

## MAT 534: HOMEWORK 1

DUE THU, SEPT.



Problems marked by asterisk (\*) are optional.

Notation:

$\mathbb{Z}$  – integer numbers

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  – congruence classes modulo  $n$  (considered as a group with respect to addition)

For some problems might need the following basic result from number theory (we will prove it later): an integer  $k$  has a multiplicative inverse modulo  $n$  if and only if  $k, n$  are relatively prime.

1. Construct the isomorphism between the dihedral group  $D_6$  (all symmetries of equilateral triangle) and the symmetric group  $S_3$
2. Let  $D_{2n}$  be the group of all symmetries of a regular  $n$ -gon. Let  $r \in D_{2n}$  be the counterclockwise rotation by  $2\pi/n$  and let  $s \in D_{2n}$  be a reflection around one of the lines of symmetry. Prove the following results:
  - (a)  $r^n = e$  (where  $e$  is the group unit)
  - (b)  $s^2 = e$
  - (c)  $rs = sr^{-1}$
  - (d) Any reflection  $s' \in D_{2n}$  can be written in the form  $s' = r^k s r^{-k}$ , for some  $k \in \mathbb{Z}$
3. Construct a bijection between the coset space  $S_n/S_k \times S_{n-k}$  and the set  $B$  of all sequences of  $k$  zeroes and  $n - k$  ones. (Hint: applying an element of  $S_n$  to the sequence  $00\dots 0111\dots 1$  produces a new sequence).
4. Prove that any subgroup of index 2 is normal.
5. Describe all subgroups of symmetric group  $S_3$ . For each of them, say whether it is normal; if it is, describe the quotient.
6. Prove that any subgroup in  $\mathbb{Z}$  must be of the form  $H = a \cdot \mathbb{Z}$  for some  $a \in \mathbb{Z}$  (hint: choose the smallest positive number in  $H$ ).
7. Let  $p$  be a prime number and  $\mathbb{Z}_p^\times$  – the group of all non-zero remainders modulo  $p$  (with respect to multiplication). Deduce from Lagrange theorem that for any integer  $a$  not divisible by  $p$ , we have  $a^{p-1} \equiv 1 \pmod p$ .
8. (a) Prove that an element  $k \in \mathbb{Z}_n$  is a generator of  $\mathbb{Z}_n$  if and only if  $k$  is relatively prime with  $n$ .  
(b) A complex number  $\zeta$  is called a primitive root of unity of order  $n$  if  $\zeta^n = 1$ , but for all  $k = 1, 2, \dots, n-1$ , we have  $\zeta^k \neq 1$ . How many primitive roots of unity of order 15 are there? Describe them all.