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QUANTIZATION

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Dedicated to the memory of
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Abstract. *In this article we propose a general definition for the quantization of classical mechanics with an arbitrary phase space. We consider the case where the phase space is a complex Kählerian manifold. As an example we consider uniform bounded regions in* C^n *with a Bergman metric, and also the two-dimensional cylinder and torus.*

Introduction

The term "quantization" arose over a span of twelve years in the literature of physics, and from the outset was used in two ways. The first meaning referred to the discretization of the set of values of some physical quantity. The second meaning referred to a construction, starting from the classical mechanics of a system, of a quantum system which had the classical system as its limit as $h \to 0$, where h is Planck's constant(1). *In this paper the term "quantization" is used in this second sense. In this way the problem of quantization, from the point of view of physics, is fundamentally nonunique:* since quantum mechanics describes nature in so much more detail than is done classically, it is possible to alter any quantum system without changing its classical limit.

The role of quantization, if we approach it in this way, consists in rendering help to physical intuition in making a proper comparison of mathematical objects with natural phenomena. The purely mathematical significance of quantization is that it provides a source of important constructs. In this connection we point to the theory of pseudodifferential operators, at whose foundation lies the quantization of the classical mechanics of systems with a linear phase space, and also to the work of A. A. Kirillov, B. Kostant and their successors who have used ideas from quantization in representation theory for Lie groups (see [3], [4], [5] and references given there).

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(1) The constant h has dimensions (equal to the product of the dimensions of energy by time). Therefore the numerical value of h depends on the choice of the system of units. *Letting h tend to zero means going from a system of units in which a quantized object is described, to a system more and more appropriate to a classical description.*

In the literature of physics, quantization is always understood intuitively. Apparently, the first precise mathematical description of quantization is due to Hermann Weyl [1]. Weyl's quantization is only applicable to classical systems which have a plane phase space, since it is based on the use of canonical coordinates p_k, q_k. The quantization of systems with curved phase space (more precisely, systems possessing interconnections) is the subject of the well-known book [2] of P. A. M. Dirac. This problem is an active one in contemporary quantum field theory. One should not consider that it has been completely solved in [2].

In this article we propose a more general definition for quantization, as well as carry out the quantization for the case where the phase space for the classical system is a complex Kählerian manifold. Although the construction appears to be almost universal, that it is meaningful is only established for the case where the manifold is C^n or a uniform bounded region of C^n. In the latter case we find a curious circumstance: Planck's constant cannot take on all nonnegative values, but is bounded from above: $h \leq c$, where c is a constant which depends only on the region. This situation is investigated in detail elsewhere. For the case of C^n our construction leads to the well-known Wick quantization.

At the end of the paper we consider quantization on a cylinder and a torus, which provides an example of Weyl quantization. The basic results of this paper were announced in [6].

§1. General problem of quantization

1. *Classical mechanics*. Let $\mathfrak{M}$ be a differentiable manifold on which there is given a second rank skew-symmetric tensor field ω, which in local coordinates has differentiable components $\omega^{ik}(x)$ and satisfies the equation

$$\sum_k \left[\omega^{\gamma k} \frac{\partial \omega^{\alpha\beta}}{\partial x^k} + \omega^{\beta k} \frac{\partial \omega^{\gamma\alpha}}{\partial x^k} + \omega^{\alpha k} \frac{\partial \omega^{\beta\gamma}}{\partial x^k} \right] = 0. \tag{1.1}$$

The components of the field ω will always be denoted by letters with upper indices. Let $A(\mathfrak{M})$ denote the algebra (with respect to the usual operations of linear combination and multiplication) of differentiable complex-valued functions on $\mathfrak{M}$ and define a Poisson bracket for elements of $A(\mathfrak{M})$:

$$[f, g] = \sum \omega^{ik} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^k}. \tag{1.2}$$

It is easily verified that if the functions f_1, f_2, f_3 are twice differentiable, then (1.1) is equivalent to the Jacobi identity for the Poisson bracket (1.2):

$$[f_1, [f_2, f_3]] + [f_3, [f_1, f_2]] + [f_2, [f_3, f_1]] = 0. \tag{1.3}$$

(The latter, among other things, shows that the condition (1.1) does not depend on the choice of coordinate system, which is not immediately obvious.)

In what follows, the manifold $\mathfrak{M}$ together with the algebra $A(\mathfrak{M})$ supplied with the Poisson bracket (1.2) will be called a classical mechanics and be denoted $(\mathfrak{M}, \omega)$.

Examples. 1. $\mathfrak{M} = R^{2n}$ is $2n$-dimensional affine space. For the coordinates x^i in R^{2n} we introduce the notation $x^i = p_i$ for $i = 1, \cdots, n$ and $x^i = q_{i-n}$ for $i = n+1, \cdots, 2n$. We set $\omega^{i,i+n} = 1$, $i = 1, \cdots, n$, and $\omega^{ik} = 0$ for $|i-k| \neq n$. The Poisson bracket (1.2) has the classical form

$$[f, g] = \sum_l \left(\frac{\partial f}{\partial p_l} \frac{\partial g}{\partial q_l} - \frac{\partial f}{\partial q_l} \frac{\partial g}{\partial p_l} \right). \tag{1.4}$$

2. Let $\mathfrak{G}$ be a Lie algebra, e^i a basis in $\mathfrak{G}$, and C_k^{ij} the structure constants which correspond to e^i. As $\mathfrak{M}$ we take the space of linear forms on $\mathfrak{G}$, and let x^i denote the coordinates in $\mathfrak{M}$ which correspond to the basis e^i (if $\xi = \sum \alpha_i e^i \in \mathfrak{G}$ and $x \in \mathfrak{M}$, then $x(\xi) = \sum x^i \alpha_i$). We set

$$\omega^{ik}(x) = \sum C_s^{ik} x^s, \tag{1.5}$$

where C_s^{ik} are structure constants. The property (1.1) is equivalent to the Jacobi identity for the structure constants:

$$\sum_k [C_s^{ik} C_k^{\alpha\beta} + C_s^{\beta k} C_k^{i\alpha} + C_s^{\alpha k} C_k^{\beta l}] = 0.$$

The Poisson bracket (1.2) has the form

$$[f, g] = \sum_{i,k,s} C_s^{ik} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^k} x^s. \tag{1.6}$$

This example was considered previously in [7].

In particular, take $\mathfrak{G}$ to be the Heisenberg-Weyl algebra, i.e. the Lie algebra with basis ξ^i, η^i, ζ, $i = 1, \cdots, n$, and relations $[\xi^i, \eta^j] = \delta_{ij}$ and $[\xi^i, \zeta] = [\eta^i, \zeta] = [\xi^i, \xi^j] = [\eta^i, \eta^j] = 0$. The coordinates in $\mathfrak{M}$ which correspond to the basis ξ^i, η^i, ζ are denoted p_i, q_i, ϵ. The Poisson bracket (1.7) has the form

$$[f, g] = \varepsilon \sum \left(\frac{\partial f}{\partial p_l} \frac{\partial g}{\partial q_l} - \frac{\partial f}{\partial q_l} \frac{\partial g}{\partial p_l} \right),$$

i.e. differs from (1.5) only by the multiplier ϵ.

3. Let $\mathfrak{M}$ be a symplectic manifold, i.e. a manifold of even dimension on which a nondegenerate closed exterior 2-form ω is given. In local coordinates

$$\omega = \sum \omega_{ik}(x) \, dx^i \wedge dx^k.$$

In distinction from the components of the tensor field introduced earlier, which were denoted by the same letter, the components of the form ω_{ik} will always be provided with lower indices. By nondegenerate we mean that $\det \|\omega_{ik}(x)\| \neq 0$ for all $x \in \mathfrak{M}$. Consequently for each x the inverse matrix $\omega^{ik}(x)$ exists. It is easily proved that the closure of the form ω is equivalent to the property (1.1) of the matrix $\omega^{ik}(x)$.(2) Therefore a classical mechanics always exists on a symplectic manifold.

Remark. If the tensor field $\omega^{ik}(x)$ is not degenerate, i.e. $\det\|\omega^{ik}(x)\| \neq 0$ for all $x \in \mathfrak{M}$, then we can consider the inverse matrix $\omega_{ik}(x)$ and by means of it the exterior form

$$\omega = \sum \omega_{ik}(x)\, dx^i \wedge dx^k.$$

Condition (1.1) is equivalent to the closure of the form ω. Thus, in this case the manifold $\mathfrak{M}$ is symplectic. However, there exist important mechanical systems for which the requisite nondegeneracy of $\omega^{ik}(x)$ is unnatural.(3)

Transformations of classical mechanics. Let $(\mathfrak{M}_i, \omega_i)$, $i = 1, 2$, be classical mechanics where $\det\|\omega_1^{ij}\| \neq 0$ and $\det\|\omega_2^{ij}\| \neq 0$. A diffeomorphism $\phi: \mathfrak{M}_1 \to \mathfrak{M}_2$ will

(2) In terms of the components ω_{ik} the condition $d\omega = 0$ has the form

$$\frac{\partial\omega_{st}}{\partial x^k} + \frac{\partial\omega_{ks}}{\partial x^t} + \frac{\partial\omega_{tk}}{\partial x^s} = 0. \qquad (*)$$

By differentiating the relation $\Sigma_\beta\, \omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha_\gamma$ we find

$$\sum_\beta \left(\frac{\partial\omega^{\alpha\beta}}{\partial x^k}\omega_{\beta\gamma} + \omega^{\alpha\beta}\frac{\partial\omega_{\beta\gamma}}{\partial x^k} \right) = 0,$$

so that

$$\frac{\partial\omega^{\alpha\beta}}{\partial x^k} = -\sum_{s,t} \omega^{\alpha s}\frac{\partial\omega_{st}}{\partial x^k}\omega^{t\beta}.$$

By inserting this expression into (1.1), we find

$$\sum_k \left[\omega^{\gamma k}\frac{\partial\omega^{\alpha\beta}}{\partial x^k} + \omega^{\beta k}\frac{\partial\omega^{\gamma\alpha}}{\partial x^k} + \omega^{\alpha k}\frac{\partial\omega^{\beta\gamma}}{\partial x^k} \right] = \sum_{k,s,t} \omega^{\gamma k}\omega^{\alpha s}\omega^{\beta t}\left(\frac{\partial\omega_{st}}{\partial x^k} + \frac{\partial\omega_{tk}}{\partial x^s} + \frac{\partial\omega_{ks}}{\partial x^t} \right).$$

Thus, when ω is nondegenerate, the conditions (1.1) and (*) are equivalent.

(3) See Example 2 in the case where $\dim \mathfrak{G} \equiv 1 \pmod 2$. In particular, this situation arises for the motion of a solid body about a fixed point. The algebra $\mathfrak{G}$ in this case is the algebra of three-dimensional rotations, and the Poisson bracket is given by the formula

$$[f, g] = \begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{vmatrix}.$$

The Hamiltonian has the form

$$H = \sum_{i,k=1}^{3} h_{ik}x_i x_k,\quad h_{ik} = \bar{h}_{ik} = h_{ki}.$$

The corresponding equations of motion $\partial x_i/\partial t = [H, x_i]$ are called the Euler equations, and the matrix h_{ik} is called the moment of inertia tensor.

be called a mapping of classical mechanics if the conjugate mapping takes the form ω_2 into ω_1; in local coordinates

$$\omega_{1,ij}(x)=\sum_{p,q}\omega_{2,pq}(y(x))\frac{\partial y^p}{\partial x^i}\frac{\partial y^q}{\partial x^j}. \tag{1.7}$$

2. *General definition of quantization.* We call an associative algebra with involution $\mathfrak{A}$ a quantization of the classical mechanics $(\mathfrak{M}, \omega)$ if it possesses the following properties.

1) *There exists a family A_h of associative algebras with involution such that*

1_1) the index h runs over a set E on the positive real axis which has 0 as a limit point (0 does not belong to E), and

1_2) the algebra $\mathfrak{A}$ is a subalgebra of the direct sum $\Sigma_{h\in E}\oplus A_h$. It is convenient to represent the elements of $\mathfrak{A}$ in the form of functions $f(h)$, $h\in E$, with values in A_h. In the usual way involution and multiplication in $\mathfrak{A}$ are related to involution and multiplication in A_h by $f^{\tilde{\sigma}}(h)=(P(h))^{\sigma}$, where $\tilde{\sigma}$ and σ denote involution in $\mathfrak{A}$ and A_h, respectively, and $(f_1 \tilde{*} f_2)(h)=f_1(h)*f_2(h)$, where $\tilde{*}$ and $*$ denote multiplication in $\mathfrak{A}$ and A_h, respectively. In the following we shall denote multiplication and involution in $\mathfrak{A}$ and A_h by the same symbols.

