MAT 534: HOMEWORK 7

DUE THU OCT 15

Problems marked by asterisk (*) are optional. Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. Which of the following rings are fields? integral domains? In each case, find all units (invertible elements).

(a) $R = \mathbb{F}[x]$.

(b) $R = \mathbb{Q}[\sqrt{d}]$, where $d \in \mathbb{Z}$ is square-free.

(c) $R = \mathbb{Z}[\sqrt{d}]$, where $d \in \mathbb{Z}$ is square-free.

(d) $R = \mathbb{R}[A] \subset \mathrm{M}_2(\mathbb{R})$, where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2. Let $\mathbb{F}[[x]]$ be the set of all formal power series in variable x with coefficients in $\mathbb{F}$. Prove that $\mathbb{F}[[x]]$ is a ring, and that $a_0 + a_1 x + a_2 x^2 + \cdots$ is a unit in this ring iff $a_0 \neq 0$.

3. Dummit and Foote, problems 11, 13 and 14 on p. 231.

4. Let $p \in \mathbb{R}[x]$ be a quadratic polynomial which has no real roots. Define $R = \mathbb{R}[x]/(p)$. Show that $R \cong \mathbb{C}$.

5. Let $I = (x - y)$, $J = (x + y)$ be ideals in $\mathbb{C}[x, y]$.

(a) Describe explicitly the rings

$$\mathbb{C}[x, y]/I,\ \mathbb{C}[x, y]/J,\ \mathbb{C}[x, y]/(I + J),\ \mathbb{C}[x, y]/IJ.$$

(Hint: you may make change of variables $x' = x+y$, $y' = x-y$). Describe each of these rings as polynomial functions on a certain subset in $\mathbb{C}^2$.

(b) Which of the ideals I, J, $I + J$, IJ is maximal? prime?

6. Dummit and Foote, problems 7 and 8 on p. 256.

***7.** Dummit and Foote, problems 33 and 34 on p. 259.