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Leon A. Takhtajan

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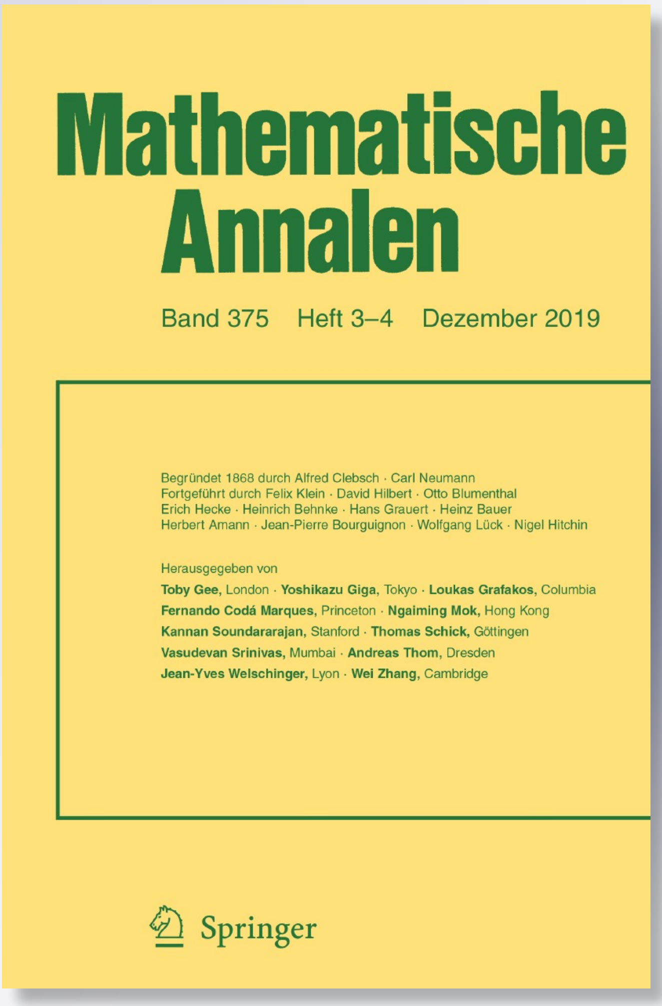

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On Kawai theorem for orbifold Riemann surfaces

Leon A. Takhtajan[1,2]

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Abstract
We prove a generalization of Kawai theorem for the case of orbifold Riemann surface. The computation is based on a formula for the differential of a holomorphic map from the cotangent bundle of the Teichmüller space to the $\mathrm{PSL}(2,\mathbb{C})$-character variety, which allows to evaluate explicitly the pullback of Goldman symplectic form in the spirit of Riemann bilinear relations. As a corollary, we obtain a generalization of Goldman's theorem that the pullback of Goldman symplectic form on the $\mathrm{PSL}(2,\mathbb{R})$-character variety is a symplectic form of the Weil–Petersson metric on the Teichmüller space.

1 Introduction

The deformation space of complex projective structures on a closed oriented genus $g \geq 2$ surface is a holomorphic affine bundle over the corresponding Teichmüller space. The choice of a Bers section identifies the deformation space with the holomorphic cotangent bundle of the Teichmüller space, a complex manifold with a complex symplectic form. Kawai's theorem [16] asserts that symplectic form on the cotangent bundle is a pulback under the monodromy map of Goldman's complex symplectic form on the corresponding $\mathrm{PSL}(2,\mathbb{C})$-character variety.

However, Kawai's proof is not very insightful. In fact, he does not use Goldman symplectic form as defined in [6], but rather uses a symplectic form on the moduli space of special rank 2 vector bundles on a Riemann surface associated with projective structures, as it is defined in [8]. The computation is highly technical and algebraic topology nature of the result gets obscured. Recently a shorter proof, relying on theorems of other authors, was given in [18]. Also in paper [4] it is proved, using special

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✉ Leon A. Takhtajan
leontak@math.sunysb.edu

1 Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA

2 Euler International Mathematical Institute, Pesochnaya Nab. 10, Saint Petersburg 197022, Russia

homological coordinates, that canonical Poisson structure on the cotangent bundle of the Teichmüller space induces the Goldman bracket on the character variety.

Here we prove a generalization of Kawai theorem for the case of orbifold Riemann surface. The computation is based on a formula for the differential of a holomorphic map from the cotangent bundle to the $\mathrm{PSL}(2,\mathbb{C})$-character variety, which allows to evaluate explicitly the pullback of Goldman symplectic form in the spirit of Riemann bilinear relations. As a corollary, we obtain a generalization of Goldman's theorem that the pullback of Goldman symplectic from on $\mathrm{PSL}(2,\mathbb{R})$-character variety is a symplectic form of the Weil–Petersson metric on the Teichmüller space.

The paper is organized as follows. In Sect. 2.1 we recall basic facts from the complex-analytic theory of Teichmüller space $\mathcal{T} = T(\Gamma)$, where Γ is a Fuchsian group of the first kind, and in Sect. 2.2 we define the holomorphic symplectic form ω on the cotangent bundle $\mathscr{M} = T^*\mathcal{T}$. In Sect. 2.3 we introduce the $\mathrm{PSL}(2,\mathbb{C})$-character variety $\mathscr{K}$ associated with the Fuchsian group Γ, and its holomorphic tangent space at $[\rho] \in \mathscr{K}$, the parabolic Eichler cohomology group $H^1_{\mathrm{par}}(\Gamma, \mathfrak{g}_{\mathrm{Ad}\rho})$. The Goldman symplectic form ω_{G} on $\mathscr{K}$ is introduced in Sect. 2.4, and the holomorphic mapping $\mathcal{Q} : \mathscr{M} \to \mathscr{K}$, as well as the map $\mathcal{F} : \mathcal{T} \to \mathscr{K}_{\mathbb{R}}$, are defined in Sect. 2.5. In Sect. 3 we explicitly compute the differential of the map $\mathcal{Q}$ in the fiber over the origin in $\mathcal{T}$. Lemma 1 neatly summarizes variational theory of the developing map in terms of the so-called Λ-operator, the classical third-order linear differential operator

$$\Lambda_q = \frac{d^3}{dz^3} + 2q(z)\frac{d}{dz} + q'(z),$$

associated with the second-order differential equation

$$\frac{d^2\psi}{dz^2} + \frac{1}{2}q(z)\psi = 0,$$

where q is a cusp form of weight 4 for Γ. Its properties are presented in **Λ1**–**Λ5** (see also, **B1**–**B3**).

The main result, Theorem 1,

$$\omega = -\sqrt{-1}\,\mathcal{Q}^*(\omega_{\mathrm{G}}),$$

is proved in Sect. 4. The proof uses Proposition 1 and explicit description of a canonical fundamental domain for Γ in Sect. 4.1. From here we obtain (see, Corollary 3)

$$\omega_{\mathrm{WP}} = \mathcal{F}^*(\omega_{\mathrm{G}}),$$

which is a generalization of Goldman theorem for orbifold Riemann surfaces.

2 The basic facts

2.1 Teichmüller space of a Fuchsian group

Here we recall the necessary basic facts from the complex-analytic theory of Teichmüller spaces (see, classic paper [1] and book [2], and also [19,23]).

2.1.1. Let Γ be, in classical terminology, a Fuchsian group of the first kind with signature $(g; n, e_1, \ldots, e_m)$, satisfying

$$2g - 2 + n + \sum_{i=1}^{m} \left(1 - \frac{1}{e_i}\right) > 0.$$

By definition, Γ is a finitely generated cofinite discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$, acting on the Lobachevsky (hyperbolic) plane, the upper half-plane

$$\mathbb{H} = \{z = x + \sqrt{-1}y : y > 0\}.$$

The group Γ has a standard presentation with $2g$ hyperbolic generators $a_1, b_1, \ldots, a_g$, b_g, m elliptic generators $c_1, \ldots, c_m$ of orders $e_1, \ldots, e_m$, and n parabolic generators $c_{m+1}, \ldots, c_{m+n}$ satisfying the relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_{m+n} = 1.$$

The group Γ can be thought of as a fundamental group of the corresponding orbifold Riemann surface $X \simeq \Gamma \backslash \mathbb{H}$.