2) *There is a homomorphism ϕ from the algebra $\mathfrak{A}$ into the algebra $A(\mathfrak{M})$.* This homomorphism must have the following properties:

2_1) For any two points $x_1, x_2\in\mathfrak{M}$ there is a function $f(x)\in\phi(\mathfrak{A})$ such that $f(x_1)\neq f_2(x_2)$.

2_2) $$\varphi\left[\frac{1}{h}(f*g-g*f)\right]=\frac{1}{i}[\varphi(f_1),\ \varphi(f_2)],$$

where $*$ denotes multiplication in $\mathfrak{A}$ and $[\cdot,\cdot]$ is the Poisson bracker in $A(\mathfrak{M})$.

2_3) $\phi(f^{\sigma})=\overline{\phi(f)}$, where $f\to f^{\sigma}$ denotes involution in $\mathfrak{A}$ and the bar denotes complex conjugation. The parameter h plays the role of Planck's constant.

Properties 2), 2_1) and 2_2) will be called the *correspondence principle*. For a quantization $\mathfrak{A}$ we shall say that it possesses the correspondence principle in *weak form* if the conditions 2), 2_1) and 2_2) are replaced by the following:

2′) In $\mathfrak{A}$ there exists a linear manifold $\tilde{\mathfrak{A}}$ on which there is defined a mapping $\phi:\tilde{\mathfrak{A}}\to A(\mathfrak{M})$ with the following properties:

$2'_0$) $\phi(f_1*f_2)=\phi(f_1)\cdot\phi(f_2)$ for $f_1, f_2, f_1*f_2\in\tilde{\mathfrak{A}}$.

2_1) $$\varphi\left(\frac{1}{h}(f_1*f_2-f_2*f_1)\right)=\frac{1}{i}[\varphi(f_1),\varphi(f_2)]$$

for $f_1, f_2, f_1*f_2, f_2*f_1\in\tilde{\mathfrak{A}}$.

$2'_2$) For any two points $x_1, x_2\in\mathfrak{M}$ there exists a function $f(x)\in\phi(\tilde{\mathfrak{A}})$ such that $f(x_1)\neq f(x_2)$.

$2'_3$) $\phi(f^{\sigma})=\overline{\phi(f)}$.

We point out that from a general point of view quantization appears as a noncommutative extension of the algebra $A(\mathfrak{M})$.

In the classical mechanics $(\mathfrak{M}, \omega)$ let there be given a dynamical system with

Hamiltonian H. For what sorts of these does there exist a quantum dynamics? From our point of view the answer is as follows. Consider the quantization $\mathfrak{A}$ of the mechanics $(\mathfrak{M}, \omega)$ and let $f(h)$ denote a preimage of H under the homomorphism ϕ. Fix h and consider a linear representation of the algebra A_h in Hilbert space. The operator $\hat{f}$ corresponding to $f(h)$ is the Hamiltonian of the corresponding quantized system. The nonuniqueness in the choice of the preimage $f(h)$ reflects the great precision of the quantum mechanical description of nature compared to the classical one, as noted in the Introduction.

We turn our attention to an important particular case of quantization.

3. *Special quantization.* We shall call by this term a quantization which possesses the following additional properties:

3) The algebra A_h consists of functions $f(x)$, $x \in \mathfrak{M}$.

4) The algebra $\mathfrak{A}$ consists of functions $f(h, x)$, $f(h, x) \in A_h$ for fixed h.

5) The homomorphism $\phi : \mathfrak{A} \to A(\mathfrak{M})$ is given by the formula

$$\varphi(f) = \lim_{h\to 0} f(h, x).$$

The quantizations studied in this section will be special quantizations. In §5 we shall give two examples—quantization on a cylinder and on a torus—which illustrate the general definition.

Besides the properties 1)–5) just enumerated, the quantizations examined here have the properties:

6) Involution in A_h is complex conjugation.

7) The algebra A_h has a unit, namely the function $f_0(x) \equiv 1$.

8) The algebra A_h has the trace

$$\operatorname{tr} f = \int f(x)\, d\mu(x),$$

where $d\mu(x)$ is some measure on $\mathfrak{M}$.(4)

We point out that if the tensor field $\omega^{ij}(x)$ is not degenerate, i.e. if there exists on $\mathfrak{M}$ a closed second-degree exterior form $\omega = \sum \omega_{ij} dx^i \wedge dx^j$, then $\mathfrak{M}$ has the natural measure $d\mu(x) = \omega^{n/2}$.(5)

4. *Adjoined elements.* Let a special quantization be given and let $f_t(x) \in A_h$ be a one-parameter semigroup. The function $\phi(x) = \lim_{t\to 0} df_t(x)/dt$ (if it exists in some sense) will be called an element adjoined to the algebra A_h. Let ${}^{\phi}A_h$ and A_h^{ϕ} be the subsets of A_h for which the limits $\lim_{t\to 0} f_t * g \in A_h$ and $\lim_{t\to 0} g * f_t \in A_h$ exist, respectively. We reserve for these limits the notation $\phi * g$ and $g * \phi$. (In concrete cases,

(4) The trace $\operatorname{tr} f$ is a linear functional which is defined on some submanifold $\tilde{A}_h \subset A_h$. The sets $\tilde{A}_h$ and $\operatorname{tr} f$ should possess the following properties:

1) If $f_1 * f_2 \in \tilde{A}_h$, then $f_2 * f_1 \in \tilde{A}_h$.

2) If $f_1 * f_2 \in \tilde{A}_h$, then $\operatorname{tr}(f_1 * f_2) = \operatorname{tr}(f_2 * f_1)$.

It is clear that $\operatorname{tr} f$ is defined only up to a constant multiple.

(5) In case $\mathfrak{M}$ is a Kähler manifold, other natural measures are also possible. See below.

as we shall see, the multiplication $f * g$ in A_h is defined by means of an integral of the form

$$(f * g)(x) = \int G_h(x_1, x_2 \mid x) f(x_1) g(x_1) \, d\mu(x_1) \, d\mu(x_2);$$

$\phi * g$ and $g * \phi$ are given by similar integrals.)

The function $\phi(h, x)$ is called an element adjoined to the algebra $\mathfrak{A}$ if, for each given h, $\phi(h, x)$ is an element adjoined to A_h.

We shall let ${}^{\phi}\mathfrak{A}$ and $\mathfrak{A}^{\phi}$ denote the sets of functions $f(h, x) \in \mathfrak{A}$ such that, for fixed h, $f(h, x) \in {}^{\phi}A_h$ and A_h^{ϕ}, respectively.

An element adjoined to $\mathfrak{A}$ is called proper if $\phi(0, x) = \lim_{h \to 0} \phi(h, x)$ exists and the set $\widetilde{\mathfrak{A}} \subset {}^{\phi}\mathfrak{A} \cap \mathfrak{A}^{\phi}$ is sufficiently rich so that for $f \in \widetilde{\mathfrak{A}}$

$$(\varphi * f)(h, x) = \varphi(0, x) f(0, x) + o(1),$$
$$(f * \varphi)(h, x) = \varphi(0, x) f(0, x) + o(1),$$
$$\frac{1}{h}(f * \varphi - \varphi * f)(h, x) = \frac{1}{i}[f, \varphi](0, x) + o(1).$$

We shall say that the adjoined element ϕ has the *quasiclassical property* if in place of the last relation we have the equation

$$\frac{1}{h}(\varphi * f - f * \varphi)(h, x) = \frac{1}{i}[\varphi, f](h, x).$$

We shall not make the words "sufficiently rich" more precise, thus leaving ourselves some extra freedom. In the case where the algebra $\mathfrak{A}$ is topological, it is natural to require that $\widetilde{\mathfrak{A}}$ be dense in $\mathfrak{A}$. In other cases it may be more natural to require that for any $x_1, x_2 \in \mathfrak{M}$ an element $f(h, x) \in \widetilde{\mathfrak{A}}$ exists such that $f(0, x_1) \neq f(0, x_2)$.

5. *Symbols*. Consider a linear representation $f \to \hat{f}$ of the algebra A_h in Hilbert space. By definition, the element f is a symbol for the operator $\hat{f}$ (with various additional indices, related to the details of its construction). The adjoined element ϕ will be called a symbol for the infinitesimal operator $\hat{\phi} = \lim_{t \to 0} d\hat{f}_t/dt$.

6. *Quantization functor*. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be algebras constructed similarly to quantizations: there exist algebras $B_h^{(1)}$, $h \in E_1$, and $B_h^{(2)}$, $h \in E_2$, such that $\mathfrak{B}_i$ consists of functions $f(h)$, $h \in E_i$, which have values in $B_h^{(i)}$. A homomorphism $\psi: \mathfrak{B}_1 \to \mathfrak{B}_2$ of such algebras is called admissible if it is generated by homomorphisms ψ_h of the algebras $B_h^{(i)}: (\psi f)(h) = \psi_h f(h)$. (In order that an admissible homomorphism $\mathfrak{B}_1 \to \mathfrak{B}_2$ exist, it is necessary that $E_1 \subset E_2$.) In the same way we define admissible homomorphisms for the algebras $\mathfrak{B}_i$.

We fix a class of classical mechanics $\mathfrak{S}$ and a category K of mappings of the elements of $\mathfrak{S}$. Let a quantization $\mathfrak{A}$ be associated with each classical mechanics $(\mathfrak{M}, \omega) \in \mathfrak{S}$.

We call the association $(\mathfrak{M}, \omega) \rightsquigarrow \mathfrak{A}$ the quantization functor, if for any pair of classical mechanics $(\mathfrak{M}_i, \omega_i)$, $i = 1, 2$, related in the manner

$$(\mathfrak{M}_2, \omega_2) = \Phi(\mathfrak{M}_1, \omega_1), \quad \Phi \in \mathscr{K},$$

there is an admissible homomorphism $\tilde{\Phi}$ such that the diagram

$$\begin{array}{ccccc} (\mathfrak{M}_1, \omega_1) & \rightsquigarrow & \mathfrak{A}_1 & \rightarrow & A(\mathfrak{M}_1) \\ \Phi\downarrow & & \tilde{\Phi}\uparrow & & \Phi^*\uparrow \\ (\mathfrak{M}_2, \omega_2) & \rightsquigarrow & \mathfrak{A}_2 & \rightarrow & A(\mathfrak{M}_2) \end{array} \tag{1.8}$$

is commutative, where ϕ_1 and ϕ_2 are the homomorphisms involved in the definition of a quantization and Φ^* is the mapping of functions conjugate to the diffeomorphism Φ.

We denote the quantization functor by Q. If the association $(\mathfrak{M}, \omega) \rightsquigarrow \mathfrak{A}$ is a functor, we shall denote this by writing $\mathfrak{A} = Q(\mathfrak{M}, \omega)$.

The quantization functor Q is called special if all the quantizations $\mathfrak{A} = Q(\mathfrak{M}, \omega)$ are special.

7. *The natural property.* For objects associated with special quantizations there occurs the important property of being natural. Let $\mathfrak{K}$ be some category of mappings of classical mechanics, and let $(\mathfrak{M}_i, \omega_i)$ and $\mathfrak{A}_i$ be two classical mechanics and their quantizations.

The admissible homomorphism $\psi: \mathfrak{A}_1 \to \mathfrak{A}_2$ is called natural with respect to $\mathfrak{K}$ if there exists a mapping $\Phi \in \mathfrak{K}$, $\Phi(\mathfrak{M}_2, \omega_2) = (\mathfrak{M}_1, \omega_1)$, such that

$$(\psi f)(h, x) = f(h, \Phi(x)), \quad x \in \mathfrak{M}_2. \tag{1.9}$$

The homomorphism ψ related to the mapping Φ by (1.9) will be denoted $\psi = \Phi^*$.

The special quantization functor Q is called natural with respect to $\mathfrak{K}$ if in the diagram (1.8) $\tilde{\Phi} = \Phi^*$.

The special quantization $\mathfrak{A} = Q(\mathfrak{M}, \omega)$, where Q is a special functor, will be called natural with respect to the category $\mathfrak{K}$, if Q has this property. In §8 it will be shown that there is no special quantization functor Q which is natural with respect to the category of all mappings of classical mechanics.

