normal operator).