NORMAL MATRIX MODELS, $\bar{\partial}$-PROBLEM, AND ORTHOGONAL POLYNOMIALS ON THE COMPLEX PLANE

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ABSTRACT. We introduce a $\bar{\partial}$-formulation of the orthogonal polynomials on the complex plane, and hence of the related normal matrix model, which is expected to play the same role as the Riemann-Hilbert formalism in the theory of orthogonal polynomials on the line and for the related Hermitian model. We propose an analog of Deift-Kriecherbauer-McLaughlin-Venakides-Zhou asymptotic method for the analysis of the relevant $\bar{\partial}$-problem, and indicate how familiar steps for the Hermitian model, e.g. the g-function "undressing", might look like in the case of the normal model. We use the particular model considered recently by P. Elbau and G. Felder as a case study.

1. Introduction

In these notes we attempt to develop for the normal matrix model a formalism analogous to the Riemann-Hilbert method in the theory of Hermitian matrix model. As in the latter case, the starting point is proper analytical characterization of the relevant orthogonal polynomials. Unlike the Hermitian matrix model, the orthogonality condition for the polynomials associated with the normal model is formulated with respect to a measure on the plane. This, as we will see below, leads to the replacement of the Riemann-Hilbert problem of [4] by a certain $\bar{\partial}$-problem. We shall present in detail the setting of the $\bar{\partial}$-problem for the case of what we will call in these notes the *Elbau-Felder model.* This model arises as a natural regularization of the normal matrix model of P. Wiegmann and A. Zabrodin in [7, 8] by restricting the matrix integral of the latter to normal matrices whose eigenvalues lie in a compact domain D of the complex plane. Using the Elbau-Felder model as a case study, we shall also outline a possible $\bar{\partial}$-version of the Deift-Kriecherbauer-McLaughlin-Venakides-Zhou (DKMVZ) asymptotic model. The DKMVZ method proved to be very efficient in the asymptotic analysis of the oscillatory Riemann-Hilbert problems appearing in Hermitian matrix model. We have not yet succeeded in providing complete generalization of the DKMVZ scheme for the orthogonal polynomials on the plane; in fact, we rather highlighted the challenging difficulties to be overcome. We hope, however, that these notes might stimulate further development of the analog DKMVZ asymptotic method for the orthogonal polynomials on the plane and related normal matrix models.

2. Preliminaries

2.1. **Normal matrix models and orthogonal polynomials.** Let D be a bounded domain on the complex plane $\mathbb{C}$ containing the origin, and let $V(z)$ be a real-valued smooth function on $\mathbb{C}$. Following P. Elbau and G. Felder [3], we shall study the normal matrtix model characterized by the partition function Z_N defined by the following N-fold integral,

$$Z_N = \underset{D^N}{\int \cdots \int} \prod_{i\neq j} |z_i - z_j|^2 e^{-N\sum_{k=1}^N V(z_k)} d^2 z_1 \cdots d^2 z_N.$$

Let χ_D be the characteristic function of the domain D.

Definition 1. Orthogonal polynomials on $\mathbb{C}$ with respect to the measure $e^{-NV(z)}\chi_D(z)d^2z$ are polynomials $P_n(z) = z^n + a_{n-1n}z^{n-1} + \cdots + a_{0n}$, satisfying

$$\iint_D P_n(z)\overline{P_m(z)}e^{-NV(z)}d^2z = h_n\delta_{mn} \quad \text{for all} \quad m,n = 0,1,2\dots. \tag{2.1}$$

The following lemma is standard.

Lemma 2.

$$Z_N = N!\prod_{n=0}^{N-1} h_n.$$

The proof is exactly the same as in the case of the Hermitian model (see e.g. [1]). As in the case of the Hermitian model, Lemma 2 reduces the question of the asymptotic analysis of the partition function Z_N as $N\to\infty$ to the asymptotic analysis of the orthogonal polynomials $P_n(z)$ as $n, N\to\infty$.

3. Matrix $\bar\partial$-problem

Using the orthogonal polynomials on the line as an analogy (see [4], [1]), we set

$$Y_n(z) = \begin{pmatrix} P_n(z) & \frac{1}{\pi}\iint_D \frac{\overline{P_n(z')}}{z'-z}e^{-NV(z')}d^2z' \\ -\frac{\pi}{h_{n-1}}P_{n-1}(z) & -\frac{1}{h_{n-1}}\iint_D \frac{\overline{P_{n-1}(z')}}{z'-z}e^{-NV(z')}d^2z' \end{pmatrix}. \tag{3.1}$$

It follows from the formula

$$\frac{\partial}{\partial\bar z}\frac{1}{z-z'} = \pi\delta(z-z'),$$

understood in the distributional sense, that

$$\frac{\partial}{\partial\bar z}Y_n(z) = \overline{Y_n(z)}(I - G(z)), \tag{3.2}$$

where I is 2×2 identity matrix and

$$G(z)=\begin{pmatrix}1 & e^{-NV(z)}\chi_D(z)\\ 0 & 1\end{pmatrix} \tag{3.3}$$

The following proposition is central (cf. the case of the orthogonal polynomials on the line).