2.1.2. Let $\mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ be the space of Beltrami differentials for Γ—a complex Banach space of $\mu \in L^\infty(\mathbb{H})$ satisfying

$$\mu(\gamma z) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \quad \text{for all} \quad \gamma \in \Gamma,$$

with the norm

$$\|\mu\|_\infty = \sup_{z \in \mathbb{H}} |\mu(z)|.$$

For a Beltrami coefficient for Γ, $\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ with $\|\mu\|_\infty < 1$, denote by w^μ the solution of the Beltrami equation

$$\begin{aligned} w^\mu_{\bar{z}} &= \mu\, w^\mu_z, \quad z \in \mathbb{H}, \\ w^\mu_{\bar{z}} &= 0, \qquad\quad z \in \mathbb{C} \backslash \mathbb{H}, \end{aligned}$$

that fixes 0, 1, ∞, and put $\mathbb{H}^\mu = w^\mu(\mathbb{H})$, $\Gamma^\mu = w^\mu \circ \Gamma \circ (w^\mu)^{-1}$. The Teichmüller space $T(\Gamma)$ of a Fuchsian group Γ is defined by

$$T(\Gamma) = \{\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma) : \|\mu\|_\infty < 1\}/\sim,$$

where $\mu \sim \nu$ if and only if $w^\mu|_{\mathbb{R}} = w^\nu|_{\mathbb{R}}$. Equivalently, $\mu \sim \nu$ if and only if $w_\mu\big|_{\mathbb{R}} = w_\nu|_{\mathbb{R}}$, where w_μ is a q.c. homeomorphism of $\mathbb{H}$ satisfying the Beltrami equation

$$(w_\mu)_{\bar{z}} = \mu(w_\mu)_z, \quad z \in \mathbb{H}.$$

We denote by $[\mu]$ the equivalence class of a Beltrami coefficient μ.

Teichmüller space $T(\Gamma)$ is a complex manifold of complex dimension

$$d = 3g - 3 + m + n.$$

The holomorphic tangent and cotangent spaces $T_0T(\Gamma)$ and $T_0^*T(\Gamma)$ at the base point, the origin $[0] \in T(\Gamma)$, are identified, respectively, with $\Omega^{-1,1}(\mathbb{H}, \Gamma)$—the vector space of harmonic Beltrami differentials for Γ, and with $\Omega^2(\mathbb{H}, \Gamma)$—the vector space of cusp forms of weight 4 for Γ. The corresponding pairing $T_0^*T(\Gamma) \otimes T_0T(\Gamma) \to \mathbb{C}$ is given by the absolutely convergent integral

$$\iint_F \mu(z)q(z)dxdy,$$

where F is a fundamental domain for Γ. There is a complex anti-linear isomorphism $\Omega^2(\mathbb{H}, \Gamma) \xrightarrow{\sim} \Omega^{-1,1}(\mathbb{H}, \Gamma)$ given by $q(z) \mapsto \mu(z) = y^2\overline{q(z)}$. Together with the pairing, it defines the Petersson inner product in $T_0T(\Gamma)$,

$$(\mu_1, \mu_2)_{\mathrm{WP}} = \iint_F \mu_1(z)\overline{\mu_2(z)}y^{-2}dxdy.$$

There is a natural isomorphism between the Teichmüller spaces $T(\Gamma)$ and $T(\Gamma_\mu)$, where $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$ is a Fuchsian group. For every $[\mu] \in T(\Gamma)$ it allows us to identify $T_{[\mu]}T(\Gamma)$ with $\Omega^{-1,1}(\mathbb{H}, \Gamma_\mu)$ and $T_{[\mu]}^*T(\Gamma)$ with $\Omega^2(\mathbb{H}, \Gamma_\mu)$. The conformal mapping

$$h_\mu = w_\mu \circ (w^\mu)^{-1} : \mathbb{H}^\mu \to \mathbb{H},$$

establishes natural isomorphisms

$$\Omega^{-1,1}(\mathbb{H}, \Gamma_\mu) \xrightarrow{\sim} \Omega^{-1,1}(\mathbb{H}^\mu, \Gamma^\mu) \quad \text{and} \quad \Omega^2(\mathbb{H}, \Gamma_\mu) \xrightarrow{\sim} \Omega^2(\mathbb{H}^\mu, \Gamma^\mu).$$

According to the isomorphism $T(\Gamma) \simeq T(\Gamma_\mu)$, the choice of a base point is inessential and we will use the notation $\mathcal{T}$ for $T(\Gamma)$.

The Petersson inner product in the tangent spaces determines the Weil–Petersson Kähler metric on $\mathcal{T}$. Its Kähler (1, 1)-form is a symplectic form ω_{WP} on $\mathcal{T}$,

$$\omega_{\mathrm{WP}}(\mu_1, \bar{\mu}_2) = \frac{\sqrt{-1}}{2} \iint_F \left(\mu_1(z)\overline{\mu_2(z)} - \overline{\mu_1(z)}\mu_2(z) \right) y^{-2} dxdy, \qquad (1)$$

where $\mu_1, \mu_2 \in T_0\mathcal{T}$.

2.1.3. Explicitly the complex structure on $\mathcal{T}$ is described as follows. Let $\mu_1, \dots, \mu_d$ be a basis of $\Omega^{-1,1}(\mathbb{H}, \Gamma)$. Bers' coordinates $(\varepsilon_1, \dots, \varepsilon_d)$ in the neighborhood U of the origin in $\mathcal{T}$ are defined by $\|\mu\|_\infty < 1$, where $\mu = \varepsilon_1\mu_1 + \cdots + \varepsilon_d\mu_d$. For the corresponding vector fields we have

$$\left.\frac{\partial}{\partial \varepsilon_i}\right|_\mu = \boldsymbol{P}_{-1,1}\left(\left(\frac{\mu_i}{1-|\mu|^2} \frac{w_z^\mu}{\overline{w_z^\mu}} \right) \circ (w^\mu)^{-1} \right) \in \Omega^{-1,1}(\mathbb{H}^\mu, \Gamma^\mu),$$

where $\boldsymbol{P}_{-1,1}$ is a projection on the subspace of harmonic Beltrami differentials. Let $p_1, \dots, p_d$ be the basis in $\Omega^2(\mathbb{H}, \Gamma)$, dual to the basis $\mu_1, \dots, \mu_d$ for $\Omega^{-1,1}(\mathbb{H}, \Gamma)$. For the holomorphic 1-forms $d\varepsilon_i$, dual to the vector fields $\frac{\partial}{\partial \varepsilon_i}$ on U, we have $d\varepsilon_i|_\mu = p_i^\mu$, where the basis $p_1^\mu, \dots, p_d^\mu$ in $\Omega^2(\mathbb{H}^\mu, \Gamma^\mu)$ has the property

$$\boldsymbol{P}_2\left(p_i^\mu \circ w^\mu \, (w_z^\mu)^2 \right) = p_i,$$

with $\boldsymbol{P}_2$ being a projection on $\Omega^2(\mathbb{H}, \Gamma)$.

2.2 Holomorphic symplectic form

Let $\mathscr{M} = T^*\mathcal{T}$ be the holomorphic cotangent bundle of $\mathcal{T}$ with the canonical projection $\pi : \mathscr{M} \to \mathcal{T}$. It is a complex symplectic manifold with canonical (2, 0)-holomorphic symplectic form $\omega = d\vartheta$, where ϑ is the Liouville 1-form (also called a tautological 1-form). At a point $(q, [\mu]) \in \mathscr{M}$ it is defined as follows (e.g., see, [3])

$$\vartheta(v) = q(\pi_* v), \quad v \in T_{(q,[\mu])}\mathscr{M}.$$

For the points in the fiber $\pi^{-1}(0)$ the symplectic form ω is given explicitly by

$$\omega((q_1, \mu_1), (q_2, \mu_2)) = \iint_F (q_1(z)\mu_2(z) - q_2(z)\mu_1(z)) dxdy, \qquad (2)$$

where $(q_1, \mu_1), (q_2, \mu_2) \in T_{(q,0)}\mathscr{M} \simeq T_0^*\mathcal{T} \oplus T_0\mathcal{T}$.

2.2.1. Let $\theta(t)$ be a smooth curve in $\mathscr{M}$ starting at $(q, 0) \in \mathscr{M}$ and lying in T^*U, where U is a Bers neighborhood of the origin in $\mathcal{T}$. Correspondingly, $\mu(t) = \pi(\theta(t))$ is a smooth curve in U satisfying $\mu(0) = 0$, and without changing the tangent vector to $\theta(t)$ at $t = 0$ we can assume that $\mu(t) = t\mu$ for some $\mu \in \Omega^{-1,1}(\mathbb{H}, \Gamma)$. We have

$$\theta(t) = \sum_{i=1}^d u^i(t) \, d\varepsilon_i|_{t\mu},$$

for small t and

$$\theta(0) = \sum_{i=1}^{d} u^i(0) p_i = q \in \Omega^2(\mathbb{H}, \Gamma).$$

The tangent vector to $\theta(t)$ at $t = 0$ is $(\dot{\theta}, \mu) \in T_{(q,0)}\mathscr{M}$, where

$$\dot{\theta} = \sum_{i=1}^{d} \dot{u}^i(0) p_i.$$

Here and in what follows the 'over-dot' denotes the derivative with respect to t at $t = 0$.

