MAT 534: HOMEWORK 13

Problems marked by asterisk (*) are optional.

Throughout this assignment, F is an arbitrary field.

***1.** Find all eigenvectors of the operator T on $\mathbb{C}^2$ with the following matrix in the standard basis:

$$T = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}.$$

Is it diagonalizable?

***2.** Let $T \in \mathrm{End}_F V$ be such that its characteristic polynomial $c_T(x)$ completely decomposes in the field F.

(a) Prove that T is diagonalizable iff there exists a polynomial p without multiple roots such that $p(T) = 0$.

(b) Let T be diagonalizable and let $W \subseteq V$ be a subspace of V which is stable under T: $T(W) \subseteq W$. Prove that then the restriction of T to W is also diagonalizable.

(c) Prove that if a square matrix A satisfies $A^k = I$, then it is diagonalizable.

(d) Prove that if a square matrix A satisfies $A^3 = A$, then it is diagonalizable.

***3.** Let $A, B \in \mathrm{End}_F V$ be commuting operators: $AB = BA$.

(a) Show that if $V(\lambda)$ is the generalized eigenspace for A (that is, $V(\lambda) = \ker(A - \lambda I)^k$, where k is the multiplicity of root λ of $c_A(x)$), then $BV(\lambda) \subseteq V(\lambda)$.

(b) Show that if A and B are both diagonalizable, then they can be diagonalized simultaneously: there is a basis of V in which both A and B are diagonal.

(c) Let $C : \mathbb{C}^n \to \mathbb{C}^n$ be an operator defined by

$$C \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{x_n + x_2}{2} \\ \frac{x_1 + x_3}{2} \\ \vdots \\ \frac{x_{n-1} + x_1}{2} \end{bmatrix}.$$

Diagonalize the operator C. (*Hint*: It commutes with the cyclic permutation of x_i).

***4.** Use Jordan canonical form to prove that if A is a real $n \times n$ matrix such that all its eigenvalues are real and positive, then there exists a real matrix B such that $B^2 = A$.

***5.** Consider the following matrix

$$A = \begin{pmatrix} 9 & 4 & 5 \\ -4 & 0 & -3 \\ -6 & -4 & -2 \end{pmatrix}.$$

(a) Find the rational canonical form for C and determine a matrix P which conjugates C to its rational canonical form.

(b) Using part (a), determine the Jordan canonical form for C and the matrix which conjugates C to its Jordan canonical form.

***6.** Dummit and Foote, exercises 11, 17, 18 on pp. 500–501.