Proposition 3. *The matrix $Y_n(z)$ is the unique solution of the $\bar\partial$-problem (3.2)–(3.3) with the normalization*

$$Y_n(z)=\left(I+O\left(\frac{1}{z}\right)\right)\begin{pmatrix}z^n & 0\\ 0 & z^{-n}\end{pmatrix}\equiv\left(I+O\left(\frac{1}{z}\right)\right)z^{n\sigma_3}, \tag{3.4}$$

as $|z|\to\infty$, where $\sigma_3=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

Proof. It follows from the geometric series expansion

$$\frac{1}{z-z'}=\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z'}{z}\right)^k$$

as $|z|\to\infty$, and the property (2.1), rewritten as

$$\iint_D P_n(z)\bar z^m e^{-NV(z)}d^2z=h_n\delta_{mn}, \tag{3.5}$$

that the matrices (3.1) satisfy normalization (3.4).

Conversely, suppose that the matrix $Y(z)$ solves the $\bar\partial$-problem (3.2) with the asymptotics (3.4). It follows from the special form (3.3) of the matrix $G(z)$ that $(Y_n)_{11}(z)=P_n(z)$ and $(Y_n)_{21}(z)=Q_{n-1}(z)$ — polynomials of orders n and $n-1$ respectively, and

$$\begin{aligned}\frac{\partial}{\partial\bar z}(Y_n)_{12}(z)&=-\overline{P_n(z)}e^{-NV(z)}\chi_D(z),\\ \frac{\partial}{\partial\bar z}(Y_n)_{22}(z)&=-\overline{Q_{n-1}(z)}e^{-NV(z)}\chi_D(z).\end{aligned}$$

Now it follows from normalization (3.4) that the leading coefficient of polynomial $P_n(z)$ is 1 and

$$\begin{aligned}(Y_n)_{12}(z)&=\frac{1}{\pi}\iint_D\frac{\overline{P_n(z')}}{z'-z}e^{-NV(z')}d^2z',\\ (Y_n)_{22}(z)&=\frac{1}{\pi}\iint_D\frac{\overline{Q_{n-1}(z')}}{z'-z}e^{-NV(z')}d^2z'.\end{aligned}$$

Using geometric series and normalization (3.4) once again, we obtain that polynomials P_n and Q_{n-1} satisfy

$$\iint\limits_D P_n(z)\bar{z}^m e^{-NV(z)} d^2z = 0 \qquad \text{for} \quad m < n,$$

$$-\iint\limits_D Q_{n-1}(z)\bar{z}^{m-1} e^{-NV(z)} d^2z = \pi\delta_{mn} \quad \text{for} \quad m \leq n.$$

From here it follows that

$$\iint\limits_D P_n(z)\overline{P_m(z)} e^{-NV(z)} d^2z = 0 \quad \text{for all} \quad m < n,$$

which is sufficient to conclude that $P_n(z)$ are orthogonal polynomials on $\mathbb{C}$ with the weight $e^{-NV(z)}\chi_D(z)$. Finally, polynomials $Q_n(z)$ satisfy

$$\iint\limits_D Q_{n-1}(z)\overline{P_{m-1}(z)} e^{-NV(z)} d^2z = -\pi\delta_{mn} \quad \text{for all} \quad m \leq n,$$

so that $Q_n(z) = -\dfrac{\pi}{h_n} P_n(z)$. □

Similar to the case of the usual orthogonal polynomials, Proposition 3 reduces the asymptotic analysis of the orthogonal polynomials (2.1) to the asymptotic analysis of the solution of the $\bar{\partial}$-problem (3.2)-(3.4).

4. Elbau-Felder potential. Towards a normal matrix version of the DKMVZ asymptotic approach.

We will consider the matrix model with the weight $e^{-NV(z)}\chi_D(z)$, where $V(z)$ is the Elbau-Felder [3] potential

$$V(z) = \frac{1}{t_0}\left(|z|^2 - 2\,\mathrm{Re}\sum_{k=1}^{n+1} t_k z^k\right), \tag{4.1}$$

where $t_1 = 0$, $|t_2| < 1/2$ and $t_0V(z)$ is positive on $D \setminus \{0\}$. Using again the Hermitian matrix model analogy, we shall expect that a fundamental role in the asymptotic analysis of the $\bar{\partial}$-problem (3.2)-(3.4) will be played by the *equilibrium measure*.