Equivalently, the curve $\theta(t)$ is given by the smooth family $q^t \in \Omega^2(\mathbb{H}^{t\mu}, \Gamma^{t\mu})$ with $q^0 = q$, and so

$$u^i(t) = \left(q^t, \frac{\partial}{\partial \varepsilon_i}\bigg|_{t\mu} \right) = \iint_F q(t)\mu_i \, dxdy,$$

where

$$q(t) = q^t \circ w^{t\mu} (w_z^{t\mu})^2, \tag{3}$$

is a pull-back of the cusp form q^t on $\mathbb{H}^{t\mu}$ to $\mathbb{H}$ by the map $w^{t\mu}$. It is a smooth family of forms of weight 4 for Γ and

$$\dot{u}^i(0) = \iint_F \dot{q}\mu_i \, dxdy, \quad i = 1, \ldots, d,$$

so that

$$\dot{\theta} = \boldsymbol{P}_2(\dot{q}).$$

2.2.2. To summarize, the value of the symplectic form (2) on tangent vectors $(\dot{\theta}_1, \mu_1)$ and $(\dot{\theta}_2, \mu_2)$ to the curves $\theta_1(t)$ and $\theta_2(t)$ at $t = 0$, is given by the following expression

$$\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) = \iint_F (\dot{q}_1\mu_2 - \dot{q}_2\mu_1) dxdy. \tag{4}$$

Remark 1 Though $\dot{q}$ is a non-holomorphic form of weight 4 for Γ, it decays exponentially at the cusps. Indeed, by conjugation it is sufficient to consider the cusp ∞. Since $w^{t\mu}(z+1) = w^{t\mu}(z) + c(t)$, we have $q^t(z + c(t)) = q^t(z)$ and

$$q(t)(z) = \sum_{n=1}^{\infty} a_n(t) e^{2\pi\sqrt{-1} n w^{t\mu}(z)/c(t)} w_z^{t\mu}(z)^2,$$

where $a_n(t)$ are corresponding Fourier coefficients of $q^t(z)$. Therefore

$$\dot{q}(z) = \sum_{n=1}^{\infty} \dot{a}_n e^{2\pi\sqrt{-1}nz} + 2q(z)\dot{w}^{\mu}_{z} + q'(z)(\dot{w}^{\mu}(z) - \dot{c}),$$

where prime always denotes the derivative with respect to z. Since $q(z)$ and $q'(z)$ decay exponentially as $y \to \infty$, we obtain

$$\dot{q}(z) = O(e^{-\pi y}) \quad \text{as} \quad y \to \infty.$$

2.3 The character variety

Here we recall necessary basic facts on the $\mathrm{PSL}(2,\mathbb{C})$-character variety for the fundamental group of the orbifold Riemann surface $X \simeq \Gamma\backslash\mathbb{H}$.

2.3.1. Let $\boldsymbol{G}$ be a Lie group $\mathrm{PSL}(2,\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ be its Lie algebra. As in [6, §2.3], we identify $\mathfrak{g}$ with the Lie algebra of vector fields $P(z)\dfrac{\partial}{\partial z}$ on $\mathbb{H}$, where $P(z) \in \mathscr{P}_2$ is a quadratic polynomial. Explicitly,

$$\mathfrak{g} \ni \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto (cz^2 - 2az - b)\frac{\partial}{\partial z} \in \mathscr{P}_2\frac{\partial}{\partial z}.$$

Let $\langle\ ,\ \rangle$ denote a 1/4 of the Killing form[1] of $\mathfrak{g}$. In terms of the standard basis $\{1, z, z^2\}$ of $\mathscr{P}_2$ the Killing form $\langle\ ,\ \rangle$ is given by the matrix

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where $C_{ij} = \langle z^{i-1}, z^{j-1}\rangle$, $i, j = 1, 2, 3$. In general, for $P_1, P_2 \in \mathscr{P}_2$

$$\langle P_1, P_2\rangle = -\frac{1}{2}B_0[P_1, P_2](z), \tag{5}$$

where for arbitrary smooth functions F and G,

$$B_0[F, G] = F_{zz}G + FG_{zz} - F_zG_z. \tag{6}$$

Note that the right hand side of (5) does not depend on z.

2.3.2. As in [6,7], let $\mathscr{K}$ be the $\boldsymbol{G}$-character variety of an orbifold Riemann surface X,

$$\mathscr{K} = \mathrm{Hom}_0(\Gamma, \boldsymbol{G})/\boldsymbol{G},$$

[1] Representing $\mathfrak{g}$ by 2×2 traceless matrices over $\mathbb{C}$ gives $\langle x, y\rangle = \mathrm{tr}\, xy$.

which consists of irreducible homomorphisms $\rho : \Gamma \to \boldsymbol{G}$, modulo conjugation, that preserve traces of parabolic and elliptic generators of Γ. The character variety $\mathscr{K}$ is a complex manifold of complex dimension $2d = 6g - 6 + 2m + 2n$, and the holomorphic tangent space $T_{[\rho]}\mathscr{K}$ at $[\rho]$ is naturally identified with the parabolic Eichler cohomology group

$$H^1_{\text{par}}(\Gamma, \mathfrak{g}_{\text{Ad}\rho}) = Z^1_{\text{par}}(\Gamma, \mathfrak{g}_{\text{Ad}\rho})/B^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho}).$$

Here $\mathfrak{g}$ is understood as a left Γ-module with respect to the action $\text{Ad}\rho$, and a 1-cocycle $\chi \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$ is a map $\chi : \Gamma \to \mathscr{P}_2$ satisfying

$$\chi(\gamma_1\gamma_2) = \chi(\gamma_1) + \rho(\gamma_1)\cdot\chi(\gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma, \tag{7}$$

where dot stands for the adjoint action of $\boldsymbol{G}$ on $\mathfrak{g} \simeq \mathscr{P}_2\dfrac{\partial}{\partial z}$,

$$(g\cdot P)(z) = \frac{P(g^{-1}(z))}{(g^{-1})'(z)}, \quad g \in \boldsymbol{G}, \;\; P \in \mathscr{P}_2. \tag{8}$$

The parabolic condition, introduced in [21], means that the restriction of a 1-cocycle $\chi \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$ to a parabolic subgroup Γ_α of Γ—the stabilizer of a cusp α for Γ—is a coboundary: there is some $P_\alpha(z) \in \mathscr{P}_2$ such that

$$\chi(\gamma) = \rho(\gamma)\cdot P_\alpha - P_\alpha, \quad \gamma \in \Gamma_\alpha.$$

We denote by $[\chi]$ the cohomology class of a 1-cocycle χ.

Remark 2 It is well-known (see, [21]) that the restriction of χ to a finite cyclic subgroup of Γ is a coboundary. Indeed, if $\gamma^n = 1$, then it follows from (7) that

$$0 = \chi(\gamma^n) = (1 + \rho(\gamma) + \cdots + \rho(\gamma^{n-1}))\cdot\chi(\gamma). \tag{9}$$

Using the unit disk model of the Lobachevsky plane, we can assume that $\gamma(u) = \zeta u$, where $\zeta^n = 1$ and $|u| < 1$. It follows from (8) and (9) that

$$\chi(\gamma)(u) = au^2 + b,$$

and there is $P \in \mathscr{P}_2$ with the property

$$\chi(\gamma)(u) = \zeta P(u/\zeta) - P(u).$$

2.4 The Goldman symplectic form

2.4.1. In case $X \simeq \Gamma\backslash\mathbb{H}$ is a compact Riemann surface (the case $m = n = 0$), Goldman [6] introduced a complex symplectic form on the character variety $\mathscr{K}$. At a point $[\rho] \in \mathscr{K}$ it is defined as

$$\omega_{\mathrm{G}}([\chi_1],[\chi_2]) = \langle [\chi_1] \cup [\chi_2] \rangle ([X]), \quad \text{where} \quad [\chi_1],[\chi_2] \in T_{[\rho]}\mathscr{K}. \tag{10}$$

Here $[X]$ is the fundamental class of X under the isomorphism $H_2(X,\mathbb{Z}) \simeq H_2(\Gamma,\mathbb{Z})$, and $\langle [\chi_1] \cup [\chi_2] \rangle \in H^2(\Gamma,\mathbb{R})$ is a composition of the cup product in cohomology and of the Killing form. At a cocycle level it is given explicitly by

$$\langle \chi_1 \cup \chi_2 \rangle (\gamma_1,\gamma_2) = \langle \chi_1(\gamma_1), \mathrm{Ad}\rho(\gamma_1)\cdot \chi(\gamma_2) \rangle, \quad \gamma_1,\gamma_2 \in \Gamma.$$

Since the right-hand side in (10) does not depend on the choice of representatives $\chi_1,\chi_2 \in Z^1(\Gamma,\mathfrak{g}_{\mathrm{Ad}\rho})$ of the cohomology classes $[\chi_1],[\chi_2] \in H^1(\Gamma,\mathfrak{g}_{\mathrm{Ad}\rho})$, we will use the notation $\omega_{\mathrm{G}}(\chi_1,\chi_2)$.