8. *Groups of actions.* Let $(\mathfrak{M}, \omega)$ be a classical mechanics and let G be a group of transformations of $\mathfrak{M}$ which belongs to some category $\mathfrak{K}$ of mappings of classical mechanics. The last circumstance means, in particular, that the transformation τ_g in $A(\mathfrak{M})$:

$$(\tau_g f)(x) = f(g^{-1}x) \tag{1.10}$$

is a Lie algebra automorphism with respect to the Poisson bracket. Let $\mathfrak{A}$ be a quantization of the classical mechanics $(\mathfrak{M}, \omega)$ which is natural with respect to the category $\mathfrak{K}$. It follows immediately from the above definitions that the transformation (1.10) is an automorphism for all the algebras A_h.

Consider the one-parameter semigroup $\tau_{g(t)}$ of automorphisms (1.10) of the algebra A_h. Suppose that these are represented in the form $\tau_{g(t)} f = \sigma_{t/h} * f * \sigma_{-t/h}$, where $\sigma_t \in A_h$ is a one-parameter group and $g = \lim_{t\to 0} d\sigma_t/dt$ is a proper adjoined element, and the set $\mathfrak{A}$ consists of functions which do not depend on h. In this case the element g possesses the quasiclassical property. Indeed, it follows from (1.10) that

$$\lim_{t\to 0} \frac{d}{dt} f(g^{-1}(t)x) = \frac{1}{h}(g * f - f * g) = \frac{1}{i}[g, f] + o(1).$$

For $f \in \mathfrak{A}$ the left-hand side of this equation does not depend on h; consequently the right-hand side has the same property, so that $o(1) = 0$.

9. *Equivalence of quantizations.* Let $\mathfrak{A}$ be a quantization of the classical mechanics $(\mathfrak{M}, \omega)$, E a set of values of h, and let s some one-to-one transformation of E. The mapping s generates an automorphism s^* of the algebra $\mathfrak{A} : (s^* f)(h) = f(sh)$.

Now let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be quantizations of one and the same classical mechanics $(\mathfrak{M}, \omega)$. $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are said to be equivalent if 1) there exist an isomorphism $\Phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ and an isomorphism $s^* : \mathfrak{A}_1 \to \mathfrak{A}_2$ of the sort described above such that the isomorphism $\Phi s^* : \mathfrak{A}_1 \to \mathfrak{A}_2$ *is admissible*, and 2) the diagram

$$\begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{\Phi s^*} & \mathfrak{A}_2 \\ & \varphi_1 \searrow \quad \swarrow \varphi_2 & \\ & A(\mathfrak{M}) & \end{array} \tag{1.11}$$

is commutative, where ϕ_i are the homomorphisms which occur in the definition of the quantizations.

In the case where a transformation group G acts on $\mathfrak{M}$, which belongs to the category with respect to which the quantizations $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are natural, we can introduce the concept of natural equivalence: in addition to (1.11) the diagram

$$\begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{\Phi s^*} & \mathfrak{A}_2 \\ \tau_g \downarrow & & \downarrow \tau_g \\ \mathfrak{A}_1 & \xrightarrow{\Phi s^*} & \mathfrak{A}_2 \end{array} \tag{1.12}$$

must also be commutative, where τ_g is an isomorphism of the form (1.10).

10. *Commentary.* 1) The motivation for introducing the notion of a special quantization is as follows. According to physical notions, quantization is a linear assignment $q : f(x) \to \hat{f}$ of an operator in some Hilbert space to a function.

We assume that the mapping q is one-to-one on the set $\mathcal{X} \subset A(\mathfrak{M})$ and that the image of $\mathcal{X}$ is some operator algebra A. In this case we can carry over into $\mathcal{X}$ multiplication from $A : (f_1 * f_2)(x) = q^{-1}(q(f_1)q(f_2))$. The set $\mathcal{X}$ provided with this multiplication is precisely the algebra A_h in the definition of a special quantization.

From the viewpoint of the ideology of quantum mechanics, the manifold $\mathfrak{M}$ of the classical mechanics should appear in the limit as $h \to 0$. Therefore the special quantization in which $\mathfrak{M}$ takes part from the outset can scarcely be appropriate in all cases. The more general definition is free from this deficiency and is, apparently, universal.

2) The algebraic and topological properties of the algebras A_h and $\mathfrak{A}$, and also the properties of the functions $f(x) \in A_h$ and $f(h, x) \in \mathfrak{A}$ have not been circumscribed in the general definition in order that we may fix these properties in the most convenient way in concrete cases.

3) The correspondence principle fixes only the asymptotic behavior of the algebras A_h as $h \to 0$. The requirements of being a functor and of being natural restrict this arbitrariness.

For the case $\mathfrak{M} = C^n$ we prove in §7 the uniqueness up to equivalence of a natural quantization, under several additional conditions of an algebraic and topological nature.

§2. Quantization in Kähler manifolds

1. *The algebra of covariant symbols.* We shall construct the algebras A_h for a Kähler manifold with the aid of covariant symbols for the operators.

We recall some basic definitions [8].

Let H be a Hilbert space and let M be some set with the measure $d\alpha$. A system of vectors $e_\alpha \in H$, $\alpha \in M$, is called complete if for any $f, g \in H$ Parseval's identity

$$(f, g) = \int (f, e_\alpha)(e_\alpha, g)\, d\alpha \tag{2.1}$$

is valid.

Let $d\sigma(\alpha)$ denote the following measure, which is absolutely continuous with respect to $d\alpha$:

$$d\sigma(\alpha) = (e_\alpha, e_\alpha)\, d\alpha.$$

Note that a complete system generates an inclusion of H into the space $L^2(M)$ by the formula

$$f \mapsto f(\alpha) = (f, e_\alpha). \tag{2.2}$$

Starting now, we shall assume that the space H is realized as a subspace of $L^2(M)$.

In particular, it follows from (2.2) that

$$e_\alpha(\beta) = (e_\alpha, e_\beta) = \overline{(e_\beta, e_\alpha)} = \overline{e_\beta(\alpha)}. \tag{2.3}$$

Let $\hat{P}_\alpha$ be the orthogonal projection onto e_α. The covariant symbol for an operator $\hat{A}$ in H is the function

$$A(\alpha) = \operatorname{tr}(\hat{A}\hat{P}_\alpha) = \frac{(\hat{A}e_\alpha, e_\alpha)}{(e_\alpha, e_\alpha)}. \tag{2.4}$$

From the definition it is clear that a covariant symbol is uniquely defined for every operator for which e_α is in the domain of definition for all $\alpha \in M$, and, in particular, for any bounded operator. Note that the covariant symbol for a bounded operator is bounded: $|A(\alpha)| \le \|A\|$. It can happen that several operators have the same covariant symbol. We shall see below that this does not occur in cases of interest to us: there is a one-to-one correspondence between operators and covariant symbols. Let A be a bounded operator in H and let $A(\alpha)$ be its covariant symbol. It is shown in [8] that the action of the operator on a vector is defined by

$$(\hat{A}f)(\alpha) = \int f(\beta) A(\beta, \alpha) \frac{(e_\beta, e_\alpha)}{(e_\beta, e_\beta)}\, d\sigma(\beta), \tag{2.5}$$

where

$$A(\alpha, \beta) = \frac{(\hat{A}e_\alpha, e_\beta)}{(e_\alpha, e_\beta)} \tag{2.6}$$

is the extension of the function $A(\alpha)$ to $M \times M$.

It follows rapidly from (2.5) that if $\hat{A} = \hat{A}_1\hat{A}_2$, and A, A_1 and A_2 are the covariant symbols for the corresponding operators, then

$$A(\alpha) = \int A_1(\gamma, \alpha) A_2(\alpha, \gamma) \frac{(e_\alpha, e_\gamma)(e_\gamma, e_\alpha)}{(e_\alpha, e_\alpha)(e_\gamma, e_\gamma)} d\sigma(\gamma). \tag{2.7}$$

The algebra with the multiplication in (2.7), which consists of covariant symbols for bounded operators, is basic for the remainder of this paper. The role of the set M can always be taken by the manifold $\mathfrak{M}$ which is the phase space for the underlying classical mechanics. It follows from the definition of a covariant symbol that if the operator $\hat{A}$ has the covariant symbol $A(\alpha)$, then the Hermitian conjugate operator $\hat{A}^*$ corresponds to the complex conjugate of the symbol, $\overline{A(\alpha)}$.

In the general theory developed in [8], along with covariant symbols, the contravariant symbols for operators also play an important part. A function $\mathring{A}(\alpha)$ is called the contravariant symbol for the operator $\hat{A}$ if this operator can be represented as the weak integral

$$\hat{A} = \int \mathring{\alpha}\,(A) \hat{P}_\alpha d\sigma(\alpha).$$

Between the covariant and contravariant symbols of a given operator we have the relation

$$A(\alpha) = \int \mathring{A}(\beta) \frac{(e_\alpha, e_\beta)(e_\beta, e_\alpha)}{(e_\alpha, e_\alpha)(e_\beta, e_\beta)} d\sigma(\beta). \tag{2.8}$$

Note that the multiplication given in (2.7) and the relation between covariant and contravariant symbols are given by means of the single function

$$G(\alpha, \beta) = \frac{(e_\alpha, e_\beta)(e_\beta, e_\alpha)}{(e_\alpha, e_\alpha)(e_\beta, e_\beta)}. \tag{2.9}$$

The integral operator (2.8), for which (2.9) is the kernel, plays an important role in this theory. We shall denote it by T.

For completeness we present formulas for the trace of an operator and the trace of a product of operators:(6)

$$\mathrm{tr}\,\hat{A} = \int A(\alpha)\, d\sigma(\alpha) = \int \mathring{A}(\alpha)\, d\sigma(\alpha), \tag{2.10}$$

$$\mathrm{tr}\,(\hat{A}\hat{B}^*) = \int A(\alpha) \overline{\mathring{B}}(\alpha)\, d\sigma(\alpha). \tag{2.11}$$

2. *Kähler manifolds.* Let $\mathfrak{M}$ be a Kähler manifold, so that in local coordinates

$$ds^2 = \Sigma g_{i\bar{k}} dz^i d\bar{z}^k, \quad \omega = \Sigma g_{i\bar{k}} dz^i \wedge d\bar{z}^k.$$

By the definition of a Kähler manifold, $d\omega = 0$ and $\det \| g_{i\bar{k}} \| \neq 0$. Consequently $\mathfrak{M}$ is

(6) It is shown in [8] that the existence of a kernel for $\hat{A}$ follows from the existence of the second integral in (2.10); the existence of the first integral is a consequence of the existence of this kernel. In particular, we note that the covariant and contravariant symbols for the operator $\hat{A} = I$ are $A(\alpha) = \mathring{A}(\alpha) \equiv 1$. Therefore the finiteness of the integral $\int d\sigma(\alpha)$ is equivalent to the finite dimensionality of the space H, and $\dim H = \mathrm{tr}\, I = \int d\sigma(\alpha)$.

a symplectic manifold. The classical mechanics $(\mathfrak{M}, \omega)$ always exists on $\mathfrak{M}$. The Poisson bracket in complex local coordinates has the form

$$[f_1, f_2] = i \sum g^{\bar{i}k} \left(\frac{\partial f_1}{\partial z^k} \frac{\partial f_2}{\partial \bar{z}^i} - \frac{\partial f_1}{\partial \bar{z}^i} \frac{\partial f_2}{\partial z^k} \right). \tag{2.12}$$

In the following there will frequently occur functions on $\mathfrak{M} \times \mathfrak{M}$ which are analytic with respect to the first argument and antianalytic with respect to the second. We shall denote such functions by $f = f(z, \bar{v})$.

In this subsection we shall describe a general scheme for the quantization of the classical mechanics $(\mathfrak{M}, \omega)$.

We recall that the potential of a Kähler metric is a function $\Phi(z, \bar{z})$ which satisfies the system of equations

$$\frac{\partial^2 \Phi}{\partial z^i \partial \bar{z}^k} = g_{i\bar{k}}. \tag{2.13}$$

The Kähler condition $d\omega = 0$ is also the condition for local solvability of the system (2.13). Equations (2.13) have a real solution and, without restriction, we shall assume that Φ is a real function. We shall also assume that the potential Φ exists globally on $\mathfrak{M}$ and that it possesses an analytic extension $\Phi(z, \bar{v})$ to some neighborhood of the diagonal in $\mathfrak{M} \times \mathfrak{M}$.

Consider the Hilbert space F_h of functions on $\mathfrak{M}$ with scalar product

$$(f, g) = c(h) \int f(z) \overline{g(z)} e^{-\frac{1}{h}\Phi(z,\bar{z})} d\mu(z, \bar{z}), \tag{2.14}$$

where

$$d\mu(z, \bar{z}) = \omega^n = \det \| g_{i\bar{k}} \| \prod \frac{dz^k \wedge d\bar{z}^k}{2\pi i}. \text{(7)}$$

Here h plays the role of Planck's constant and the function $c(h)$ is defined below. The integral in (2.14), as all similar integrals below for which the domain of integration is not indicated, is assumed to extend over all $\mathfrak{M}$.