Definition 4. An equilibrium measure for V on D is a Borel probability measure μ on D without point masses so that

$$I(\mu) = \inf I(\nu), \quad \nu \subset \mathbb{M}(D),$$

where $\mathbb{M}(D)$ is the set of all Borel probability measures μ on D without point masses, and the functional $I(\nu)$ is defined by the equation,

$$I(\nu) := \int V(z)d\nu(z) + \int\int_{z\neq\zeta} \log|z-\zeta|^{-1} d\nu(z)d\nu(\zeta).$$

Theorem 5 (Elbau-Felder [3]). *There is $\delta > 0$ such that for all $0 < t_0 < \delta$ the unique equilibrium measure $d\mu$ exists and is given by*

$$d\mu(z) = \frac{1}{\pi t_0}\chi_{D_+}(z)d^2z,$$

where the domain $D_+ \subset D$ contains the origin and has the property that

$$t_0 = \frac{1}{\pi}\iint\limits_{D_+} d^2z,$$

$$t_k = -\frac{1}{\pi k}\iint\limits_{\mathbb{C}\setminus D_+} z^{-k}d^2z = \frac{1}{2\pi i k}\oint\limits_{\partial D_+} \bar{z}z^{-k}dz, \quad k = 1, \dots, n+1,$$

$$t_k = 0, \quad j > n+1.$$

These relations determines D_+ uniquely. In fact, the boundary Γ of D_+ is a polynomial curve of degree n, i.e. Γ is a smooth simple closed curve in the complex plane with a parametrization $h : S^1 \subset \mathbb{C} \to \mathbb{C}$ of the form,

$$h(w) = rw + a_0 + a_1w^{-1} + ... + a_nw^{-n}, \quad |w| = 1,$$

with $r > 0$ and $a_n \neq 0$. The equilibrium measure has the following properties. Set

$$E(z) = V(z) + 2\iint \log|z-\zeta|^{-1}d\mu(\zeta).$$

1. *$E(z) = E_0$ — a constant — for $z \in D_+$.*
2. *$E(z) \geq E_0$ — for $z \in D \setminus D_+$.*

4.1. **A naive DKMVZ scheme.** Suppose that there is an analytic function $g(z)$ with the following properties.

1. $g(z) = \log z + O\left(\frac{1}{z}\right)$ as $|z| \to \infty$.
2. $V(z) - g(z) - \overline{g(z)} = E_0$ on D_+.
3. $V(z) - g(z) - \overline{g(z)} > E_0$ on $\mathbb{C} \setminus D_+$.

Such function $g(z)$ could be used to study the asymptotics of the matrix $Y_n(z)$ in the limit,

$$n, N \to \infty, \quad \frac{n}{N} = \gamma, \quad \gamma \text{ is fixed}, \tag{4.2}$$

in exactly the same manner is it is done in the case of the orthogonal polynomials in the line (see [2] and [1]). Namely, set

$$V_\gamma(z) = \frac{1}{\gamma}V(z)$$

and consider corresponding equilibrium measure $d\mu_\gamma$ (assuming that $0 < \gamma t_0 < \delta$) with the domain $D_+(\gamma)$ and the corresponding function $g_\gamma(z)$ satisfying properties **1–3** (with D_+ and E_0 replaced by $D_+(\gamma)$ and $E_0(\gamma)$). Then we can "undress" the $\bar{\partial}$-problem (3.2)–(3.3) with the normalization (3.4) by setting

$$Y_n(z) = e^{-\frac{nE_0(\gamma)}{2}\sigma_3}\Psi_n(z)e^{ng_\gamma(z)\sigma_3 + \frac{nE_0(\gamma)}{2}\sigma_3}.$$

The resulting matirx $\Psi_n(z)$ satisfies the simplified $\bar\partial$-problem

$$\frac{\partial}{\partial \bar z}\Psi_n(z) = \overline{\Psi_n(z)}\begin{pmatrix} 0 & -e^{-n(V_\gamma(z)-g_\gamma(z)-\overline{g_\gamma(z)}-E_0(\gamma))} \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

with the standard normalization

$$\Psi_n(z) = I + O\left(\frac{1}{z}\right) \quad \text{as} \quad |z| \to \infty, \tag{4.4}$$

which follows from property **1** of the function $g_\gamma(z)$.