According to [6, Proposition 3.9],[2] the fundamental class $[X]$ in terms of the group homology is realized by the following 2-cycle

$$c = \sum_{k=1}^{g} \left\{ \left(\frac{\partial R}{\partial a_k}, a_k \right) + \left(\frac{\partial R}{\partial b_k}, b_k \right) \right\} \in H_2(\Gamma,\mathbb{Z}), \tag{11}$$

where $R = R_g$,

$$R_k = \prod_{i=1}^{k} a_i b_i a_i^{-1} b_i^{-1}, \quad k = 1,\ldots,g,$$

and by the Fox free differential calculus

$$\frac{\partial R}{\partial a_k} = R_{k-1} - R_k b_k, \quad \frac{\partial R}{\partial b_k} = R_{k-1} a_k - R_k. \tag{12}$$

In these notations (10) takes the form

$$\omega_{\mathrm{G}}(\chi_1,\chi_2) = -\sum_{k=1}^{g} \left\langle \chi_1\!\left(\# \frac{\partial R}{\partial a_k} \right), \chi_2(a_k) \right\rangle + \left\langle \chi_1\!\left(\# \frac{\partial R}{\partial b_k} \right), \chi_2(b_k) \right\rangle, \tag{13}$$

where a cocycle χ extends from a map on Γ to a linear map defined on the integral group ring $\mathbb{Z}[\Gamma]$, and # denotes the natural anti-involution on $\mathbb{Z}[\Gamma]$,

$$\# \left(\sum n_j \gamma_j \right) = \sum n_j \gamma_j^{-1}.$$

Remark 3 We have

$$\# \frac{\partial R}{\partial a_k} = R_{k-1}^{-1}(1-\alpha_k) \quad \text{and} \quad \# \frac{\partial R}{\partial b_k} = R_k^{-1}(1-\beta_k),$$

[2] See also, exercises 4(b) and 4(c) on p. 46 in [5].

where $\alpha_k = R_k b_k^{-1} R_k^{-1}$ and $\beta_k = R_k a_k^{-1} R_{k-1}^{-1}$, are dual generators of the group Γ (see, Sect. 4.1.1), and expression (13) takes the form

$$\omega_{\mathrm{G}}(\chi_1, \chi_2) = -\sum_{k=1}^{g} \langle \chi_1(\alpha_k), \rho(R_{k-1})\cdot\chi_2(a_k)\rangle + \langle \chi_1(\beta_k), \rho(R_k)\cdot\chi_2(b_k)\rangle\,.$$

2.4.2. In case $m+n>0$, we define R_k, $k=1,\dots,g$, as before and put

$$R_{g+i} = R_g c_1\cdots c_i,\quad i=1,\dots,m+n;\qquad R = R_{g+m+n}.$$

According to [10,11,14,17], the Goldman symplectic form ω_{G} on the character variety $\mathscr{K}$ associated with the fundamental group of an orbifold Riemann surface is defined as follows

$$\omega_{\mathrm{G}}(\chi_1, \chi_2) = -\sum_{k=1}^{g}\left\langle \chi_1\left(\#\frac{\partial R}{\partial a_k}\right), \chi_2(a_k)\right\rangle + \left\langle \chi_1\left(\#\frac{\partial R}{\partial b_k}\right), \chi_2(b_k)\right\rangle$$
$$-\sum_{i=1}^{m+n}\left\langle \chi_1\left(\#\frac{\partial R}{\partial c_i}\right), \chi_2(c_i)\right\rangle - \sum_{i=1}^{m+n}\langle \chi_1(c_i^{-1}), P_{2i}\rangle, \qquad (14)$$

where

$$\frac{\partial R}{\partial c_i} = R_{g+i-1}, \qquad (15)$$

and $P_{2i}\in\mathscr{P}_2$ are given by

$$\chi_2(\gamma) = \rho(\gamma)\cdot P_{2i} - P_{2i},\quad \gamma\in\Gamma_i = \langle c_i\rangle,\quad i=1,\dots,m+n.$$

As in the previous case, the right-hand side of (14) depends only on cohomology classes $[\chi_1], [\chi_2] \in H^1_{\mathrm{par}}(\Gamma, \mathfrak{g}_{\mathrm{Ad}\rho})$. For details and the proof that it defines a symplectic form on $\mathscr{K}$ we refer to [10,11,14,17].

2.5 The holomorphic map $\mathcal{Q}:\mathscr{M}\to\mathscr{K}$

The holomorphic map $\mathcal{Q}:\mathscr{M}\to\mathscr{K}$ is defined as follows. Let $(q,[\mu])\in\mathscr{M}$, where $q\in\Omega^2(\mathbb{H}^\mu,\Gamma^\mu)$. On $\mathbb{H}^\mu = w^\mu(\mathbb{H})$ consider the Schwarz equation

$$\mathscr{S}(f) = q,$$

where $\mathscr{S}$ stands for the Schwarzian derivative,

$$\mathscr{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Its solution, the developing map $f : \mathbb{H}^\mu \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, satisfies

$$f \circ \gamma^\mu = \rho(\gamma) \circ f \quad \text{for all} \quad \gamma^\mu = w^\mu \circ \gamma \circ (w^\mu)^{-1} \in \Gamma^\mu,$$

and determines $[\rho] \in \mathrm{Hom}_0(\Gamma, \boldsymbol{G})/\boldsymbol{G}$.

Indeed, f can be obtained as a ratio of two linearly independent solutions of the differential equation

$$\psi'' + \frac{1}{2}q(z)\psi = 0. \tag{16}$$

Since q is a cusp form of weight 4 for Γ^μ, a simple application of the Frobenius method (e.g., see, [15]) to (16) at cusps and elliptic fixed points shows that ρ preserves traces of parabolic and elliptic generators of Γ. Namely, the substitution $\zeta = e^{2\pi\sqrt{-1}z}$ sends the cusp ∞ to $\zeta = 0$ and transforms (16) to a second order linear differential equation with regular singular point at $\zeta = 0$. The characteristic equation has a double root $r = 0$, which corresponds to a parabolic monodromy, and similar analysis applies to elliptic fixed points.

Since the representation ρ is irreducible [9,20], we have $[\rho] \in \mathscr{K}$, which allows us to define the holomorphic map $\mathcal{Q}$ by

$$\mathscr{M} \ni (q, [\mu]) \mapsto \mathcal{Q}(q, [\mu]) = [\rho] \in \mathscr{K}.$$

Remark 4 Besides the holomorphic embedding $\mathcal{T} \hookrightarrow \mathscr{M}$ given by the zero section, there is a smooth non-holomorphic embedding $\iota : \mathcal{T} \to \mathscr{M}$, given by

$$\mathcal{T} \ni [\mu] \mapsto (\mathscr{S}(h_\mu), [\mu]) \in \mathscr{M},$$

where $h_\mu = w_\mu \circ (w^\mu)^{-1}$ (see, Sect. 2.1.2). The image of the smooth curve $\{[t\mu]\}$ on $\mathcal{T}$ under the map $\mathcal{F} = \mathcal{Q} \circ \iota$—the curve $\{\Gamma_{t\mu}\}$ on $\mathscr{K}$—lies in the real subvariety $\mathscr{K}_\mathbb{R}$ of $\mathscr{K}$, the character variety for $\boldsymbol{G}_\mathbb{R} = \mathrm{PSL}(2, \mathbb{R})$.

3 Differential of the map $\mathcal{Q}$

3.1 The set-up

Consider a smooth curve $\theta(t)$ on $\mathscr{M}$, defined in Sect. 2.2.1. Its image under the map $\mathcal{Q}$ is a smooth curve on $\mathscr{K}$, given by the family $\{[\rho^t]\}$, where $[\rho^0] = [\rho] = \mathcal{Q}(q, 0) \in \mathscr{K}$. According to Sect. 2.5,

$$\rho^t(\gamma) = f^t \circ \gamma^{t\mu} \circ (f^t)^{-1} \quad \text{for all} \quad \gamma^{t\mu} \in \Gamma^{t\mu}.$$

The maps $f^t : \mathbb{H}^{t\mu} \to \mathbb{P}^1$ are defined by

$$\mathscr{S}(f^t) = q^t, \tag{17}$$

where $f^0 = f : \mathbb{H} \to \mathbb{P}^1$ satisfies

$$\mathscr{S}(f) = q$$

and

$$f \circ \gamma = \rho(\gamma) \circ f \quad \text{for all} \quad \gamma \in \Gamma.$$

Put $g^t = f^t \circ w^{t\mu} : \mathbb{H} \to \mathbb{P}^1$. It follows from (17) that

$$\mathscr{S}(g^t) = \mathscr{S}(f^t) \circ w^{t\mu}(w_z^{t\mu})^2 + \mathscr{S}(w^{t\mu}) = q(t) + \mathscr{S}(w^{t\mu}), \tag{18}$$

where $q(t)$ is a non-holomorphic form of weight 4 for Γ, given by (3). Differentiating with respect to t at $t = 0$ the equation

$$g^t \circ \gamma = \rho^t(\gamma) \circ g^t,$$

we get

$$\dot{g} \circ \gamma = \dot{\rho}(\gamma) \circ f + \rho(\gamma)' \circ f \, \dot{g},$$

and using the equation

$$\rho(\gamma)' \circ f \, f' = f' \circ \gamma \, \gamma',$$

we obtain

$$\frac{1}{\gamma'} \frac{\dot{g}}{f'} \circ \gamma = \frac{\dot{g}}{f'} + \frac{1}{f'} \frac{\dot{\rho}(\gamma)}{\rho(\gamma)'} \circ f.$$

For the corresponding cocycle χ, representing a tangent vector to the curve $[\rho^t]$ at $t = 0$, we have

$$\chi(\gamma) = \dot{\rho}(\gamma) \circ \rho(\gamma)^{-1} = -\frac{\dot{\rho}(\gamma^{-1})}{(\rho(\gamma)^{-1})'},$$

so that

$$\frac{1}{f'} \chi(\gamma^{-1}) \circ f = \frac{\dot{g}}{f'} - \frac{1}{\gamma'} \frac{\dot{g}}{f'} \circ \gamma. \tag{19}$$

Indeed, it immediately follows from (19) that $\chi \in Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\rho})$. To show that χ is a parabolic cocycle, it is sufficient to check it for the subgroup Γ_∞ generated by $\tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which corresponds to the cusp at ∞. We can assume that the maps f^t fix ∞, so that the maps $g^t = f^t \circ w^{t\mu}$ also have this property,

$$g^t(z + 1) = g^t(z) + c(t).$$

Thus $\dot{g}(z+1) = \dot{g}(z) + \dot{c}$ and $\chi(\tau) = \dot{c}$. Whence there is $P \in \mathscr{P}_2$ such that $\chi(\tau) = P \circ \tau - P$.

