MAT 534: HOMEWORK 12

DUE TU NOV 24

Problems marked by asterisk (*) are optional.

Throughout this assignment, all vector spaces are over a field F.

1. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible, i.e. has both left and right inverse, and $U = T^{-1}$. Show that this is false when V is not finite-dimensional. (*Hint*: Let $T = D$ be the differentiation operator on the space of polynomials.)

2. Let V_1 and V_2 be subspaces of the vector space V. Verify that $V_1 \cap V_2$ and $V_1 + V_2$ are also subspaces.

(a) Prove that $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$.

(b) If V is finite-dimensional, prove that it is possible to choose a basis $\{v_i\}_{i \in I}$ in V and two subsets $I_1, I_2 \subseteq I$ such that $\{v_i\}_{i \in I_1}$ is a basis of V_1, $\{v_i\}_{i \in I_2}$ is a basis of V_2, and $\{v_i\}_{i \in I_1 \cup I_2}$ is a basis of $V_1 + V_2$.

3. Let $P : V \to V$ be a linear operator on a finite-dimensional space V such that $P^2 = P$. Prove that $V = V_1 \oplus V_2$, where $P|_{V_1} = \mathrm{id}$ and $P|_{V_2} = 0$, so that P is a projection operator.

4. Let A, B be commuting linear operators on V such that $A^2 = A$, $B^2 = B$. Prove that then $\ker(AB) = \ker(A) + \ker(B)$.

***5.** Prove the formula for Cauchy and Vandermonde determinants.

6. Compute $\det A$, where A is the following $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \dots & \dots & \dots & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

(2 on the diagonal, -1 immediately below and above the diagonal and zeros elsewhere).

7. Let $A : V \to V$ be a linear operator on a finite-dimensional space V. Define e^A by the standard power series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots,$$

which is convergent in the natural topology on $\mathrm{End}(V)$.

(a) Prove that if $P \in \mathrm{End}(V)$ is invertible, then $Pe^A P^{-1} = e^{PAP^{-1}}$.

(b) Prove that if A and B commute, then $e^{A+B} = e^A e^B$.

(c) Prove that if A is antisymmetric, $A = -A^t$, then $O = e^A$ is orthogonal, i.e., $OO^t = I$.