It is easy to pass to limit (4.2) in the $\bar\partial$-problem (4.3)–(4.4). Indeed, it follows from properties **2-3** of the function $g_\gamma(z)$ that

$$\lim_{n,N\to\infty} \Psi_n(z) = \Psi_0(z),$$

where the matrix $\Psi_0(z)$ satisfies the following *model $\bar\partial$-problem.*

$$\frac{\partial}{\partial \bar z}\Psi_0(z) = \overline{\Psi_0(z)}\begin{cases} \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) & z \in D_+(\gamma) \\ 0 & z \notin D_+(\gamma) \end{cases} \tag{4.5}$$

with the standard normalization

$$\Psi_0(z) = I + O\left(\frac{1}{z}\right) \quad \text{as} \quad |z| \to \infty. \tag{4.6}$$

This model $\bar\partial$-problem is easily solved explicitly,

$$\Psi_0(z) = \begin{pmatrix} 1 & \frac{1}{\pi}\iint\limits_{D_+(\gamma)} \frac{1}{\zeta - z} d^2\zeta \\ 0 & 1 \end{pmatrix}.$$

4.2. **Function $g(z)$.** Of course, the main assumption that there is an analytic function function $g(z)$ satisfying the properties **1-3** is not correct. Firstly, it follows from the property **1** that $g(z)$ in the neighborhood of infinity is defined up to an integer multiple of $2\pi i$, which is not a drawback since $e^{ng(z)\sigma_3}$ si well-defined for integer n. Secondly, the property **2** implies that the function $V(z)$ is harmonic in D_+, which clearly contradicts (4.1). Adding to the confusion is the formal manipulation

$$\log|z-\zeta|^2 = \log(z-\zeta) + \log(\bar z - \bar\zeta),$$

which suggests that

$$g(z) = \iint \log(z-\zeta)d\mu(\zeta) = \frac{1}{\pi t_0}\iint\limits_{D_+} \log(z-\zeta)d^2\zeta \tag{4.7}$$

satisfies properties **1-3**. However, this is not so since we need to treat carefully the branches of log in order to define the integral in (4.7) and investigate its analytic properties.

For this aim, consider the logarithmic potential given by the uniform distribution of charges in the domain D,

$$V_0(z) = \iint\limits_D \log|z-w|^2 d^2w.$$

Let $\Gamma = \partial D$ with fixed point $\zeta_0 \in \Gamma$. For $z \in D$ denote by $D_\varepsilon(z)$ domain obtained by removing the disk of radius ε around z, so that

$$\partial D_\varepsilon(z) = \Gamma \cup -C_\varepsilon(z),$$

where $C_\varepsilon(z)$ is the circle $|w-z| = \varepsilon$ oriented counter-closkwise, and the minus sign denotes negative orientation. Since

$$\log|w-z|^2 dw \wedge d\bar{w} = -d(\log|w-z|^2 \bar{w}dw) - \frac{\bar{w}}{\bar{w}-\bar{z}}\, dw \wedge d\bar{w},$$

by Stokes' theorem, we have

$$\begin{aligned}
V_0(z) &= \frac{i}{2}\iint\limits_D \log|z-w|^2 dw \wedge d\bar{w} \\
&= \frac{i}{2}\lim_{\varepsilon\to 0}\iint\limits_{D_\varepsilon(z)} \left(-d(\log|w-z|^2\bar{w}dw) - \frac{\bar{w}}{\bar{w}-\bar{z}}\, dw \wedge d\bar{w}\right) \\
&= \frac{1}{2i}\lim_{\varepsilon\to 0}\iint\limits_{D_\varepsilon(z)} d\left(\log|w-z|^2\bar{w}dw + \frac{\bar{w}}{\bar{w}-\bar{z}}\, wd\bar{w}\right) \\
&= \frac{1}{2i}\lim_{\varepsilon\to 0}\oint\limits_{\partial D_\varepsilon(z)} \left(\log|\zeta-z|^2\bar{\zeta}d\zeta + \frac{\bar{\zeta}}{\bar{\zeta}-\bar{z}}\,\zeta d\bar{\zeta}\right).
\end{aligned}$$

Now

$$\lim_{\varepsilon\to 0}\int\limits_{C_\varepsilon(z)} \log|\zeta-z|^2\bar{\zeta}d\zeta = 0, \qquad \lim_{\varepsilon\to 0}\int\limits_{C_\varepsilon(z)} \frac{\bar{\zeta}}{\bar{\zeta}-\bar{z}}\,\zeta d\bar{\zeta} = -2\pi i|z|^2,$$

so that

$$V_0(z) = \pi|z|^2 + \frac{1}{2i}\oint\limits_\Gamma \left(\log|\zeta-z|^2\bar{\zeta}d\zeta + \frac{\bar{\zeta}}{\bar{\zeta}-\bar{z}}\,\zeta d\bar{\zeta}\right).$$

