MAT 534: HOMEWORK 10

DUE THU NOV 14

Throughout this assignment, R is a ring with 1, all modules are left R-modules and all vector spaces are over a field F. Problems marked by asterisk (*) are optional.

1. (a) Let N be a submodule of M. Show that if both N and M/N are finitely generated, so is M.

(b) Let M be a finitely generated module over a Noetherian ring. Prove that M is *Noetherian module*, that is, every its submodule is finitely generated. (*Hint*: Prove it first for free modules using (a) and induction in rank).

2. A module is called *irreducible* or *simple*, if it has no nonzero proper submodules.

(a) Prove that every irreducible module is cyclic with every nonzero element as its generator.

(b) Prove that every irreducible module is isomorphic to R/I, where I is a maximal left ideal.

(c) Describe all irreducible modules over $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

3. Let T be a linear operator on a finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$, where I is the identity operator. Prove that T is invertible, i.e. has both left and right inverse, and $U = T^{-1}$. Show that this is false when V is not finite-dimensional. (Hint: Let $T = D$ be the differentiation operator on the space of polynomials.)

4. Let V_1 and V_2 be subspaces of the vector space V. Verify that $V_1 \cap V_2$ and $V_1 + V_2$ are also subspaces and prove that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

5. Let $P : V \to V$ be a linear operator on a finite-dimensional space V such that $P^2 = P$. Prove that

$$V = V_1 \oplus V_2,$$

where $P|_{V_1} = \text{id}$ and $P|_{V_2} = 0$, so that P is a projection operator.

6. Let A and B be commuting linear operators on a finite-dimensional vector space V such that $A^2 = A$ and $B^2 = B$. Prove that then

$$\ker AB = \ker A + \ker B.$$