3.2 Differential equation and the Λ-operator

From (18) it is easy to obtain a differential equation for $\dot{g}$. Namely, differentiate equation (18) with respect to t at $t = 0$. Using $g^0 = f$ and $\dot{w}^{\mu}_{zzz} = 0$ for $\mu \in \Omega^{-1,1}(\mathbb{H}, \Gamma)$, which follows from classic Ahlfors' formula in [1], we get

$$\dot{q} = \frac{d}{dt}\bigg|_{t=0} \mathscr{S}(g^t) = \frac{\dot{g}_{zzz}}{f'} - 3\frac{f''}{f'^2}\dot{g}_{zz} + \left(3\frac{f''^2}{f'^3} - \frac{f'''}{f'^2}\right)\dot{g}_z.$$

Since $q = \mathscr{S}(f)$, a simple computation shows that this equation can be written neatly as follows

$$\Lambda_q\left(\frac{\dot{g}}{f'}\right) = \dot{q}, \tag{20}$$

where Λ_q is the following linear differential operator of the third order,

$$\Lambda_q(F)(z) = F_{zzz} + 2q(z)F_z + q'(z)F.$$

In case $q = 0$ the operator Λ_0 is just a third derivative operator. The Λ-operator is classical and goes back to Appell (see, [22, Example 10 in Sect. 14.7]). Its basic properties are summarized below.

Λ1. If ψ_1 and ψ_2 are solutions of the ordinary differential Eq. (16), then

$$\Lambda_q(\psi_1\psi_2) = 0.$$

Since for $q = \mathscr{S}(f)$ one can always choose $\psi_1 = \frac{1}{\sqrt{f'}}$ and $\psi_2 = \frac{f}{\sqrt{f'}}$,

$$\Lambda_q\left(\frac{P \circ f}{f'}\right) = 0$$

for every $P \in \mathscr{P}_2$.

Λ2. If a function h satisfies $\Lambda_0(h) = p$ and f is holomorphic and locally schlicht, then $H = \dfrac{h \circ f}{f'}$ satisfies

$$\Lambda_q(H) = P,$$

where $q = \mathscr{S}(f)$ and $P = p \circ f(f')^2$.

Λ3. If $q \circ \gamma\, (\gamma')^2 = q$ for some $\gamma \in \boldsymbol{G}$, then

$$\Lambda_q\left(\frac{F \circ \gamma}{\gamma'}\right) = \Lambda_q(F) \circ \gamma\, (\gamma')^2.$$

Λ4. The general solution of the equation

$$\Lambda_q(G) = Q,$$

where $q = \mathscr{S}(f)$ and Q is holomorphic on $\mathbb{H}$, is given by

$$G(z) = \frac{1}{2}\int_{z_0}^{z} \frac{(f(z)-f(u))^2}{f'(z)f'(u)} Q(u)du + \frac{1}{f'(z)}(af(z)^2 + bf(z) + c),$$

where a, b, c are arbitrary anti-holomorphic functions of z.

Λ5.

$$\Lambda_q(F)G + F\Lambda_q(G) = (B_q[F,G])_z,$$

where the bilinear form B_q is given by

$$B_q[F,G] = F_{zz}G + FG_{zz} - F_zG_z + 2q(z)FG.$$

All these properties are well-known and can be verified by direct computation. In particular, property **Λ4**, according to **Λ2**, follows from case $q = 0$, when the equation $\Lambda_0(G) = Q$ is readily solved by

$$G(z) = \frac{1}{2}\int_{z_0}^{z} (z-u)^2 Q(u)du + az^2 + bz + c.$$

Bilinear form B_q, introduced in **Λ5**, will play an important role in our approach. It has the following properties.

B1. We have

$$B_q\left[\frac{F\circ f}{f'}, \frac{G\circ f}{f'}\right] = B_0[F,G]\circ f,$$

where $q = \mathscr{S}(f)$. In general,

$$\left(B_{\mathscr{S}(f_1)}[F,G]\right)\circ f_2 = B_{\mathscr{S}(f_1\circ f_2)}\left[\frac{F\circ f_2}{f_2'}, \frac{G\circ f_2}{f_2'}\right].$$

B2. If $q\circ\gamma\,(\gamma')^2 = q$ for some $\gamma \in \boldsymbol{G}$, then

$$B_q[F,G]\circ\gamma = B_q\left[\frac{F\circ\gamma}{\gamma'}, \frac{G\circ\gamma}{\gamma'}\right].$$

B3. If $(F\circ\gamma)\dfrac{\overline{\gamma'}}{\gamma'} = F$ for some $\gamma \in \boldsymbol{G}$, then

$$B_q[F,G] - B_q[F,G]\circ\gamma\,\overline{\gamma'} = B_q[F,H], \quad \text{where} \quad H = G - \frac{G\circ\gamma}{\gamma'}.$$

3.3 The differential

We summarize the obtained results in the following statement.

Lemma 1 *Let $(\dot{\theta}, \mu) \in T_{(q,0)}\mathscr{M}$, where $\dot{\theta} = P_2(\dot{q})$, be a tangent vector corresponding to a curve $\{q^t\}$. For a representative χ of the cohomology class*

$$[\chi] = d\mathcal{Q}|_{(q,0)}\,(\dot{\theta}, \mu) \in H^1_{\text{par}}(\Gamma, \mathfrak{g}_{\mathrm{Ad}\rho}),$$

we have

$$\frac{1}{f'}\,\chi(\gamma^{-1}) \circ f = \frac{\dot{g}}{f'} - \frac{1}{\gamma'}\,\frac{\dot{g}}{f'} \circ \gamma,$$

where $\dfrac{\dot{g}}{f'}$ satisfies

$$\Lambda_q\left(\frac{\dot{g}}{f'}\right) = \dot{q}, \quad \frac{\partial}{\partial \bar{z}}\left(\frac{\dot{g}}{f'}\right) = \mu.$$

Proof It remains only to check the last equation. Since $g^t = f^t \circ w^{t\mu}$, it follows from the Beltrami equation for $w^{t\mu}$ that on $\mathbb{H}$ the function g^t satisfies

$$g^t_{\bar{z}} = t\mu\, g^t_z,$$

and therefore

$$\dot{g}_{\bar{z}} = \mu\, f',$$

i.e.,

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\dot{g}}{f'}\right) = \mu. \tag{21}$$

□

Remark 5 We have

$$\Lambda_q(\mu) = \dot{q}_{\bar{z}},$$

which is a compatibility condition of Eqs. (20) and (21). It can be also verified directly by differentiating the equation

$$\left(\frac{\partial}{\partial \bar{z}} - t\mu\frac{\partial}{\partial z} - 2t\mu_z\right) q(t) = 0$$

at $t = 0$,

$$\dot{q}_{\bar{z}} = 2q\mu_z + q'\mu = \Lambda_q(\mu).$$

Corollary 1 *The function* $\frac{\dot{g}}{f'}$ *is given by the following formula*

$$\frac{\dot{g}(z)}{f'(z)} = \dot{w}(z) + \frac{1}{2}\int_{z_0}^{z} \frac{(f(z)-f(u))^2}{f'(z)f'(u)}\tilde{q}(u)du + \frac{P(f(z))}{f'(z)},$$

where $P \in \mathscr{P}_2$ *and* $\tilde{q} = \dot{q} - \Lambda_q(\dot{w}) = \dot{q} - 2q\dot{w}_z - q'\dot{w}$.