Set $\omega = \bar{w}dw$ and define a function Ω on $\Gamma \setminus \{\zeta_0\}$ by

$$\Omega(\zeta) = \int_{\zeta_0}^{\zeta} \omega,$$

where the integration is along the oriented path in Γ connecting points ζ_0 and ζ. We have $\Omega_-(\zeta_0) = 0$ for a path consisting of a single point ζ_0, and

$$\Omega_+(\zeta_0) = \oint\limits_\Gamma \bar{w}dw = \iint\limits_D d\bar{w}\wedge dw = 2iA(D),$$

for the loop Γ starting and ending at ζ_0, where $A(D)$ is the area of D. Thus

$$\oint_\Gamma \log|\zeta - z|^2 \zeta d\zeta = \oint_\Gamma \log|\zeta - z|^2 d\Omega(\zeta) = \Delta(\log|z-\zeta|^2\Omega(\zeta))\Big|_{\zeta_0}^{\zeta_0}$$

$$- \oint_\Gamma \Omega(\zeta)\left(\frac{d\zeta}{\zeta - z} + \frac{d\bar\zeta}{\bar\zeta - \bar z}\right)$$

$$= 2iA(D)\log|z-\zeta_0|^2 - \oint_\Gamma \Omega(\zeta)\left(\frac{d\zeta}{\zeta - z} + \frac{d\bar\zeta}{\bar\zeta - \bar z}\right),$$

so that

$$V_0(z) = \pi|z|^2 + A(D)\log|z-\zeta_0|^2 + \frac{i}{2}\oint_\Gamma\left(\frac{\Omega(\zeta)}{\zeta - z}\,d\zeta + \frac{\Omega(\zeta)}{\bar\zeta - \bar z}\,d\bar\zeta - \frac{\bar\zeta}{\bar\zeta - \bar z}\,\zeta d\bar\zeta\right).$$

Since the potential V_0 is real-valued, we have

$$V_0(z) = \pi|z|^2 + A(D)\log|z-\zeta_0|^2 + \frac{i}{4}\oint_\Gamma\left(\frac{\Omega(\zeta)}{\zeta - z}\,d\zeta - \frac{\overline{\Omega(\zeta)}}{\bar\zeta - \bar z}\,d\bar\zeta\right.$$

$$\left. - \frac{\overline{\Omega(\zeta)}}{\zeta - z}\,d\zeta + \frac{\Omega(\zeta)}{\bar\zeta - \bar z}\,d\bar\zeta + \frac{\zeta}{\zeta - z}\,\bar\zeta d\zeta - \frac{\bar\zeta}{\bar\zeta - \bar z}\,\zeta d\bar\zeta\right)$$

$$= \pi|z|^2 + A(D)\log|z-\zeta_0|^2 + \frac{i}{4}\oint_\Gamma\left(\frac{\Omega(\zeta) - \overline{\Omega(\zeta)} + |\zeta|^2}{\zeta - z}\,d\zeta\right.$$

$$\left. - \frac{\overline{\Omega(\zeta)} - \Omega(\zeta) + |\zeta|^2}{\bar\zeta - \bar z}\,d\bar\zeta\right).$$

Finally, observing that $\omega + \bar\omega = d|w|^2$, we get

$$\Omega(\zeta) + \overline{\Omega(\zeta)} = \int_{\zeta_0}^{\zeta} d|w|^2 = |\zeta|^2 - |\zeta_0|^2,$$

so that

$$\Omega(\zeta) - \overline{\Omega(\zeta)} + |\zeta|^2 = 2\Omega(\zeta) + |\zeta_0|^2,$$

and we obtain

(4.8)

$$V_0(z) = \pi(|z|^2 - |\zeta_0|^2) + A(D)\log|z-\zeta_0|^2 + \frac{i}{2}\oint_\Gamma\left(\frac{\Omega(\zeta)}{\zeta - z}\,d\zeta - \frac{\overline{\Omega(\zeta)}}{\bar\zeta - \bar z}\,d\bar\zeta\right),$$

where $z \in D$. This is a desired representation of the area potential $V_0(z)$ as the real part of the first derivative of a single layer potential.