Let $f_k(z)$ be an orthogonal basis of normalized vectors in F_h.

Theorem 2.1. 1) *In each coordinate neighborhood the series*

$$L_h(z, \bar{z}) = \sum f_k(z) \overline{f_k(z)} \tag{2.15}$$

is absolutely and uniformly convergent.

2) *The function* $L_h(z, \bar{z})$ *is independent of the choice of orthonormal basis* f_k.

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(7) ω^n, $n = \dim_C \mathfrak{M}$, is one of the possible natural measures on a Kähler manifold. Another possible variant is $d\mu = \omega_1^n$, where

$$\omega_1 = \sum \frac{\partial^2 \ln g}{\partial z^i \partial \bar{z}^k} dz^i \wedge d\bar{z}^k, \quad g = \det \| g_{i\bar{k}} \|.$$

The reason that ω^n was chosen is that only in this case can one prove the correspondence principle. See the following section.

The proof of the first statement duplicates the standard proof of the existence of a Bergman kernel function (see [9], for example) and will therefore be omitted.(8) By applying the Cauchy-Bunjakovskiĭ inequality we find as a consequence of the first statement that the series

$$L_h(z, \bar{v}) = \sum f_k(z)\overline{f_k(v)} \tag{2.16}$$

is absolutely and uniformly convergent in each coordinate neighborhood in $\mathfrak{M} \times \mathfrak{M}$. We shall denote by $L_h^2(\mathfrak{M})$ the Hilbert space of all measurable functions on $\mathfrak{M}$ which are square summable with respect to the measure $c(h)e^{-\Phi/h}d\mu$, and by P the orthogonal projection of $L_h^2(\mathfrak{M})$ onto F_h. It is clear that

$$(Pf)(z) = c(h)\int f(v, \bar{v})\, L_h(z, \bar{v})\, e^{-\frac{1}{h}\Phi(v, \bar{v})}\, d\mu(v, \bar{v}).$$

Hence $L_h(z, \bar{v})$ is independent of the choice of basis.

From the Cauchy-Bunjakovskiĭ inequality we have the important inequality

$$|L_h(z, \bar{v})|^2 \leqslant L_h(z, \bar{z})\, L_h(v, \bar{v}). \tag{2.17}$$

Let us set

$$\Phi_{\bar{v}}(z) = L_h(z, \bar{v}). \tag{2.18}$$

We note that

$$c(h)\int |\Phi_{\bar{v}}(z)|^2 e^{-\frac{1}{h}\Phi(z, \bar{z})}\, d\mu(z, \bar{z}) = \sum f_k(v)\overline{f_k(v)} = L_h(v, \bar{v}).$$

Consequently $\Phi_{\bar{v}} \in F_h$. From (2.18) and (2.16) it follows that for any $f \in F_h$

$$(f, \Phi_{\bar{v}}) = f(v). \tag{2.19}$$

(2.1) obviously follows from (2.19), where the role of α is played by the point v of the manifold $\mathfrak{M}$, and

$$d\alpha = c(h)\, e^{-\frac{1}{h}\Phi(z, \bar{z})}\, d\mu(z, \bar{z}), \quad d\sigma(\alpha) = L_h(z, \bar{z})\, d\alpha. \tag{2.20}$$

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(8) The proof of the existence of a kernel function is based on the inequality

$$|f(z_0)|^2 \leqslant a\int_U |f(z)|^2 \prod \frac{dz^k \wedge d\bar{z}^k}{2\pi i}, \tag{$*$}$$

where U is a coordinate neighborhood centered at z_0. We set

$$\rho(z, \bar{z}) = \exp\left\{-\frac{1}{h}\Phi(z, \bar{z})\right\} \det \| g_{i\bar{k}}(z, \bar{z}) \|.$$

It is clear that $\rho(z, \bar{z}) \geq a_0 = a_0(U) > 0$ with $z \in U$. Therefore from (*) we have the inequality

$$|f(z_0)|^2 \leqslant a a_0^{-1} \int_U |f(z)|^2 \rho(z, \bar{z}) \prod \frac{dz^k \wedge d\bar{z}^k}{2\pi i}. \tag{$**$}$$

The first assertion of the theorem follows from (**) in exactly the same way as the existence of a kernel function follows from (*).

Thus the function $\Phi_{\bar{v}}(z)$ forms a complete system in F_h. We form the algebra A_h out of covariant symbols for bounded operators in F_h. We point out that the covariant symbol $A(z, \bar{z})$ of the operator $\hat{A}$ is the diagonal $z = v$ of the function

$$A(z, \bar{v}) = \frac{(A\Phi_{\bar{v}}, \Phi_{\bar{z}})}{(\Phi_{\bar{v}}, \Phi_{\bar{z}})}, \tag{2.21}$$

defined on $\mathfrak{M} \times \mathfrak{M}$. The specialization of (2.5), (2.7) and (2.10) to the case being considered gives

$$(\hat{A}f)(z) = c(h) \int A(z, \bar{v}) f(v) L_h(z, \bar{v}) e^{-\frac{1}{h}\Phi(v, \bar{v})} d\mu(v, \bar{v}), \tag{2.22}$$

$$(A_1 * A_2)(z, \bar{z}) = c(h) \int A_1(z, \bar{v}) A_2(v, \bar{z}) \frac{L_h(z, \bar{v}) L_h(v, \bar{z})}{L_h(z, \bar{z}) L_h(v, \bar{v})} \cdot \frac{L_h(v, \bar{v})}{e^{\frac{1}{h}\Phi(v, \bar{v})}} d\mu(v, \bar{v}), \tag{2.23}$$

$$\operatorname{tr} \hat{A} = c(h) \int A(z, \bar{z}) \frac{L_h(z, \bar{z})}{e^{\frac{1}{h}\Phi(z, \bar{z})}} d\mu(z, \bar{z}). \tag{2.24}$$

Remarks. 1. According to (2.22), the operator $\hat{A}$ is generated by the function $A(z, \bar{v})$ which is the analytic continuation of $A(z, \bar{z})$ from the diagonal in $\mathfrak{M} \times \mathfrak{M}$ to all of $\mathfrak{M} \times \mathfrak{M}$. Also in our case the operator $\hat{A}$ is uniquely generated by its covariant symbol $A(z, \bar{z})$. The correspondence between operators and covariant symbols is one to one.

2. Consider the family of Kähler metrics which differ from one another by a constant factor $\lambda : ds^2 = \lambda^{-1} ds_0^2$. The potentials of the metrics ds^2 and ds_0^2 are connected by the relation $\Phi(z, \bar{z}) = \lambda^{-1}\Phi_0(z, \bar{z})$. Thus the scale factor λ in this theory plays essentially the role of Planck's constant.[9]

3. *Auxiliary assumptions and hypotheses.* We consider the function

$$\varphi(z, \bar{z} \mid v, \bar{v}) = \Phi(z, \bar{v}) + \Phi(v, \bar{z}) - \Phi(v, \bar{v}) - \Phi(z, \bar{z}). \tag{2.25}$$

By our general assumptions, the function $\Phi(z, \bar{v})$ is defined in a neighborhood of the diagonal of the manifold $\mathfrak{M} \times \mathfrak{M}$. Thus the function ϕ is also defined in a neighborhood of the diagonal of $\mathfrak{M} \times \mathfrak{M}$. We shall assume in addition that ϕ possesses an analytic continuation to the entire manifold $\mathfrak{M} \times \mathfrak{M}$ (as will be seen in due course, this is possible without the function $\Phi(z, \bar{v})$ having to possess the same property).

We shall call the point $z \in \mathfrak{M}$ proper if for any neighborhood U of the point z there exists an $\alpha(U) > 0$ such that $\phi(z, \bar{z} | v, \bar{v}) \geq -\alpha(U)$ for $v \notin U$.

The following lemma serves as the basis of the proof of the correspondence principle.

Lemma 2.1. *Let z_0 be a proper point, let $f(v, \bar{v})$ be a triply continuously differentiable function and let the integral*

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(9) Note, in connection with this, that if $\mathfrak{M}$ is an irreducible symmetric space, then an invariant metric ds^2 is defined on it up to a constant multiplier.

$$\mathcal{I}(h) = h^{-n} \int f(v, \bar{v})\, e^{\frac{1}{h}\varphi(v,\bar{v} \mid z_0, \bar{z}_0)}\, d\mu(v, \bar{v}) \tag{2.26}$$

exist and be absolutely convergent for $h = h_0$. *Then* $\mathcal{I}(h)$ *exists for all* $0 < h \le h_0$, *and as* $h \to 0$

$$\mathcal{I}(h) = f(z_0, \bar{z}_0) + h(\Delta f(z_0, \bar{z}_0) + \sigma f(z_0, \bar{z}_0)) + o(h), \tag{2.27}$$

where Δ *is the Laplace-Beltrami operator on* $\mathfrak{M}$, *and*

$$\sigma = \sigma(z_0, \bar{z}_0) = \frac{3}{2}\Delta \ln g(z_0, \bar{z}_0), \quad g(z, \bar{z}) = \det \| g_{i\bar{k}}(z, \bar{z}) \|.$$

Proof. We note that $\phi(v, \bar{v} \mid z_0, \bar{z}_0) \le 0$ and that

$$e^{\frac{1}{h}\varphi} = e^{\frac{1}{h_0}\varphi} \cdot e^{\left(\frac{1}{h} - \frac{1}{h_0}\right)\varphi}. \tag{2.28}$$

Consequently $e^{\phi/h} \le e^{\phi/h_0}$ for $0 < h \le h_0$, and the integral (2.26) exists for $0 < h \le h_0$. Now let U be a neighborhood of z_0 such that for $v_1, v_2 \in U$ the function $\Phi(v_1, \bar{v}_2)$ is analytic with respect to v_1 and antianalytic with respect to v_2. We break the integral $\mathcal{I}(h)$ up into the sum $\mathcal{I}(h) = \mathcal{I}_1(h) + \mathcal{I}_2(h)$, where $\mathcal{I}_1(h)$ extends over U and $\mathcal{I}_2(h)$ over $\mathfrak{M} \setminus U$. We find from (2.28) that for $0 < h \le h_0$

$$e^{\frac{1}{h}\varphi} \leqslant e^{\frac{1}{h}\varphi} e^{-\left(\frac{1}{h} - \frac{1}{h_0}\right)\delta}.$$

Thus $\mathcal{I}_2(h) \le \mathcal{I}_2(h_0) e^{-(1/h - 1/h_0)\delta}$, and consequently $\mathcal{I}_2(h) \to 0$ as $h \to 0$ faster than any power of h.

We turn to the integral $\mathcal{I}_1(h)$. Introduce coordinates t^i into U so that the coordinates of z_0 are $t^i = 0$. We decompose ϕ into its Taylor series. This is conveniently carried out by means of the operators $L_t = \Sigma\, t^i \partial/\partial \tau^i$ and $\bar{L}_t = \Sigma\, \bar{t}^i \partial/\partial \bar{\tau}^i$:

$$\varphi(v, \bar{v} \mid z_0, \bar{z}_0) = \psi(t, \bar{t}) = \Phi(0, \bar{t}) + \Phi(t, 0) - \Phi(0, 0) - \Phi(t, \bar{t})$$

$$= -\left[L_t\bar{L}_t + \frac{1}{2}(L_t^2\bar{L}_t + L_t\bar{L}_t^2) + \frac{1}{4}L_t^2\bar{L}_t^2 + \frac{1}{6}(L_t^3\bar{L}_t + L_t\bar{L}_t^3)\right]\Phi(\tau, \bar{\tau})\big|_{\tau=\bar{\tau}=0} + R,$$

where

$$R = \sum_{|p|+|q|=6} t^p \bar{t}^q R_{p,q}(t, \bar{t}),$$

$R_{p,q}(t, \bar{t})$ is an analytic function in $U \times U$, and p and q are multi-indices. We now make use of Lemma A.1 of the Appendix. According to this lemma, in the coordinates t^i as $h \to 0$ we have

$$\mathcal{I}_1(h) = f(0, 0) + h(\Delta f(0, 0) + \sigma f(0, 0)) + o(h),$$

where $\sigma = 3\Delta \ln g/2$, Δ is the Laplace-Beltrami operator and $g = \det\|g_{i\bar{k}}\|$. The lemma is proved.