Proof It follows from properties $\mathbf{\Lambda 1}$ and $\mathbf{\Lambda 4}$, since the holomorphic function $\frac{\dot{g}}{f'} - \dot{w}$ satisfies

$$\Lambda_q\left(\frac{\dot{g}}{f'} - \dot{w}\right) = \tilde{q}.$$

□

Remark 6 Similarly to Wolpert's formulas [24] for Bers and Eichler–Shimura cocycles, from Corollary 1 one can obtain an explicit formula for the parabolic cocycle $\chi \in Z^1_{\mathrm{par}}(\Gamma, \mathfrak{g}_{\mathrm{Ad}\rho})$.

Corollary 2 *For every cusp* α *for* Γ *there is* $P_\alpha \in \mathscr{P}_2$ *such that*

$$\frac{\dot{g}(z)}{f'(z)} = \frac{P_\alpha(f(z))}{f'(z)} + O(e^{-c_\alpha \operatorname{Im}\sigma_\alpha z}) \quad as \quad \operatorname{Im}\sigma_\alpha z \to \infty,$$

where $\sigma_\alpha \in \mathrm{PSL}(2,\mathbb{R})$ *is such that* $\sigma_\alpha(\alpha) = \infty$ *and* $c_\alpha > 0$.

Proof It follows from Remark 1 and Lemma 1 (or from Corollary 1). □

Remark 7 For the family $q^t = \mathscr{S}(h_{t\mu})$, introduced in Remark 4, we have $g^t = w_{t\mu}$ and $\dot{q} = \dot{g}_{zzz}$. It follows from classic Ahlfors' formula in [1] that

$$\dot{q} = -\frac{1}{2}q, \quad \text{where} \quad \mu = y^2\bar{q}.$$

Thus

$$d\imath|_0(\mu) = (-\tfrac{1}{2}q, \mu) \in T_0\mathscr{M},$$

and it follows from (1) that

$$\imath^*(\omega) = \sqrt{-1}\,\omega_{\mathrm{WP}}.$$

4 Computation of the symplectic form

4.1 The fundamental domain

Here we recall the definition of a canonical fundamental domain for the Fuchsian group Γ (see, [13] and references therein).

4.1.1. In case $m = n = 0$ choose $z_0 \in \mathbb{H}$ and standard generators $a_k, b_k, k = 1, \ldots, g$. The oriented canonical fundamental domain F with the base point z_0 is a topological $4g$-gon whose ordered vertices are given by the consecutive quadruples

$$(R_k z_0, R_k a_{k+1} z_0, R_k a_{k+1} b_{k+1} z_0, R_k a_{k+1} b_{k+1} a_{k+1}^{-1} z_0), \quad k = 0, \ldots, g-1.$$

Corresponding A and B edges of F are analytic arcs $A_k = (R_{k-1} z_0, R_{k-1} a_k z_0)$ and $B_k = (R_k z_0, R_k b_k z_0)$, $k = 1, \ldots, g$, and corresponding dual edges are $A'_k = (R_k b_k z_0, R_k b_k a_k z_0)$ and $B'_k = (R_{k-1} a_k z_0, R_k b_k a_k z_0)$ (see, Fig. 1 for a typical fundamental domain for a group Γ of genus 2).
We have

$$\partial F = \sum_{k=1}^{g} (A_k - B_k - A'_k + B'_k).$$

Here

$$A_k = \alpha_k(A'_k) \quad \text{and} \quad B_k = \beta_k(B'_k),$$

where $\alpha_k = R_{k-1} b_k^{-1} R_k^{-1}$ and $\beta_k = R_k a_k^{-1} R_{k-1}^{-1}$. They satisfy

$$[\alpha_k, \beta_k] = R_{k-1} R_k^{-1},$$

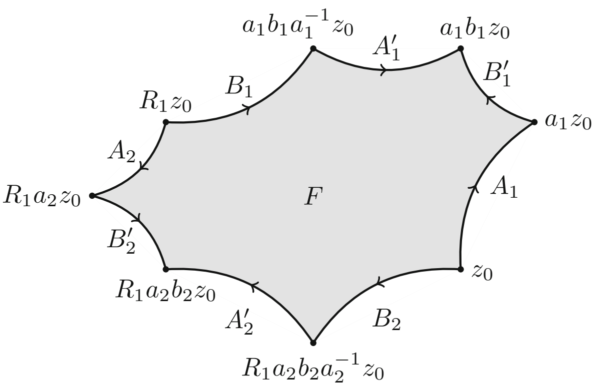

Fig. 1 Fundamental domain for a group Γ of genus 2

so that

$$\mathcal{R}_k=\prod_{i=1}^{k}[\alpha_i,\beta_i]=R_k^{-1}\quad\text{and}\quad\prod_{k=1}^{g}\alpha_k\beta_k\alpha_k^{-1}\beta_k^{-1}=1.$$

The generators α_k, β_k, $k=1,\ldots,g$, are dual generators of Γ, introduced by A. Weil [21] (see also, [12]), and

$$a_k^{-1}=\mathcal{R}_k\beta_k\mathcal{R}_{k-1}^{-1},\quad b_k^{-1}=\mathcal{R}_{k-1}\alpha_k\mathcal{R}_k^{-1}.$$

We have $A_k=(\mathcal{R}_{k-1}^{-1}z_0, \beta_k^{-1}\mathcal{R}_k^{-1}z_0)$, $B_k=(\mathcal{R}_k^{-1}z_0, \alpha_k^{-1}\mathcal{R}_{k-1}^{-1}z_0)$ and

$$\partial F=\sum_{i=1}^{2g}(S_l-\lambda_i(S_i)),$$

where $S_k=A_k$, $S_{k+g}=-B_k$ and $\lambda_k=\alpha_k^{-1}$, $\lambda_{k+g}=\beta_k^{-1}$, $k=1,\ldots,g$.

Remark 8 The ordering of vertices of F for the dual generators corresponds to the opposite orientation, so that (cf. (11))

$$c=-\sum_{k=1}^{g}\left\{\left(\frac{\partial\mathcal{R}}{\partial\alpha_k},\alpha_k\right)+\left(\frac{\partial\mathcal{R}}{\partial\beta_k},\beta_k\right)\right\}.$$

4.1.2. In general case $m+n>0$, oriented canonical fundamental domain F with the base point z_0 is a $(4g+2m+2n)$-gon whose ordered vertices are given by the consecutive quadruples

$$(R_kz_0, R_ka_{k+1}z_0, R_ka_{k+1}b_{k+1}z_0, R_ka_{k+1}b_{k+1}a_{k+1}^{-1}z_0),\quad k=0,\ldots,g-1,$$

followed by the consecutive triples $(R_{g+i-1}z_0, z_i, R_{g+i}z_0)$, $i=1,\ldots,m+n$. Here $z_i\in\mathbb{H}$, $i=1,\ldots,m$, are fixed points of the elliptic elements

$$\gamma_i=R_{g+i-1}c_i^{-1}R_{g+i-1}^{-1},$$

and $z_{m+j}\in\mathbb{R}$, $j=1,\ldots,n$, are fixed points of the parabolic elements

$$\gamma_{m+j}=R_{g+m+j-1}c_{m+j}^{-1}R_{g+m+j-1}^{-1}$$

(see, Fig. 2 for a typical fundamental domain of group Γ of signature $(1;1,6)$, where z_1 is elliptic fixed point of order 6 and z_2 is a cusp).
We have

$$\partial F=\sum_{k=1}^{g}(A_k-B_k-A_k'+B_k')+\sum_{i=1}^{m+n}(C_i-C_i'),$$

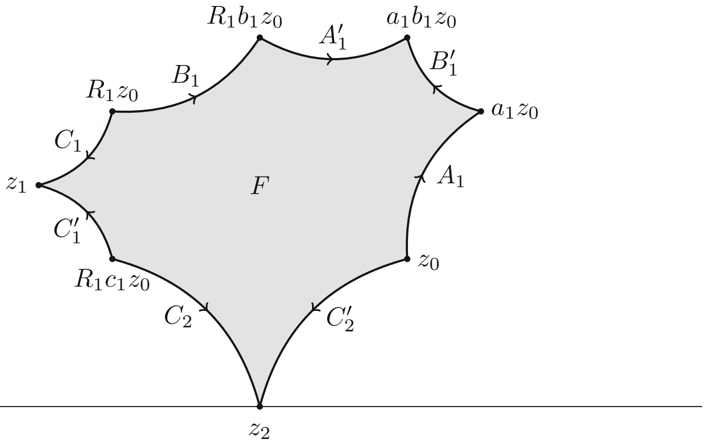

Fig. 2 Fundamental domain for a group Γ of signature (1;1,6)

where

$$C_i = (R_{g+i-1}z_0, z_i), \quad C_i' = (R_{g+i}z_0, z_i), \quad C_i = \gamma_i(C_i'), \quad i = 1, \ldots, m+n.$$

The generators $\alpha_k, \beta_k, k = 1, \ldots, g$, and $\gamma_i, i = 1, \ldots, m+n$, are dual generators of Γ satisfying

$$\mathcal{R}_g \gamma_1 \cdots \gamma_{m+n} = 1.$$

We have $C_i = (\mathcal{R}_{g+i-1}^{-1} z_0, z_i)$ and

$$\partial F = \sum_{k=1}^{N} (S_k - \lambda_k(S_k)), \quad N = 2g + m + n, \tag{22}$$

where $S_{2g+i} = C_i, \lambda_{2g+i} = \gamma_i^{-1}, i = 1, \ldots, m+n$.