The same computation for $z \in \mathbb{C} \setminus \bar{D}$ gives

$$V_0(z) = A(D) \log |z - \zeta_0|^2 + \frac{i}{2} \oint_\Gamma \left(\frac{\Omega(\zeta)}{\zeta - z} \, d\zeta - \frac{\overline{\Omega(\zeta)}}{\bar{\zeta} - \bar{z}} \, d\bar{\zeta} \right), \tag{4.9}$$

Returning to Elbau-Felder potential $V(z)$ and setting $D = D_+$, $\Gamma = \partial D_+$, we get $E(z) = V(z) - \frac{1}{\pi t_0} V_0(z)$, so that

(4.10)

$$V(z) - \frac{1}{t_0}(|z|^2 - |\zeta_0|^2) - \log |z - \zeta_0|^2 - \frac{i}{2\pi t_0} \oint_\Gamma \left(\frac{\Omega(\zeta)}{\zeta - z} \, d\zeta - \frac{\overline{\Omega(\zeta)}}{\bar{\zeta} - \bar{z}} \, d\bar{\zeta} \right) = E_0$$

when $z \in D_+$, and

$$V(z) - \log |z - \zeta_0|^2 - \frac{i}{2\pi t_0} \oint_\Gamma \left(\frac{\Omega(\zeta)}{\zeta - z} \, d\zeta - \frac{\overline{\Omega(\zeta)}}{\bar{\zeta} - \bar{z}} \, d\bar{\zeta} \right) = E(z) \tag{4.11}$$

when $z \in D_-$.

Remark 6. We note that equation (4.10), i.e. the statement that the l.h.s. of (4.10) is constant when $z \in D_+$, is equivalent to the moment equations of Theorem 5 which determine the contour Γ (cf.[3], p.12, Lemma 6.3).

Now we are ready to introduce the function $g(z)$. Namely, set

$$g(z) = \log(z - \zeta_0) + \frac{i}{2\pi t_0} \oint_\Gamma \frac{\Omega(\zeta)}{\zeta - z} \, d\zeta. \tag{4.12}$$

The function $g(z)$ is holomorphic in $\mathbb{C} \setminus \Gamma$, is multi-valued with periods $2\pi i\mathbb{Z}$ (singe-valued on the plane with the outside cut starting from ζ_0) and has the asymptotics

$$g(z) = \log z + O(z^{-1}) \quad \text{as} \quad z \to \infty.$$

The function $e^{ng(z)}$ is single-valued for $n \in \mathbb{Z}$. The function $g(z)$ is discontinuous on Γ (by Sokhotski-Plemelj formula).

We summarize this as the following statement.

Proposition 7. *The Elbau-Felder potential $V(z)$ has the following representations*

(i) *For $z \in D_+$,*

$$V(z) - g(z) - \overline{g(z)} = E_0 + \frac{1}{t_0}(|z|^2 - |\zeta_0|^2).$$

(ii) *For $z \in D_-$,*

$$V(z) - g(z) - \overline{g(z)} = E(z).$$

(iii) *For $z \in D \setminus D_+$,*

$$V(z) - g(z) - \overline{g(z)} = E(z) > E_0. \tag{4.13}$$

4.3. **A first possible version of the DKMVZ scheme.** The correct strategy is now the following. Let $g_\gamma(z)$, $D_+(\gamma)$, $E_0(\gamma)$, etc. denote the respective objects associated with the potential $V_\gamma(z)$. We set

$$Y_n(z) = e^{-\frac{nE_0(\gamma)}{2}\sigma_3+\frac{n|\zeta_0|^2}{2\gamma t_0}\sigma_3}\Psi_n(z)e^{ng_\gamma(z)\sigma_3+\frac{nE_0(\gamma)}{2}\sigma_3-\frac{n|\zeta_0|^2}{2\gamma t_0}\sigma_3}. \tag{4.14}$$

The resulting matirx $\Psi_n(z)$ satisfies the $\bar\partial$-problem (the correct version of (4.3))

$$\frac{\partial}{\partial\bar z}\Psi_n(z) = \overline{\Psi_n(z)}\begin{pmatrix}0 & -e^{-\frac{n|z|^2}{\gamma t_0}}\\ 0 & 0\end{pmatrix}, \quad z\in D_+,$$

(4.15)

$$\frac{\partial}{\partial\bar z}\Psi_n(z) = \overline{\Psi_n(z)}\begin{pmatrix}0 & -e^{-n\left(E(z)-E_0(\gamma)+\frac{|\zeta_0|^2}{\gamma t_0}\right)}\chi_{D_+}(z)\\ 0 & 0\end{pmatrix}, \quad z\in\mathbb{C}\setminus D_+,$$

$$\Psi_{n+}(z) = \Psi_{n-}(z)e^{\frac{n}{\gamma t_0}\Omega(z)\sigma_3}, \quad z\in\Gamma\equiv\Gamma(\gamma),$$

with the standard normalization

$$\Psi_n(z) = I + O\left(\frac{1}{z}\right) \quad \text{as} \quad |z|\to\infty. \tag{4.16}$$

