MAT 534: HOMEWORK 10

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Throughout this assignment, R is a ring with 1 and all modules are left R-modules. Problems marked by asterisk (*) are optional.

1. (a) Let N be a submodule of M. Show that if both N and M/N are finitely generated, so is M.

(b) Let M be a finitely generated module over a Noetherian ring. Prove that M is *Noetherian module*, that is, every its submodule is finitely generated. (*Hint*: Prove it first for free modules using (a) and induction in rank).

2. A module is called *irreducible* or *simple*, if it has no nonzero proper submodules.

(a) Prove that every irreducible module is cyclic with every nonzero element as its generator.

(b) Prove that every irreducible module is isomorphic to R/I, where I is a maximal left ideal.

(c) Describe all irreducible modules over $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

***3.** Dummit and Foote, exercises 8 and 9 on p. 344.

***4.** Dummit and Foote, exercise 2 on p. 356.

5. Let R be a P.I.D. and let M be an R-module annihilated by some nonzero $a \in R$, that is, $am = 0$ for any $m \in M$. Suppose that $a = a_1 \cdots a_n$, where a_i are pairwise relatively prime. Prove that

$$M = M_1 \oplus \cdots \oplus M_n, \quad \text{where} \quad M_i = \{m \in M : a_i M = 0\}.$$

(*Hint*: First prove it for $n = 2$ and then use induction.)

6. Dummit and Foote, exercise 11 on p. 356.

7. Let M be irreducible module. Prove that $D = \mathrm{End}_R(M)$ is a *division ring* (also called a *skew-field*) — a ring with $1 \neq 0$ such that every non-zero element has a multiplicative inverse. Show that M is a vector space over D.

***8.** (Wedderburn's Theorem). Let R be a ring and let M be an irreducible, *faithful* R-module. The latter means that if $rm = 0$ for $r \in R$ all $m \in M$, then $r = 0$. Let $D = \mathrm{End}_R(M)$ and suppose that M is finitely generated over D. Then $R = \mathrm{End}_D(M)$ — a matrix algebra with elements in D.