Corollary. *Let the integral* (2.26) *exist for* $h = h_0$ *and* $f(v, \bar{v}) \equiv 1$. *For* $0 < h \le h_0$ *define the function* $\tilde{c}(h) = \tilde{c}(h \mid z, \bar{z})$ *by the equation*

$$\tilde{c}^{-1}(h) = \int e^{\frac{1}{h}\varphi(z,\bar{z}|v,\bar{v})} d\mu(v,\bar{v}). \tag{2.29}$$

Then $\tilde{c}(h) = \tilde{c}_1(h)/h^n$, *where* $\tilde{c}_1(h)$ *is a function which is continuous in* $0 \le h \le h_0$ *and differentiable at* $h = 0$, *where*

$$\tilde{c}_1(0) = 1, \quad \tilde{c}_1'(0) = -\sigma = -\frac{3}{2}\ln g(z,\bar{z}). \tag{2.30}$$

The proof is obvious.

We make the following important definition. The point $z_0 \in \mathfrak{M}$ is called *distinguished* if $\Phi(z_0, \bar{v}) = 0$ when v runs over some neighborhood U of the point z_0.

Note that the potential defined by the system of equations (2.13) is not unique, but is given only up to terms of the form $\psi(z) + \overline{\psi(z)}$, where $\psi(z)$ is an arbitrary analytic function. By selecting the function $\psi(z)$ in a suitable fashion, we can make any point of $\mathfrak{M}$ distinguished. Conversely, by singling out some point we determine the potential up to an additive constant.

We now formulate the assumptions under which we shall construct a quantization on Kähler manifolds.

Hypothesis A. *There exists in* R^1_+ *a set* E *having* 0 *as a limit point and such that for* $h \in E$

$$L_h(z,\bar{z}) = \lambda e^{\frac{1}{h}\Phi(z,\bar{z})}. \tag{2.31}$$

Hypothesis B. *A point* $h_0 \in E$ *exists such that for* $0 < h \le h_0$, $h \in E$, *the functions* $f(z) \in F_h$ *separate points of* $\mathfrak{M}$ (*i.e. for any* $z_1, z_2 \in \mathfrak{M}$ *there is an* $f(z) \in F_h$ *such that* $f(z_1) \ne f(z_2)$).

Hypothesis C. *Let* z_n *be a sequence of points of* $\mathfrak{M}$ *having the property that for* h_0 *defined under hypothesis* B, *for any* $f \in F_h$ *the limit* $\lim_{n\to\infty} f(z_n)$ *exists and is finite. Then the sequence* z_n *has a limit* $z_0 \in \mathfrak{M}$.

Hypothesis D. *There exists a distinguished point on* $\mathfrak{M}$.

We shall deduce a number of consequences from these hypotheses. Let F_{h,z_0} denote the space F_h constructed by means of the potential $\Phi_{z_0}(z,\bar{z})$ having the distinguished point z_0, which is normalized by the condition $\Phi_{z_0}(z_0,\bar{z}_0) = 0$.

Lemma 2.2. *Let condition* A *hold for the potential* Φ, *and let* $L_h(z,\bar{v}) \ne 0$ *for any* $z, v \in \mathfrak{M}$ *and* $h \in E$. *Then the following assertions are true:*

1) *For any point* $z_0 \in \mathfrak{M}$ *there exists a potential* $\Phi_{z_0}(z,\bar{z})$ *which produces the same metric as* $\Phi(z,\bar{z})$ *and which has* z_0 *as a distinguished point.*

2) *The spaces* F_h *and* F_{h,z_0} *are isomorphic under the isomorphism* $U: F_{h,z_0} \to F_h$ *given by*

$$(Uf)(z) = \frac{f(z)L_h(z,\bar{z}_0)}{\sqrt{L_h(z_0,\bar{z}_0)}}.$$

Proof. Consider a segment on the negative part of the real axis and select a single-valued branch of the logarithm by means of the condition $\ln 1 = 0$. Set

$$\Phi_{z_0}(z, \bar{z}) = \Phi(z, \bar{z}) - h \ln (L_h(z, \bar{z}_0) L_h(z_0, \bar{z})) + h \ln L_h(z_0, \bar{z}_0).$$

According to condition A, in a sufficiently small neighborhood of z_0

$$\ln L_h(z, \bar{z}_0) = \frac{1}{h} \Phi(z, \bar{z}_0) + \ln\lambda.$$

Therefore

$$\Phi_{z_0}(z, \bar{z}) = \Phi(z, \bar{z}) - \Phi(z, \bar{z}_0) - \Phi(z_0, z) + \Phi(z_0, \bar{z}_0)$$

gives the same metric as $\Phi(z, \bar{z})$. It is clear that z_0 is a distinguished point for the potential $\Phi_{z_0}(z, \bar{z})$.

A scalar product in F_{h,z_0} is given by the formula

$$(f, f) = c(h) \int |f|^2 \frac{L_h(z_0, \bar{z}) L_h(z, \bar{z}_0)}{L_h(z_0, \bar{z}_0) L_h(z, \bar{z})} d\mu(z, \bar{z}).$$

By setting

$$g(z) = (Uf)(z) = \frac{f(z) L_h(z, \bar{z}_0)}{\sqrt{L_h(z_0, \bar{z}_0)}},$$

we obtain the isomorphism $U : F_{h,z_0} \to F_h$.

Remark. $L_h(z, \bar{v}) \neq 0$ for any $z, v \in \mathfrak{M}$ in the case where the set E contains an interval. Indeed, let $\Delta \subset E$ be an interval, $\alpha \in E$. It follows from hypothesis A that $L_h(z, \bar{v}) = [L_\alpha(z, \bar{v})]^{\alpha/h}$. From analyticity with respect to z and the uniqueness of $L_h(z, \bar{v})$ for $h \in \Delta$ it follows that $L_h(z, \bar{v}) \neq 0$.

Lemma 2.3. *Let hypothesis* A *be valid. Then the following assertions are true:*

1) *For $h \in E$ the function $c(h)$ which normalizes the scalar product* (2.14) *can be defined uniquely by the condition*

$$L_h(z, \bar{z}) = e^{\frac{1}{h}\Phi(z, \bar{z})}. \tag{2.32}$$

2) *This value of $c(h)$ is given by the formula*

$$c^{-1}(h) = \int \frac{L_h(z, \bar{v}) L_h(v, \bar{z})}{L_h(z, \bar{z}) L_h(v, \bar{v})} d\mu(v, \bar{v}) \tag{2.33}$$

(*the integral on the right is independent of z and $\bar{z}$*).

3) *The function $\phi(z, \bar{z} | v, \bar{v})$ has a unique analytic continuation in the real sense from a neighborhood of the diagonal in $\mathfrak{M} \times \mathfrak{M}$ to all of $\mathfrak{M} \times \mathfrak{M}$.*

4) *If, besides hypothesis A, hypothesis* D *is also valid, then* (2.33) *can be replaced by the simpler expression*

$$c^{-1}(h) = \int \frac{d\mu(v, \bar{v})}{L_h(v, \bar{v})}. \tag{2.33'}$$

Proof. For the moment, let $L_h^{(1)}(z, \bar{z})$ denote the function L_h which corresponds to $c(h) \equiv 1$, and let $L_h^{(c)}(z, \bar{z})$ denote the function which corresponds to an arbitrary $c(h)$. It is clear that $L_h^{(c)} = c^{-1}(h) L_h^{(1)}$. Hence (2.32) defines $c(h)$ uniquely.

We now apply (2.23) in the case where the operators $\hat{A}_1$ and $\hat{A}_2$ are unitary. In this case $A_1 \equiv A_2 \equiv A_1 * A_2 \equiv 1$. By using (2.32) we obtain (2.33). Formula (2.33′) is obtained from (2.33) with $z = z_0$, where z_0 is the distinguished point.

Let $h_0 \in E$. According to condition A, in some neighborhood of the diagonal in $\mathfrak{M} \times \mathfrak{M}$ we have the equation

$$e^{\frac{1}{h}\varphi(z,\bar{z}\mid v,\bar{v})} = \left[\frac{L_{h_0}(z,\bar{v})\, L_{h_0}(v,\bar{z})}{L_{h_0}(z,\bar{z})\, L_{h_0}(v,\bar{v})}\right]^{\frac{h_0}{h}}. \tag{2.34}$$

The expression in brackets on the right-hand side is defined for all $z, v \in \mathfrak{M}$ and is positive. Consequently the right-hand side of (2.34) is an analytic function in the real sense on $\mathfrak{M} \times \mathfrak{M}$. By taking the logarithm and multiplying by h we find the required analytic continuation for ϕ. The lemma is proved.

Corollary. *For $h \in E$*

$$\Phi_{\bar{v}}(z) = e^{\frac{1}{h}\Phi(z,\bar{v})}. \tag{2.35}$$

Proof. According to (2.18), $\Phi_{\bar{v}}(z)$ is a single-valued analytic function on $\mathfrak{M} \times \mathfrak{M}$. According to (2.32) the left and right sides of (2.35) are equal for $z = v$.

We point out that although $\Phi_{\bar{v}}(z)$ is a single-valued analytic function on the entire manifold $\mathfrak{M} \times \mathfrak{M}$, it does not follow from (2.35) that $\Phi(z, \bar{v})$ has the same property.

Lemma 2.4. *Let conditions* A, B, C *and* D *be satisfied. Then* 1) *each point of the manifold $\mathfrak{M}$ is proper, and* 2) *the integral* (2.26) *exists for $f(v, \bar{v}) \equiv 1$ and $h \in E$.*

Proof. It follows from (2.34) that for $h = h_0 \in E$ and $f \equiv 1$ the integral (2.26), up to a constant multiplier, is equal to the right-hand side of (2.23), where $A_1 \equiv A_2 \equiv 1$, i.e. when the operators $\hat{A}_1$ and $\hat{A}_2$ are unitary. Hence the integral (2.26) exists for $f \equiv 1$, $h \in E$.

We turn now to the first assertion. Recall that $\Phi_{\bar{v}}(z) = L_h(z, \bar{v}) \in F_h$ for any v and for any h. In particular, this is true for $v = v_0$ (a distinguished point) and $h \in E$. By hypothesis A, in this case

$$\Phi_{\bar{v}}(z) = e^{\frac{1}{h}\Phi(z,\bar{v})} \equiv 1$$

(see (2.35)). Thus $f_0(z) \equiv 1 \in F_h$ for $h \in E$.

We fix a point z_0 and a neighborhood U of this point. From (2.17) and (2.34) it follows that $e^{\phi/h} \leq 1$ for $z = z_0$ and $v \notin U$. Thus if the point z_0 is not proper, then there exists a point $v_0 \notin U$ such that

$$\frac{L_{h_0}(z_0, \bar{v}_0)\, L_{h_0}(v_0, \bar{z}_0)}{L_{h_0}(z_0, \bar{z}_0)\, L_{h_0}(v_0, \bar{v}_0)} = 1, \tag{2.36}$$

or there exists a sequence of points $v_n \notin U$ such that

$$\lim_{n\to\infty} \frac{L_{h_0}(z_0, \bar{v}_n)\, L_{h_0}(v_n, \bar{z}_0)}{L_{h_0}(z_0, \bar{z}_0)\, L_{h_0}(v_n, \bar{v}_n)} = 1. \tag{2.37}$$

Consider the first possibility. We show that in the case of (2.36) we have $f(z_0) = f(v_0)$ for any function $f \in F_h$. Let $f_0(z) \equiv 1$ and $f(z) \in F_h$. We set $f_1(z) = \|f_0\|^{-1/2} f_0$ and $f_2(z) = \alpha f(z) + \beta$, where α and β are chosen by the condition $(f_1, f_2) = 0$ and $(f_2, f_2) = 1$. We augment the pair of functions to form a basis f_k in F_h, and use this basis in (2.16). From the properties of the Cauchy-Bunjakovskiĭ inequality it follows from (2.36) that $f_k(z_0) = \lambda f_k(v_0)$. By applying this relation for $k = 1$ and $k = 2$; we find that $\lambda = 1$ and that $f(z_0) = f(v_0)$. According to hypothesis B, from this we have $v_0 = z_0$, which contradicts the condition $v_0 \notin U$.