By virtue of condition (4.13), we expect that the limiting function $\Psi^0_n(z)$ satisfies the model problem

$$\frac{\partial}{\partial\bar z}\Psi^0_n(z) = \overline{\Psi^0_n(z)}\begin{cases}\begin{pmatrix}0 & -e^{-\frac{n|z|^2}{\gamma t_0}}\\ 0 & 0\end{pmatrix} & z\in D_+(\gamma)\\ 0 & z\notin D_+(\gamma)\end{cases} \tag{4.17}$$

$$\Psi^0_{n+}(z) = \Psi^0_{n-}(z)e^{\frac{n}{\gamma t_0}\Omega(z)\sigma_3}, \quad z\in\Gamma,$$

with the standard normalization

$$\Psi_0(z) = I + O\left(\frac{1}{z}\right) \quad \text{as} \quad |z|\to\infty. \tag{4.18}$$

The open questions are now the following.

(1) How to solve this model problem? The "unfortunate" thing is the presence of complex conjugation in (4.17). Indeed, if we neglect the jump across the contour Γ, the $\bar\partial$-problem alone can be of course solved explicitly,

$$\Psi_0(z) = \begin{pmatrix}1 & \frac{1}{\pi}\iint\limits_{D_+(\gamma)}\frac{e^{-\frac{n|\zeta|^2}{\gamma t_0}}}{\zeta - z}d^2\zeta\\ 0 & 1\end{pmatrix}, \tag{4.19}$$

and the solution won't have any jumps. If not for the complex conjugation, the function $\Psi_0(z)$ could be used to undress in the usual way problem (4.17) and reduce it to a pure Riemann-Hilbert problem.

(2) The arguments that led us to the model problem (4.17) and which are based on inequality (4.13), even on the formal level, are not very convincing: the real part of $\Omega(z)$ is $|z|^2 - |\zeta_0|^2 \neq 0$ so that the diagonal jump matrix on Γ is not pure oscillatory.

4.4. **A second possible version of the DKMVZ scheme.** The above deficiency of the proposed analog of the DKMVZ scheme can be partially overcome by performing the following modification. Let us replace the function $\Omega(\zeta)$ by the function,

$$\Omega_0(\zeta) = \frac{1}{2}\int_{\zeta_0}^{\zeta} (\bar{w}dw - wd\bar{w}).$$

The function $\Omega_0(\zeta)$ is pure imaginary on Γ and it is related to the function $\Omega(\zeta)$ by the equation,

$$\Omega_0(\zeta) = \Omega(\zeta) - \frac{|\zeta|^2 - |\zeta_0|^2}{2}.$$

By using again Stokes' theorem, we observe that

$$\frac{i}{4\pi t_0}\oint_{\Gamma} (|\zeta|^2 - |\zeta_0|^2)\left(\frac{d\zeta}{\zeta - z} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}}\right)$$
$$= -\frac{1}{2\pi t_0}\iint_{D_+}\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}}\right) d^2\zeta - \frac{1}{t_0}(|z|^2 - |\zeta_0|^2).$$

This allows to re-write equations (4.10) and (4.11) in the form,

$$V(z) - \log|z - \zeta_0|^2 - \frac{i}{2\pi t_0}\oint_{\Gamma}\left(\frac{\Omega_0(\zeta)}{\zeta - z}\,d\zeta - \frac{\overline{\Omega_0(\zeta)}}{\bar{\zeta} - \bar{z}}\,d\bar{\zeta}\right)$$
$$+\frac{1}{2\pi t_0}\iint_{D_+}\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}}\right) d^2\zeta = E_0 \tag{4.20}$$

when $z \in D_+$, and

$$V(z) - \log|z - \zeta_0|^2 - \frac{i}{2\pi t_0}\oint_{\Gamma}\left(\frac{\Omega_0(\zeta)}{\zeta - z}\,d\zeta - \frac{\overline{\Omega_0(\zeta)}}{\bar{\zeta} - \bar{z}}\,d\bar{\zeta}\right)$$
$$+\frac{1}{2\pi t_0}\iint_{D_+}\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}}\right) d^2\zeta = E(z) \tag{4.21}$$

when $z \in D_-$. These formulae in turn yield the following modification of the definition (4.12) of the g-function

$$g(z) = \log(z - \zeta_0) + \frac{i}{2\pi t_0}\oint_{\Gamma}\frac{\Omega_0(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi t_0}\iint_{D_+}\frac{\zeta}{\zeta - z}d^2\zeta. \tag{4.22}$$

Note that the function $g(z)$ is not holomorphic in $\mathbb{C} \setminus \Gamma$ anymore! In fact[1],

$$\frac{\partial}{\partial \bar{z}} g(z) = \frac{z}{2t_0} \chi_{D_+}(z). \tag{4.23}$$