4.2 The main formula

Here we obtain another representation for the symplectic form ω. Put $F^Y = \{z \in F : \operatorname{Im}(\sigma_j^{-1}) \leq Y, j = 1, \ldots, n\}$, where $\sigma_j^{-1}(x_j) = \infty$, and denote by $H_j(Y)$ corresponding horocycles in F. We have

$$\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) = \frac{\sqrt{-1}}{2} \lim_{Y \to \infty} \int_{F^Y} (\dot{q}_1 \mu_2 - \dot{q}_2 \mu_1) dz \wedge d\bar{z}.$$

Lemma 2 *The symplectic form ω, evaluated on two tangent vectors $(\dot{\theta}_1, \mu_1)$ and $(\dot{\theta}_2, \mu_2)$ corresponding to the curves $\theta_1(t)$ and $\theta_2(t)$, is given by*

$$\omega((\dot{\theta}_1,\mu_1),(\dot{\theta}_2,\mu_2)) = \frac{\sqrt{-1}}{4}\int_{\partial F}\left\{\left(\dot{q}_2\frac{\dot{g}_1}{f'}-\dot{q}_1\frac{\dot{g}_2}{f'}\right)dz+\left(B_q\left[\mu_2,\frac{\dot{g}_1}{f'}\right]-B_q\left[\mu_1,\frac{\dot{g}_2}{f'}\right]\right)d\bar{z}\right\}.$$

Proof Denote the 1-form under the integral by ϑ. We have, using Lemma 1,

$$\begin{aligned} d\vartheta &= \left(\dot{q}_{2\,\bar{z}}\frac{\dot{g}_1}{f'}+\dot{q}_2\left(\frac{\dot{g}_1}{f'}\right)_{\bar{z}}-\dot{q}_{1\,\bar{z}}\frac{\dot{g}_2}{f'}-\dot{q}_1\left(\frac{\dot{g}_2}{f'}\right)_{\bar{z}}\right)d\bar{z}\wedge dz \\ &\quad+\left(\Lambda_q(\mu_2)\frac{\dot{g}_1}{f'}+\mu_2\Lambda_q\left(\frac{\dot{g}_1}{f'}\right)-\Lambda_q(\mu_1)\frac{\dot{g}_2}{f'}-\mu_1\Lambda_q\left(\frac{\dot{g}_2}{f'}\right)\right)dz\wedge d\bar{z} \\ &= \left(\dot{q}_{2\,\bar{z}}\frac{\dot{g}_1}{f'}+\dot{q}_2\mu_1-\dot{q}_{1\,\bar{z}}\frac{\dot{g}_2}{f'}-\dot{q}_1\mu_2\right)d\bar{z}\wedge dz \\ &\quad+\left(\dot{q}_{2\,\bar{z}}\frac{\dot{g}_1}{f'}+\mu_2\dot{q}_1-\dot{q}_{1\,\bar{z}}\frac{\dot{g}_2}{f'}-\mu_1\dot{q}_2\right)dz\wedge d\bar{z} \\ &= 2(\dot{q}_1\mu_2-\dot{q}_2\mu_1)dz\wedge d\bar{z}. \end{aligned}$$

Since due to exponential decay of $\dot{q}_1,\dot{q}_2$ and μ_1,μ_2 at the cusps the integrals over horocycles $H_j(Y)$ tend to 0 as $Y\to\infty$, by Stokes' theorem we get (4). □

The line integral over ∂F in Lemma 2 can be evaluated explicitly.

Proposition 1 *We have*

$$\omega((\dot{\theta}_1,\mu_1),(\dot{\theta}_2,\mu_2)) = \frac{\sqrt{-1}}{4}\sum_{i=1}^{N}\left(B_q\left[\frac{\dot{g}_2}{f'},\frac{1}{f'}\chi_1(\lambda_i^{-1})\circ f\right]-B_q\left[\frac{\dot{g}_1}{f'},\frac{1}{f'}\chi_2(\lambda_i^{-1})\circ f\right]\right)\Bigg|_{\partial S_i(0)}^{\partial S_i(1)}.$$

Proof Using Lemma 2, formula (22), Lemma 1 and property **B3**, we get

$$\begin{aligned} &\frac{4}{\sqrt{-1}}\omega((\dot{\theta}_1,\mu_1),(\dot{\theta}_2,\mu_2)) \\ &= \sum_{i=1}^{N}\left(\int_{S_i}-\int_{\lambda_i(S_i)}\right)\left\{\left(\dot{q}_2\frac{\dot{g}_1}{f'}-\dot{q}_1\frac{\dot{g}_2}{f'}\right)dz+\left(B_q\left[\mu_2,\frac{\dot{g}_1}{f'}\right]-B_q\left[\mu_1,\frac{\dot{g}_2}{f'}\right]\right)d\bar{z}\right\} \\ &= \sum_{i=1}^{N}\int_{S_i}\left\{\left(\dot{q}_2\,\frac{1}{f'}\chi_1(\lambda_i^{-1})\circ f-\dot{q}_1\,\frac{1}{f'}\chi_2(\lambda_i^{-1})\circ f\right)dz\right. \\ &\quad\left.+\left(B_q\left[\mu_2,\frac{1}{f'}\chi_1(\lambda_i^{-1})\circ f\right]-B_q\left[\mu_1,\frac{1}{f'}\chi_2(\lambda_i^{-1})\circ f\right]\right)d\bar{z}\right\}. \end{aligned}$$

Using Lemma 1 and properties **Λ1** and **Λ5**, we obtain

$$B_q\left[\mu,\frac{1}{f'}\chi(\lambda_i^{-1})\circ f\right]=\frac{\partial}{\partial\bar{z}}B_q\left[\frac{\dot{g}}{f'},\frac{1}{f'}\chi(\lambda_i^{-1})\circ f\right]$$

and

$$\frac{\partial}{\partial z}B_q\left[\frac{\dot{g}}{f'}, \frac{1}{f'}\chi(\lambda_i^{-1})\circ f\right] = \Lambda_q\left(\frac{\dot{g}}{f'}\right)\frac{1}{f'}\chi(\lambda_i^{-1})\circ f = \dot{q}\,\frac{1}{f'}\chi(\lambda_i^{-1})\circ f.$$

Since

$$\Phi_{\bar{z}}d\bar{z} = d\Phi - \Phi_z dz,$$

we finally get (note how the signs match)

$$\begin{aligned}
&\frac{4}{\sqrt{-1}}\omega((\dot{\theta}_1,\mu_1),(\dot{\theta}_2,\mu_2))\\
&= \sum_{i=1}^{N}\int_{S_i}\left(dB_q\left[\frac{\dot{g}_2}{f'}, \frac{1}{f'}\chi_1(\lambda_i^{-1})\circ f\right] - dB_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\lambda_i^{-1})\circ f\right]\right)\\
&= \sum_{i=1}^{N}\left(B_q\left[\frac{\dot{g}_2}{f'}, \frac{1}{f'}\chi_1(\lambda_i^{-1})\circ f\right] - B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\lambda_i^{-1})\circ f\right]\right)\Bigg|_{\partial S_i(0)}^{\partial S_i(1)}.
\end{aligned}$$

According to Corollary 2, $B_q\left[\frac{\dot{g}}{f'}, \frac{1}{f'}\chi(\lambda_i^{-1})\circ f\right](z)$ has a limit as z approaches the cusps for Γ. □

4.3 Main result

Theorem 1 *The pull-back of the Goldman symplectic form on $\mathscr{K}$ by the map $\mathcal{Q}$ is $\sqrt{-1}$ times canonical symplectic form on $\mathscr{M}$,*

$$\omega = -\sqrt{-1}\mathcal{Q}^*(\omega_{\mathrm{G}}).$$

Proof Since the choice of a base point for $\mathcal{T}$ is inessential (see, Sect. 2.1.2), it is sufficient to compute the pullback only for the points in $\mathcal{Q}(q,0)$. For the convenience of the reader, consider first the case $m = n = 0$, when $N = 2g$. Using property **B2** and Eqs. (7)–(8), we have for arbitrary $\alpha, \beta \in \Gamma$,

$$\begin{aligned}
&B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\alpha)\circ f\right](\beta z_0) = B_q\left[\frac{1}{\beta'}\left(\frac{\dot{g}_1}{f'}\right)\circ\beta, \frac{1}{\beta'}\left(\frac{1}{f'}\chi_2(\alpha)\circ f\right)\circ\beta\right](z_0)\\
&= B_q\left[\frac{\dot{g}_1}{f'} - \frac{1}{f'}\chi_1(\beta^{-1})\circ f, \frac{1}{f'}\chi_2(\beta^{-1}\alpha)\circ f - \frac{1}{f'}\chi_2(\beta^{-1})\circ f\right](z_0)\\
&= B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}(\chi_2(\beta^{-1}\alpha) - \chi_2(\beta^{-1}))\circ f\right](z_0)\\
&\quad + B_0[\chi_1(\beta^{-1}), \chi_2(\beta^{-1}) - \chi_2(\beta^{-1}\alpha)](z_0).
\end{aligned}$$