Now consider the second possibility. Let H be an arbitrary Hilbert space, ξ, $\eta_n \in H$, and $|(\xi, \eta_n)|/\|\xi\|\,\|\eta_n\| \to 1$. Let $e_n = \eta_n/\|\eta_n\|$ and select a weakly convergent subsequence e_{n_k}. Let $e = \lim e_{n_k}$. Then $(\xi/\|\xi\|, e) = 1$, whence it follows that $e = \lambda\xi/\|\xi\|$, $|\lambda| = 1$. Since $\|e\| = 1$, the subsequence e_{n_k} converges, not only weakly, but also strongly. Consider as H the space l_2, and set

$$\xi = \{f_1(z_0), f_2(z_0), \ldots\}, \quad \eta_n = \{f_1(v_n), f_2(v_n), \ldots\},$$

where f_k is the basis in F_h containing the function $f_1(z) = \text{const}$. Since $f_1(z_0) = f_1(v_n) = \text{const} \neq 0$, it follows from the condition $e = \lambda\xi/\|\xi\|$ that $\overline{\lim}\|\eta_{n_k}\| < \infty$. Let n_s be a subsequence of the numbers n_k for which $\lim\|\eta_{n_s}\|$ exists. It is clear that as $s \to \infty$, $\lim f_k(v_{n_s}) = f_k$ exists and is finite, and in the strong sense $\lim \eta_{n_s} = \eta = (f_1, f_2, \cdots)$. Equation (2.37) means that $|(\xi, \eta)|/\|\xi\|\,\|\eta\| = 1$. Thus $\eta = \lambda\xi$, i.e. $f_k = \lambda f_k(z_0)$. By setting $k = 1$, we find that $\lambda = 1$. Let $f(z) = \Sigma\, c_k f_k(z)$; then

$$|f(v_{n_k}) - f(z_0)|^2 \leqslant \sum |c_k|^2 \sum |f_k(v_{n_k}) - f_k(z_0)|^2$$
$$= \|f\|^2 \|\eta - \eta_{n_k}\|^2 \to 0 \quad \text{as} \quad k \to \infty.$$

According to hypothesis C it follows from this that $v_0 = \lim v_{n_k} \in \mathfrak{M}$ exists, and according to hypothesis B $v_0 = z_0$, which contradicts the condition that $v_n \notin U$. The lemma is proved.

Lemma 2.4 shows that from hypotheses A–D follow the conditions of Lemma 2.1.

4. *Correspondence principle.* Consider the set of functions $f(h|z, \bar{z})$, $h > 0$, which are continuous with respect to all the arguments and which have the property that $f(h|z, \bar{z}) \in A_h$ for fixed h. Let E be the set described in hypothesis A. Let $\mathfrak{A}$ denote the algebra which consists of the values of the functions $f(h|z, \bar{z})$ with $h \in E$. Let $\mathfrak{B}$ denote the set of functions which can be represented in the form

$$f(h|z, \bar{z}) = f(0|z, \bar{z}) + hf_1(z, \bar{z}) + h^2 f_2(h|z, \bar{z}),$$

where $f(0|z,\bar z)$, $f_1(z,\bar z)$ and $f_2(h|z,\bar z)$ are functions which possess analytic continuations onto $\mathfrak{M}\times\mathfrak{M}$. With respect to these analytic continuations we assume that there exists an h_0 such that for $h < h_0$

$$c(h)\int (|f(0|z,\bar v)|^2+|f_1(z,\bar v)|^2+|f_2(h|z,\bar v)|^2 e^{-\frac{1}{h}\Phi(z,\bar z)}\,d\mu(z,\bar z)\leqslant r,$$

where $r = r(v)$ does not depend on h.

We show that the weak form of the correspondence principle holds for the algebra $\mathfrak{A}$ and the set $\tilde{\mathfrak{A}} = \mathfrak{A}\cap\mathfrak{B}$. Thus we shall establish that the algebra $\mathfrak{A}$ is a special quantization.

Theorem 2.2. *Let hypotheses* A, B, C *and* D *be valid, and let* $f, g\in\mathfrak{B}$. *Then as* $h\to 0$, $h\in E$,

$$(f*g)(h|z,\bar z)=f(0|z,\bar z)\,g(0|z,\bar z)+o(1), \tag{2.38}$$

$$\frac{1}{h}(f*g-g*f)(h|z,\bar z)=\frac{1}{i}[f,g](0|z,\bar z)+o(1). \tag{2.39}$$

Proof. Let $u(h|v,\bar v)=f(h|z,\bar v)g(h|v,\bar z)$, and write the left-hand side of (2.38) in the form

$$\psi(h|z,\bar z)=c(h)\int u(h|v,\bar v)\,e^{\frac{1}{h}\varphi(z,\bar z|v,\bar v)}\,d\mu(v,\bar v), \tag{2.40}$$

where ϕ is defined by (2.25) and $c(h)=\tilde c(h)$ is given in (2.29).

According to the assumptions made regarding the functions f and g, the function u can be represented in the form

$$u(h|v,\bar v)=u(0|v,\bar v)+hu_1(h|v,\bar v),$$

where

$$u(0|v,\bar v)=f(0|z,\bar v)\,g(0|v,\bar z),$$
$$u_1(h|v,\bar v)=f(0|z,\bar v)\,g_1(v,\bar z)+f_1(z,\bar v)\,g(0|v,\bar z)+hu_2(h\,v,\bar v),$$
$$u_2(h|v,\bar v)=f(0|z,\bar v)\,g_2(h|v,\bar z)+f_2(h|z,\bar v)\,g(0|v,\bar z)+f_1(z,\bar v)\,g_1(v,\bar z).$$

From these assumptions it also follows that for $h\le h_0$

$$c(h)\int |u_2|\,e^{\frac{1}{h}\varphi}\,d\mu(v,\bar v)\leqslant a,$$

where $a=a(z,\bar z)$ does not depend on h.

Now recall that according to (2.29) and (2.30) with $h\in E$ we have

$$c(h)=\tilde c(h)=\frac{1-hb(h)}{h^n},$$

with $b(0)=\sigma$. We apply Lemma 2.1 to the integral (2.40) (note that according to Lemma 2.4 the applicability of Lemma 2.1 follows from hypotheses A–D). As a consequence of some obvious calculations and estimates we obtain

$$\psi(h\,|\,z,\bar z)=f(0\,|\,z,\bar z)\,g(0\,|\,z,\bar z)+h\,(\Delta_{v\bar v}f(0\,|\,z,\bar v)\,g(0\,|\,v,\bar z)|_{\substack{v=z\\ \bar v=\bar z}}\\ +f(0\,|\,z,\bar z)\,g_1(z,\bar z)+f_1(z,\bar z)\,g(0\,|\,z,\bar z))+o(h). \tag{2.41}$$

Equation (2.38) follows immediately from (2.41). By interchanging the roles of f and g we find, in addition, that

$$\frac{1}{h}(f*g-g*f)(h\,|\,z,\bar z)\\ =\Delta_{v,\bar v}(f(0\,|\,z,\bar v)\,g(0\,|\,v,\bar z)-g(0\,|\,z,\bar v)\,f(0)\,|\,v,\bar z))_{\substack{v=z\\ \bar v=\bar z}}+o(1). \tag{2.42}$$

Let us now turn our attention to the fact that the Laplace-Beltrami operator on a Kähler manifold in local coordinates has the form (as applied to functions)

$$\Delta=\sum g^{\bar l k}\frac{\partial^2}{\partial\bar z^l\partial z^k} \tag{2.43}$$

(see Lemma 3 of the Appendix). By comparing (2.42) and (2.43) with the expression for the Poisson bracket (2.22), we see that the first term on the right-hand side of (2.42) is equal to $[f,g]/i$. The theorem is proved.

Theorem 2.3. *Let hypothesis* B *be valid. Then, for any points* $z_1, z_2 \in \mathfrak{M}$, *there exists a function* $f(h|z,\bar z)\in\mathfrak{A}\cap\mathfrak{B}$ *such that* $f(0|z_1,\bar z_1)\neq f(0|z_2,\bar z_2)$.

Proof. Let $z_1, z_2\in\mathfrak{M}$. According to hypothesis B, for some $\alpha\in E$ there exists a function $\psi_1(z)\in F_\alpha$ such that $\psi_1(z_1)\neq\psi_1(z_2)$. We shall show that there also exists in F_α a function $\psi(z)$ which has the property $|\psi(z_1)|\neq|\psi(z_2)|$. Indeed, if $\psi_1(z)$ does not possess this property, then for suitable ϵ, $|\epsilon|=1$, the function

$$\psi_2(z)=f_0(z)+\varepsilon\frac{\psi_1(z)}{|\psi_1(z_1)|},$$

certainly does, where $f_0\equiv 1\in F_\alpha$.

Consider the function $\psi_2(z)\overline{\psi_2(z)}e^{-\Phi(z,\bar z)/\alpha}$. In case it does not separate points z_1 and z_2, by replacing $\psi_2(z)$ by $\psi(z)=\psi_2(z)+\mu f_0(z)$ with suitable μ we can arrange it so that the function $\psi(z)\psi(z)e^{-\Phi(z,\bar z)/\alpha}$ will separate the points z_1 and z_2.

Consider the twice continuously differentiable function $\epsilon(h)$ possessing the properties $\epsilon(h)\geq 0$, $\epsilon(0)=1$ and $\epsilon(h)\equiv 0$ for $h>\alpha$. We shall show that the function

$$f(h\,|\,z,\bar z)=\varepsilon(h)\,\psi(z)\,\overline{\psi(z)}\,e^{-\frac{1}{\alpha}\Phi(z,\bar z)} \tag{2.44}$$

is the one desired. Let $\hat A$ be the operator in F_h with covariant symbol (2.44), $h<\alpha$ and $h\in E$. Let us estimate $\|A\|$:

$$(\hat{A}f)(z) = \varepsilon(h)\psi(z)c(h)\int \overline{\psi(v)}f(v)e^{\left(\frac{1}{h}-\frac{1}{\alpha}\right)\Phi(z,\bar{v})-\frac{1}{h}\Phi(v,\bar{v})}d\mu(v,\bar{v}). \qquad (2.45)$$

Denote the integral on the right-hand side by $\mathcal{I}(z)$ and apply the Cauchy-Bunjakovskiĭ inequality to it:

$$|\mathcal{I}(z)|^2 \leqslant c^{-1}(h)\|f\|^2\int|\psi(v)|^2 e^{\left(\frac{1}{h}-\frac{1}{\alpha}\right)(\Phi(z,\bar{v})+\Phi(v,\bar{z}))-\frac{1}{h}\Phi(v,\bar{v})}d\mu(v,\bar{v}).$$

By making use of the inequality $\Phi(z, \bar{v}) + \Phi(v, \bar{z}) - \Phi(v, \bar{v}) - \Phi(z, \bar{z}) \leq 0$ we obtain the further estimate

$$|\mathcal{I}(z)|^2 \leqslant c^{-1}(h)\|f\|^2 e^{\left(\frac{1}{h}-\frac{1}{\alpha}\right)\Phi(z,\bar{z})}\int|\psi(v)|^2 e^{-\frac{1}{\alpha}\Phi(v,\bar{v})}d\mu(v,\bar{v})$$
$$= c^{-1}(h)c^{-1}(\alpha)\|f\|^2\|\psi\|^2 e^{\left(\frac{1}{h}-\frac{1}{\alpha}\right)\Phi(z,\bar{z})}.$$

Returning to (2.45), we obtain

$$\|\hat{A}f\|^2 \leqslant \varepsilon^2(h)c(h)\,c^{-1}(\alpha)\,\|f\|^2\|\psi\|^2\int|\psi(z)|^2 e^{\left(\frac{1}{h}-\frac{1}{\alpha}\right)\Phi(z,\bar{z})-\frac{1}{h}\Phi(z,\bar{z})}d\mu(z,\bar{z})$$
$$= \varepsilon^2(h)c(h)c^{-2}(\alpha)\|\psi\|^4\|f\|^2.$$

Hence $\|A\| = \epsilon(h)c^{1/2}(h)c^{-1}(\alpha)\|\psi\|^2$, so that $f(h|z, \bar{z}) \in \mathfrak{A}$.