A slight modification is also needed in the definition of the function $\Psi_n(z)$; indeed, we should put,

$$Y_n(z) = e^{-\frac{nE_0(\gamma)}{2}\sigma_3} \Psi_n(z) e^{ng_\gamma(z)\sigma_3 + \frac{nE_0(\gamma)}{2}\sigma_3}. \tag{4.24}$$

Taking into account (4.23), the $\bar{\partial}$-problem for the matrix Ψ now reads,

$$\frac{\partial}{\partial \bar{z}} \Psi_n(z) + n \frac{z}{2\gamma t_0} \Psi_n(z)\sigma_3 = \overline{\Psi_n(z)} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad z \in D_+, \tag{4.25}$$

$$\frac{\partial}{\partial \bar{z}} \Psi_n(z) = \overline{\Psi_n(z)} \begin{pmatrix} 0 & -e^{-n(E(z)-E_0(\gamma))}\chi_{D_+}(z) \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C} \setminus D_+, \tag{4.26}$$

$$\Psi_{n+}(z) = \Psi_{n-}(z) e^{\frac{n}{\gamma t_0}\Omega_0(z)\sigma_3}, \quad z \in \Gamma \equiv \Gamma(\gamma),$$

with the standard normalization

$$\Psi_n(z) = I + O\left(\frac{1}{z}\right) \quad \text{as} \quad |z| \to \infty. \tag{4.27}$$

The function $\Omega_0(z)$ is now purely imaginary. Therefore, the arguments based on inequality (4.13) seem to be more sound than in the previous approach, and they lead us to the following new model $\bar{\partial}$-problem

$$\frac{\partial}{\partial \bar{z}} \Psi_n^0(z) + n \frac{z}{2\gamma t_0} \Psi_n^0(z)\sigma_3 = \overline{\Psi_n^0(z)} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad z \in D_+, \tag{4.28}$$

$$\frac{\partial}{\partial \bar{z}} \Psi_n^0(z) = 0, \quad z \in \mathbb{C} \setminus D_+, \tag{4.29}$$

$$\Psi_{n+}^0(z) = \Psi_{n-}^0(z) e^{\frac{n}{\gamma t_0}\Omega_0(z)\sigma_3}, \quad z \in \Gamma \equiv \Gamma(\gamma),$$

with the standard normalization

$$\Psi_n^0(z) = I + O\left(\frac{1}{z}\right) \quad \text{as} \quad |z| \to \infty. \tag{4.30}$$

Remark 8. Due to the presence of the large parameter n in the left hand side of equation (4.25), the transition to the model problem (4.28)-(4.30) is still not quite satisfactory even on the formal level.

[1]Probably this property of the function $g(z)$ reflects a major difference between the Riemann-Hilbert problem and the $\bar{\partial}$-problem.

4.5. **An important concluding remark.** In context of the theory of orthogonal polynomials, a matrix $\bar{\partial}$-problem has also appeared in the recent work of K. McLaughlin and P. Miller [6] devoted to orthogonal polynomials on the unite circle with the non-analytic weights. However, unlike the problem (3.2)-(3.4), the $\bar{\partial}$-problem of [6] is not the starting point of the analysis; indeed, the staring point of [6] is still the usual matrix Riemann-Hilbert problem and the $\bar{\partial}$-problem of McLaughlin and Miller is introduced out of the necessity to modify the "openning lenses" step of the usual DKMVZ scheme. Even more important difference between the $\bar{\partial}$-problem considered here and the $\bar{\partial}$-problem in [6] is the absence of the complex conjugation in the basic $\bar{\partial}$-relation. In one hand, this fact simplifies the implementation of the "undressing procedures" — the very important technical element of all integrable asymptotic schemes. On the other hand, as we have shown, the presence of the complex conjugation in the right hand side of (3.2) is truly essential for the incorporation into the asymptotic analysis of the concepts of equilibrium measure and g-function.

In spite of these differences we believe that using methods of [6] will help allow to overcome the indicated above obstacles in the asymptotic analysis of the $\bar{\partial}$-problem (3.2)-(3.4). Specifically, we we think that one needs to develop and then apply to the model problems (4.17)-(4.18) or (4.28)-(4.30) of the the $\bar{\partial}$-version of the "openning lenses" step in the DKMVZ method.

In conclusion, we want to point out at the $\bar{\partial}$-method in the theory of integrable systems, introduced long ago by A.S. Fokas and M.J. Ablowitz in [5], as yet another source of tools for the analysis of the $\bar{\partial}$-problem (3.2)-(3.4).

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