Using (5), (7) and Adρ invariance of the Killing form, we obtain

$$B_0[\chi_1(\beta^{-1}), \chi_2(\beta^{-1}) - \chi_2(\beta^{-1}\alpha)](z_0) = 2\langle \chi_1(\beta^{-1}), \rho(\beta^{-1})\chi_2(\alpha)\rangle \\ = -2\langle \chi_1(\beta), \chi_2(\alpha)\rangle,$$

so that

$$B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\alpha)\circ f\right](\beta z_0) \\ = B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}(\chi_2(\beta^{-1}\alpha) - \chi_2(\beta^{-1}))\circ f\right](z_0) - 2\langle \chi_1(\beta), \chi_2(\alpha)\rangle. \quad (23)$$

Now for $i = k$ using (23) for $\alpha = \alpha_k$, $\beta = \beta_k^{-1}\mathcal{R}_k^{-1}$ and $\alpha = \alpha_k$, $\beta = \mathcal{R}_{k-1}^{-1}$, we obtain

$$B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\lambda_k^{-1})\circ f\right]\Bigg|_{\partial S_k(0)}^{\partial S_k(1)} \\ = B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\left(\chi_2(\mathcal{R}_k\beta_k\alpha_k) - \chi_2(\mathcal{R}_k\beta_k) - \chi_2(\mathcal{R}_{k-1}\alpha_k) + \chi_2(\mathcal{R}_{k-1})\right)\circ f\right](z_0) \\ -2\langle \chi_1(\beta_k^{-1}\mathcal{R}_k^{-1}) - \chi_1(\mathcal{R}_{k-1}^{-1}), \chi_2(\alpha_k)\rangle. \quad (24)$$

For $i = k + g$ we use $\alpha = \beta_k$, $\beta = \mathcal{R}_k^{-1}$ and $\alpha = \beta_k$, $\beta = \alpha_k^{-1}\mathcal{R}_{k-1}^{-1}$ to compute

$$B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\lambda_{i+k}^{-1})\circ f\right]\Bigg|_{\partial S_{i+k}(0)}^{\partial S_{i+k}(1)} \\ = B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\left(\chi_2(\mathcal{R}_k\beta_k) - \chi_2(\mathcal{R}_k) - \chi_2(\mathcal{R}_{k-1}\alpha_k\beta_k) + \chi_2(\mathcal{R}_{k-1}\alpha_k)\right)\circ f\right](z_0) \\ -2\langle \chi_1(\mathcal{R}_k^{-1}) - \chi_1(\alpha_k^{-1}\mathcal{R}_{k-1}^{-1}), \chi_2(\beta_k)\rangle. \quad (25)$$

Since $\mathcal{R}_{k-1}\alpha_k\beta_k = \mathcal{R}_k\beta_k\alpha_k$ and $\mathcal{R}_g = 1$, we see that the sum over k of terms in the second lines in Eqs. (24)–(25) vanishes. Using (12)–(13) and Remark 8, we get

$$\sum_{i=1}^{2g} B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\lambda_i^{-1})\circ f\right]\Bigg|_{\partial S_i(0)}^{\partial S_i(1)} \\ = 2\sum_{k=1}^{g}\Big(\langle \chi_1(\mathcal{R}_{k-1}^{-1}) - \chi_1(\beta_k^{-1}\mathcal{R}_k^{-1}), \chi_2(\alpha_k)\rangle + \langle \chi_1(\alpha_k^{-1}\mathcal{R}_{k-1}^{-1}) - \chi_1(\mathcal{R}_k^{-1}), \chi_2(\beta_k)\rangle\Big) \\ = 2\omega_G(\chi_1, \chi_2).$$

Similarly,

$$\sum_{i=1}^{2g} B_q\left[\frac{\dot{g}_2}{f'}, \frac{1}{f'}\chi_1(\lambda_i^{-1})\circ f\right]\Bigg|_{\partial S_i(0)}^{\partial S_i(1)} = -2\omega_G(\chi_2,\chi_1)$$

and we finally obtain

$$\omega((\dot{\theta}_1,\mu_1),(\dot{\theta}_2,\mu_2)) = -\sqrt{-1}\omega_{\rm G}(\chi_1,\chi_2).$$

In general, assume that $m+n>0$. In this case

$$\sum_{i=1}^{2g} B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\lambda_i^{-1})\circ f\right]\Bigg|_{\partial S_i(0)}^{\partial S_i(1)} = -B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\mathcal{R}_g)\circ f\right](z_0)$$
$$+2\sum_{k=1}^{g}\Big(\langle\chi_1(\mathcal{R}_{k-1}^{-1})-\chi_1(\beta_k^{-1}\mathcal{R}_k^{-1}),\chi_2(\alpha_k)\rangle+\langle\chi_1(\alpha_k^{-1}\mathcal{R}_{k-1}^{-1})-\chi_1(\mathcal{R}_k^{-1}),\chi_2(\beta_k)\rangle\Big), \tag{26}$$

and we need to compute

$$\sum_{i=1}^{m+n} B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\gamma_i)\circ f\right]\Bigg|_{\mathcal{R}_{g+i-1}^{-1}z_0}^{z_i}.$$

Using (23) with $\alpha=\gamma_i$ and $\beta=\mathcal{R}_{g+i-1}^{-1}$, we get

$$B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\gamma_i)\circ f\right](\mathcal{R}_{g+i-1}^{-1}z_0)$$
$$= B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\big(\chi_2(\mathcal{R}_{g+i})-\chi_2(\mathcal{R}_{g+i-1})\big)\circ f\right](z_0)+2\langle\chi_1(\mathcal{R}_{g+i-1}^{-1}),\chi_2(\gamma_i)\rangle.$$

Since restriction of χ_2 to the stabilizer $\Gamma_i=\langle\gamma_i\rangle$ of a fixed point z_i is a coboundary, there is $P_{2i}\in\mathscr{P}_2$ such that

$$\chi_2(\gamma_i)=\rho(\gamma_i)P_{2i}-P_{2i}.$$

Using property **B2**, $\gamma_i z_i = z_i$ and (5), we get

$$B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\gamma_i)\circ f\right](z_i) = B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{(\gamma_i^{-1})'}\left(\frac{1}{f'}P_{2i}\circ f\right)\circ\gamma_i^{-1}-\frac{1}{f'}P_{2i}\circ f\right](z_i)$$
$$= B_q\left[\frac{1}{\gamma_i'}\frac{\dot{g}_1}{f'}\circ\gamma_i-\frac{\dot{g}_1}{f'}, \frac{1}{f'}P_{2i}\circ f\right](z_i)$$
$$= -B_0[\chi_1(\gamma_i^{-1}),P_{2i}](z_i) = 2\langle\chi_1(\gamma_i^{-1}),P_{2i}\rangle.$$

Thus using $\mathcal{R}_{g+m+n} = 1$ we obtain

$$\sum_{i=1}^{m+n} B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\gamma_i)\circ f\right]\Bigg|_{\mathcal{R}^{-1}_{g+i-1}z_0}^{z_i}$$
$$= B_q\left[\frac{\dot{g}_1}{f'}, \frac{1}{f'}\chi_2(\mathcal{R}_g)\circ f\right](z_0) + 2\sum_{i=1}^{m+n}\Big(\langle\chi_1(\mathcal{R}^{-1}_{g+i-1}), \chi_2(\gamma_i)\rangle + \langle\chi_1(\gamma_i^{-1}), P_{2i}\rangle\Big). \tag{27}$$

Putting together formulas (26)–(27) and using (14)–(15), we finally obtain

$$\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) = -\sqrt{-1}\,\omega_{\mathrm{G}}(\chi_1, \chi_2).$$

□

Remark 9 The above computation is a non-abelian analog of Riemann bilinear relations, which arise from the isomorphism

$$\mathcal{H}^1(X, \mathbb{C})/\mathcal{H}^1(X, \mathbb{Z}) \xrightarrow{\sim} \mathscr{K}_{\mathrm{ab}},$$

where $\mathcal{H}^1(X, \mathbb{C})$ is the complex vector space of harmonic 1-forms on X and $\mathscr{K}_{\mathrm{ab}} = (\mathbb{C}^*)^{2g}$ is the complex torus—a character variety for the abelian group $G = \mathbb{C}^*$.

Combing Theorem 1 and Remark 7, we get a a generalization of Goldman's theorem [6, Sect. 2.5] for the case of orbifold Riemann surfaces.

Corollary 3 *The pullback of the Goldman symplectic form on the character variety $\mathscr{K}_{\mathbb{R}}$ by the map $\mathcal{F}$ is a symplectic form of the Weil–Petersson metric on $\mathcal{T}$,*

$$\omega_{\mathrm{WP}} = \mathcal{F}^*(\omega_{\mathrm{G}}).$$

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