We now show that $f \in \mathfrak{B}$. Set

$$f_1(z) = \psi(z)^{-\frac{1}{\alpha}\Phi(z,\bar{v})}.$$

It is sufficient to verify that $f_1 \in F_h$ for $h < \alpha$ and v fixed, and that $\|f_1\|_h \leq r$, where $r = r(v)$ does not depend on h. Let $\chi(z, \bar{z})$ denote the function

$$\chi(z,\bar{z}) = |\psi(z)|^2 e^{-\frac{1}{\alpha}(\Phi(z,\bar{v})+\Phi(v,\bar{z}))}$$

and let z_0 be the distinguished point. In this case

$$\|f_1\|_h^2 = c(h)\int\chi(z,\bar{z})e^{\frac{1}{h}\varphi(z,\bar{z}|z_0,\bar{z}_0)}d\mu(z,\bar{z}),$$

where

$$\varphi(z,\bar{z}\,|\,z_0,\bar{z}_0) = \Phi(z,\bar{z}_0)+\Phi(z_0,\bar{z})-\Phi(z_0,\bar{z}_0)-\Phi(z,\bar{z}) = -\Phi(z,\bar{z})$$

in view of the fact that z_0 is the distinguished point. We are now in a position to apply Lemma 2.1. According to this lemma, as $h \to 0$ the function $\|f_1\|_h^2$ has the limit

$$\lim_{h\to 0}\|f_1\|_h^2 = \chi(z_0,\bar{z}_0) < \infty,$$

from which the necessary estimate follows. The theorem is proved.

Theorems 2.2 and 2.3 mean that the correspondence principle in its weak form is satisfied.

5. *Properties of the operator* T_h. In general, the operator T is given by (2.8). In our case the measure σ and the vectors ϕ_α depend on h, and therefore the operator $T = T_h$ also depends on h. By taking hypothesis A into account we see that for $h \in E$ the operator T_h is equal to

$$(T_h f)(z, \bar{z}) = \int f(v, \bar{v})\, G_h(z, \bar{z} \mid v, \bar{v})\, d\mu(v, \bar{v}), \tag{2.46}$$

where

$$G_h(z, \bar{z} \mid v, \bar{v}) = c(h) \frac{L_h(z, \bar{v})\, L_h(v, \bar{z})}{L_h(z, \bar{z})\, L_h(v, \bar{v})}.$$

We shall consider operators T_h which belong to L^p on $\mathfrak{M}$. The L^p norm of T_h is denoted by $\|T_h\|_p$.

Theorem 2.4. 1) $\|T_h\|_p \leq 1$. 2) *As* $h \to 0$, $\lim T_h = I$ *in the weak sense, where* I *is the unit operator in* L^p.

Proof. The kernel $G_h(z, \bar{z} \mid v, \bar{v})$ has the following properties: 1) $G_h \geq 0$. 2) $G_h(z, \bar{z} \mid v, \bar{v}) = G_h(v, \bar{v} \mid z, \bar{z})$. 3) $\int G_h(z, \bar{z} \mid v, \bar{v})\, d\mu(v, \bar{v}) = 1$.

(Properties 1 and 2 are obvious; property 3 follows from (2.23) with $A_1 = A_2 \equiv 1$.) Therefore the first assertion of the theorem follows from the following general fact.

Let M *be a set with measure* dx, *and let* A *be an operator in* $L^p(M)$, $p \geq 1$, *defined by*

$$(Af)(x) = \int K(x, y)\, f(y)\, dy,$$

where $K(x, y) \geq 0$ *for almost all* $x, y \in M$ *and* $\int K(x, y)\, dy = \int K(x, y)\, dx = 1$. *Then* $\|A\|_p \leq 1$.

Indeed, by Riesz's theorem [10], $\ln \|A\|_p$ is a convex function of p^{-1} for $1 \leq p \leq \infty$. Therefore it is sufficient to verify the estimate for the norm in the spaces $L^1(M)$ and $L^\infty(M)$ ($\|f\|_\infty = \sup |f(x)|$). These estimates are obvious:

$$\|Af\|_1 \leqslant \int K(x, y) |f(y)|\, dy\, dx = \int |f(y)|\, dy = \|f\|_1,$$

$$\|Af\|_\infty \leqslant \sup_x \int K(x, y)\, |f(y)|\, dy \leqslant \sup_x \sup_s |f(s)| \int K(x, y)\, dy = \sup_s |f(s)| = \|f\|_\infty.$$

We turn now to the second assertion. First let f be a continuously differentiable function which is equal to zero outside of some coordinate neighborhood U. In consideration of hypothesis A we write $T_h f$ in the form

$$(T_h f)(z, \bar{z}) = c(h) \int f(v, \bar{v})\, e^{\frac{1}{h} \varphi(v, \bar{v} \mid z, \bar{z})}\, d\mu(v, \bar{v}).$$

According to Lemma 2.3 it follows from hypotheses A, B, C and D that each point of $\mathfrak{M}$ is proper and therefore $\phi \leq 0$, so that the inequality is strict if $z \neq v$. If $z = v$, it follows immediately from (2.25) that $\phi = 0$. Consequently for fixed z the function ϕ has its maximum with respect to v at $v = z$. Let $U_1 \supset U$ be a larger coordinate neighborhood.

Let us fix $z \in U_1$ and introduce coordinates t^i into U_1 so that the coordinates of z are $t^i = 0$. By the above, in these coordinates

$$\varphi(z, \bar{z} \mid v, \bar{v}) = \psi(z, \bar{z} \mid t, \bar{t}) = \sum \varphi_{i\bar{k}} t^i \bar{t}^k + R(t, \bar{t}),$$
$$R(t, \bar{t}) = \sum_{|p|+|q|=3} t^p \bar{t}^q R_{p,q}(t, \bar{t}),$$

where $R_{p,q}(t, \bar{t})$ is a continuous function and p and q are multi-indices. We see that we can apply Lemma 3 of the Appendix, according to which, for $z \in U$,

$$|(T_h f)(z, \bar{z}) - f(z, \bar{z})| \leqslant a\sqrt{h} \max_{t \in U} \left(\left| \frac{\partial f}{\partial t^i} \right|, \left| \frac{\partial f}{\partial \bar{t}^i} \right| \right) + f(z, \bar{z}) \cdot o(1). \tag{2.47}$$

In case $z \notin U_1$, it follows from the fact that z is a proper point that

$$(T_h f)(z, \bar{z}) \leqslant a e^{-\frac{b}{h}}, \quad b > 0. \tag{2.47$'$}$$

Let $h_n \to 0$, let T_{h_n} be a weakly convergent sequence, and set $Q = \lim T_{h_n}$. From (2.47) and (2.47′) it follows that if f is a continuously differentiable function and $f = 0$ outside of some coordinate neighborhood, then $Qf = f$. Therefore $Q = I$. Thus Q cannot depend on the choice of the sequence h_n, and consequently T_h has a weak limit as $h \to 0$: $\lim T_h = I$.

6. *Discussion of the hypotheses.* The basic hypothesis concerning the existence of a potential is valid for local Kähler manifolds, i.e. for bounded regions in C^n having Kähler metrics. Kähler potentials certainly do not exist globally on compact manifolds. However, they can exist on a manifold $\widetilde{\mathfrak{M}}$ obtained from a compact manifold $\mathfrak{M}$ by removing a submanifold of smaller dimension.

There follows from the hypotheses A–D a curious differential-geometric property of the manifolds $\mathfrak{M}$.

Theorem 2.5. *Let* $g(z, \bar{z}) = \det \| g_{i\bar{k}} \|$. *When hypotheses* A–D *are valid,* $g(z, \bar{z})$ *satisfies the equation* $\Delta \ln g = \text{const}$, *where* Δ *is the Laplace-Beltrami operator on* $\mathfrak{M}$.

Proof. By Lemma 2.3, hypotheses A–D imply the applicability of Lemma 2.1 to the function $f(z, \bar{z}) \equiv 1$. In this way we also have (2.30). Note that

$$\tilde{c}_1'(0) = \tilde{c}_1'(0 \mid z, \bar{z}) = \lim_{h \to 0} \frac{\tilde{c}_1(h) - \tilde{c}_1(0)}{h} = \lim_{h \to 0} \frac{h^n \tilde{c}(h \mid z, \bar{z}) - 1}{h}. \tag{2.48}$$

Now let $h \to 0$ on the right-hand side of (2.48) by means of some sequence $h_n \in E$, where E is the set involved in condition A. With $h \in E$, $\tilde{c}(h \mid z, \bar{z}) = c(h)$ is independent of $(z, \bar{z})$ (see (2.33) and (2.33′)). Consequently $\tilde{c}_1'(0)$ possesses the same property, and, in view of (2.30),

$$\sigma(z, \bar{z}) = \frac{3}{2} \Delta \ln g(z, \bar{z}) = \text{const}.$$

The theorem is proved.

7. *Dimension of the space* F_h.

Theorem 2.6.

$$\dim F_h = c(h) \int L_h(z, \bar{z}) e^{-\frac{1}{h}\Phi(z,\bar{z})} d\mu(z, \bar{z}). \tag{2.49}$$

Proof. Equation (2.49) follows from (2.24) with $\hat{A} = I$ (see also footnote (6)).

8. *Possible generalizations.* Consider the associative algebra A which consists of those functions on $\mathfrak{M}$ which can be analytically continued to $\mathfrak{M} \times \mathfrak{M}$. In analogy with (2.23) we give the law of multiplication in A in the form

$$(A_1 * A_2)(z, \bar{z}) = \int A_1(z, \bar{v}) A_2(v, \bar{z}) G(v, \bar{v} | z, \bar{z}) d\mu(v, \bar{v}). \tag{2.50}$$

It can be shown that if A contains sufficiently many elements, then necessarily

$$G = \frac{F(z, \bar{v}) F(v, \bar{z})}{F(z, \bar{z}) \tilde{F}(v, \bar{v})}, \tag{2.51}$$

where $F(z, \bar{v})$ is an analytic function on $\mathfrak{M} \times \mathfrak{M}$ and $\tilde{F}(v, \bar{v})$ is some function of $\mathfrak{M}$.

If the algebra A possesses property 8) of quantizations (regarding the trace) and the set $\mathfrak{X}$ on which the trace is defined is sufficiently large, we necessarily have $\tilde{F}(v, \bar{v}) = cF(v, \bar{v})$.

We shall not dwell on these statements, but merely note that in the case where the region of C^n is bounded, but not homogeneous, it is natural to try to construct an algebra by means of (2.50) and (2.51) by setting $F(z, \bar{z}) = K^{1/h}(z, \bar{z})$, $\tilde{F} = cF$, where K is the Bergman kernel function. In this case the correspondence principle would follow from Lemma 2.1. However, even in this simple situation, if just hypothesis A is not valid, it cannot be proved that an algebra having the law of multiplication (2.50), (2.51) exists which contains any other function $f(z, \bar{z})$ than $f(z, \bar{z}) \equiv 0$.

§3. Quantization in homogeneous Kähler manifolds

1. *Linear representations of a local group of motions in the space* F_h. Let $\mathfrak{M}$ be a homogeneous Kähler manifold, let G be a group of motions in $\mathfrak{M}$, and let $\Phi(z, \bar{z})$ be a potential, invariant with respect to G, for a metric on $\mathfrak{M}$. We shall assume that the potential Φ exists globally on the set $\tilde{\mathfrak{M}}$ obtained from $\mathfrak{M}$ by removing a submanifold $\mathfrak{M}'$ of smaller dimension. From the invariance of the metric generated by Φ it follows that

$$\Phi(gz, \overline{gz}) = \Phi(z, \bar{z}) + \psi(g, z) + \overline{\psi(g, z)}, \tag{3.1}$$

where $\psi(g, z)$ is for fixed g an analytic function of z defined over $\tilde{\mathfrak{M}} \cap g\tilde{\mathfrak{M}}$. Relation (3.1) defines a function $\psi(g, z)$ up to a purely imaginary term. In order to remove this indeterminacy we select a point z_0 and set

$$\psi(g, z_0) = \frac{1}{2}(\Phi(gz_0, \overline{gz_0}) - \Phi(z_0, \bar{z}_0)).$$

Fix an open set $\mathfrak{M}_0 \subset \tilde{\mathfrak{M}}$. Consider a symmetric neighborhood U_G of the unit e of the group G which is sufficiently small that $g\mathfrak{M}_0 \subset \widetilde{\mathfrak{M}} \cap g\widetilde{\mathfrak{M}}$ for $g \in U_G$. It follows from (3.1) that for $g_1, g_2, g_1g_2 \in U_G$ and $z \in \mathfrak{M}_0$

$$\operatorname{Re}\psi(g_1g_2, z) = \operatorname{Re}\psi(g_1, g_2z) + \operatorname{Re}\psi(g_2, z).$$

In turn, from this under the same conditions we get the relation

$$\psi(g_1g_2, z) = \psi(g_1, g_2z) + \psi(g_2, z) + i\alpha(g_1, g_2), \tag{3.2}$$

where $\alpha(g_1, g_2)$ is a real function.

We shall show that for any $z \in \widetilde{\mathfrak{M}}$

$$\psi(e, z) = 0. \tag{3.3}$$

The relation (3.3) is correct for $z = z_0$ by the definition of the function ψ. Setting $g_1 = g_2 = e$ in (3.2) shows that $\psi(e, z) = -\alpha(e, e)$ does not depend on z. In particular, $0 = \psi(e, z_0) = -\alpha(e, e)$, so that $\psi(e, z) = -\alpha(e, e) = 0$.

Theorem 3.1. *Let hypothesis A be valid and let $g \in U_G$. Then the following assertions are true:*

1) *If $f(z) \in F_h$, then the function*

$$f(gz)\, e^{-\frac{1}{h}\psi(g,z)},$$

defined originally on $\mathfrak{M}_0$, can be continued analytically to $\widetilde{\mathfrak{M}}$ and belongs to F_h.

2) *The operators in F_h*

$$(\hat{T}_g f)(z) = f(g^{-1}z)\, e^{-\frac{1}{h}\psi(g^{-1},z)} \tag{3.4}$$

are unitary and define projective representations of the local group U_G.

3) *In the case where relation (3.2) defining α is*

$$\alpha(g_1, g_2) = \sigma(g_1g_2) - \sigma(g_1) - \sigma(g_2),$$

the operators $\hat{T}_g e^{i\sigma(g)}$ form a linear representation of U_G.

4) *The representations $\hat{T}_g$ are irreducible.*

5) *Let $\hat{A}$ be a bounded operator in F_h, let $A(z, \bar{z})$ be its covariant symbol, and let $(\tau_g A)(z, \bar{z})$ be the covariant symbol for the operator $\hat{T}_g \hat{A} \hat{T}_g^{-1}$. Then:*

5_1) *the function $A(z, \bar{z})$ originally defined on $\widetilde{\mathfrak{M}}$ can be continued analytically to all of $\mathfrak{M}$, and*

5_2) *for all $z \in \mathfrak{M}$*

$$(\tau_g A)(z, \bar{z}) = A(g^{-1}z, \overline{g^{-1}z}). \tag{3.5}$$

Proof. Using (3.1), we find from (2.35) with $v, z \in \widetilde{\mathfrak{M}} \cap g\widetilde{\mathfrak{M}}$ that

$$\Phi_{\overline{g^{-1}v}}(g^{-1}z) = \Phi_{\bar{v}}(z)\, e^{\frac{1}{h}\psi(g^{-1},z) + \frac{1}{h}\overline{\psi(g^{-1},v)}} \tag{3.6}$$

Let $w = g^{-1}v$ and use the fact that according to (3.3) and (3.2),

$$\psi(g^{-1}, gz) = -\psi(g, z) - i\alpha(g^{-1}, g).$$

As a result we have

$$\Phi_{\bar{w}}(g^{-1}z)\,e^{-\frac{1}{h}\psi(g^{-1},z)} = \Phi_{\overline{gw}}(z)\,e^{-\frac{1}{h}\overline{\psi(g,w)}+\frac{i}{h}\alpha(g^{-1},g)}. \tag{3.7}$$

Note that the right-hand side of (3.7) for fixed w can be continued analytically with respect to z to $\widetilde{\mathfrak{M}}$. Consequently the left-hand side has the same property.

Therefore the operator (4.3) is defined for the functions $\Phi_{\bar{v}}(z)$, $v \in \widetilde{\mathfrak{M}} \cap g\widetilde{\mathfrak{M}}$. We denote the operator (3.4) restricted to functions of this type by $\hat{T}'_g$. It follows from (3.7) that $T'_g\Phi_{\bar{v}} \in F_h$. By using (3.7) and (3.6) we find

$$\begin{aligned}(\hat{T}'_g\Phi_{\bar{w}}, \hat{T}'_g\Phi_{\bar{v}}) &= (\Phi_{\overline{gw}}, \Phi_{\overline{gv}})\,e^{-\frac{1}{h}[\overline{\psi(g,w)}+\psi(g,v)]} \\ &= \Phi_{\overline{gw}}(gv)\,e^{-\frac{1}{h}[\overline{\psi(g,w)}+\psi(g,v)]} = \Phi_{\bar{w}}(v) = (\Phi_{\bar{w}}, \Phi_{\bar{v}}).\end{aligned} \tag{3.8}$$

Let F'_h denote the set of finite linear combinations of the vectors $\Phi_{\bar{v}_k}(z)$, $v_k \in \widetilde{\mathfrak{M}} \cap g\widetilde{\mathfrak{M}}$. From the general formula (2.19) it follows that F'_h is dense in F_h.

Let $\hat{T}''_g$ denote the extension, by linearity, of $\hat{T}'_g$ to F'_h, and let $\hat{T}'''_g$ be the closure of $\hat{T}''_g$. It is clear that $\hat{T}''_g$ is defined by (3.4). The unitarity of $\hat{T}''_g$ follows from (3.8). Let $f_n \in F'_h$ and $f = \lim f_n$ with respect to the norm of F_h. It follows from (2.19) that $f = \lim f_n$ uniformly within any open set contained with its closure in $\widetilde{\mathfrak{M}}$. Hence it follows that $\hat{T}'''_g$ as also defined by (3.4), i.e. $\hat{T}'''_g = \hat{T}_g$. In particular, the right-hand side of (3.4) is a single-valued analytic function, and $(\hat{T}_g f, \hat{T}_g f) = (f, f)$. Hence the fact that the operators $\hat{T}_g$ form a projective representation of the local group U_G follows immediately from relation (3.2). Thus we have the third assertion of the theorem.

We prove the fifth assertion. Using (3.7), we obtain for $v \in \mathfrak{M}_0$

$$(\hat{T}_g\hat{A}\hat{T}_g^{-1}\Phi_{\bar{v}}, \Phi_{\bar{v}}) = (\hat{A}T_g^{-1}\Phi_{\bar{v}}, \hat{T}_g^{-1}\Phi_{\bar{v}}) = (\hat{A}\Phi_{\overline{g^{-1}v}}, \Phi_{\overline{g^{-1}v}})\,e^{-\frac{1}{h}[\overline{\psi(g^{-1},v)}+\psi(g^{-1},v)]}.$$

In particular, with $\hat{A} = I$

$$(\Phi_{\bar{v}}, \Phi_{\bar{v}}) = (\Phi_{\overline{g^{-1}v}}, \Phi_{\overline{g^{-1}v}})\,e^{-\frac{1}{h}[\psi(g^{-1},v)+\overline{\psi(g,^{-1}v)}]}.$$

Consequently the symbol for the operator $\hat{T}_g\hat{A}\hat{T}_g^{-1}$ with $v \in \mathfrak{M}_0$ is

$$\frac{(\hat{T}_g\hat{A}\hat{T}_g^{-1}\Phi_{\bar{v}}, \Phi_{\bar{v}})}{(\Phi_{\bar{v}}, \Phi_{\bar{v}})} = \frac{(\hat{A}\Phi_{\overline{g^{-1}v}}, \Phi_{\overline{g^{-1}v}})}{(\Phi_{\overline{g^{-1}v}}, \Phi_{\overline{g^{-1}v}})} = A(g^{-1}v, \overline{g^{-1}v}). \tag{3.9}$$

The left-hand side of (3.9) obviously has an analytic continuation to $\widetilde{\mathfrak{M}}$, so that the right-hand side has the same property. Let $v_0 \in \mathfrak{M}' = \mathfrak{M}\backslash\widetilde{\mathfrak{M}}$, and let the element $g \in U_G$ possess the property that $v_1 = gv_0 \in \widetilde{\mathfrak{M}}$.(10) The left-hand side of (3.9) is defined for $v = v_1$. This makes it possible to define the right-hand side for $v = v_0$. Thus we can define the symbol $A(z, \bar{z})$ over the entire manifold $\mathfrak{M}$. It is clear that the function $A(z, \bar{z})$ defined over all of $\mathfrak{M}$ which is obtained in this way is analytic in the real sense and that (3.9) has a meaning for all $v \in \widetilde{\mathfrak{M}}$.

(10) Such an element exists because of the assumption that $\mathfrak{M}'$ is a manifold of smallest dimensionality.

We next turn to the fourth assertion. Let $\hat{A}$ be a bounded operator which commutes with all the T_g, and let $A(z, \bar{z})$ be its symbol. It follows from (3.9) that $A(z, \bar{z}) = A(g^{-1}z, \overline{g^{-1}z})$ for $g \in U_G$. Since the group G is generated by any neighborhood of the identity, the last relation is correct for all $g \in G$. By the transitivity of the action of G on $\mathfrak{M}$ it follows from this that $A(z, \bar{z}) = a = \text{const}$. In view of the one-to-one character of the correspondence between symbols and operators we therefore have that $\hat{A} = aI$, where I is the unit operator in F_h. The theorem is completely proved.

2. *Projective representation of the total group of motions in the space* F_h.

Theorem 3.2. *The representation* (3.4) *can be extended to a unitary projective representation of the entire group* G.

Proof. Consider the algebra A_h as a linear space, and look at the representation of the local group U_G in A_h defined by (3.5). This representation can be extended to a linear representation of the whole group G, since G is generated by U_G. On the same basis, for all $g \in G$ the transformations (3.5) are automorphisms of the algebra A_h. By construction, A_h is isomorphic to the algebra of all bounded operators in F_h. It is well known [11] that the automorphisms of the algebra of all bounded operators in Hilbert space are inner. Therefore for each $g \in G$ there exists a bounded operator $\hat{U}_g$ in F_h which generates the automorphism (3.5):

$$\hat{A}^g = \hat{U}_g \hat{A} \hat{U}_g^{-1}, \tag{3.10}$$

where $\hat{A} \to \hat{A}^g$ is an automorphism of the operator algebra corresponding to r_g. The operator $\hat{U}_g$ is given by (3.10) up to a factor. It is clear that these operators form a projective representation of G.

The transformation r_g takes real functions into real functions. Consequently the automorphism (3.10) takes Hermitian operators into Hermitian operators. Hence the operator $\hat{U}_g$ differs from a unitary operator only by a factor. Thus the operators $\hat{U}_g$ can be considered as unitary operators multiplied by a factor of modulus 1. The theorem is proved.

Let $\widetilde{G}$ denote the group which consists of all the operators $\hat{U}_g$, $g \in G$. It is clear that $\widetilde{G}$ is a central extension of G. Thus the operators U_g form a unitary linear representation of $\widetilde{G}$. We denote this representation by $\hat{T}_{\tilde{g}}$. Let $\pi: \widetilde{G} \to G$ denote the homomorphism defined by the equation $\pi(\hat{U}_g) = g$. In what follows, elements of $\widetilde{G}$ will be denoted by $\tilde{g}$.

Let $U_{\widetilde{G}} \subset \widetilde{G}$ be the preimage of the neighborhood U_G under the homomorphism π, and let $\hat{T}_g$ be the representation (3.4). By definition $\hat{T}_{\tilde{g}} = \theta \hat{T}_g$ with $\tilde{g} \in U_{\widetilde{G}}$, $g = \pi(\tilde{g})$ and $|\theta| = 1$. Consequently

$$(\hat{T}_{\tilde{g}} f)(z) = f(g^{-1}z)\, e^{-\frac{1}{h}\varphi(\tilde{g}^{-1}, z)}, \tag{3.11}$$

where $\phi(\tilde{g}, z) = \psi(g, z) + i\beta(\tilde{g})$, $\beta(\tilde{g})$ being a real function defined up to a term of the form $2\pi nh$. This indeterminacy may be removed by setting $\beta(e) = 0$. In this case, by (3.3) we also have $\phi(e, z) = 0$. In the following it will be assumed that $\beta(e) = \phi(e, z) = 0$.

Theorem 3.3. *Let $f_0(z) \equiv 1 \in F_h$. Then* 1) *the function $e^{-\phi(\tilde{g},z)/h}$ can be extended to $\tilde{G} \times \tilde{\mathfrak{M}}$ with preservation of analyticity with respect to z, and* 2) *a global representation $\hat{T}_{\tilde{g}}$ of the group G is given by* (3.11).

Proof. From the fact that the operators (3.11) form a representation of the local group $U_{\tilde{G}}$, and from the condition $\phi(e, z) = 0$, it follows that for $\tilde{g}_1, \tilde{g}_2, \tilde{g}_1\tilde{g}_2 \in U_{\tilde{G}}$ and $z \in \mathfrak{M}_0$

$$\varphi(\tilde{g}_1 \tilde{g}_2, z) = \varphi(\tilde{g}_1, g_2 z) + \varphi(\tilde{g}_2, z), \quad g_2 = \pi(\tilde{g}_2). \tag{3.12}$$

Setting $f(z) \equiv 1$ in (3.11), we see that $\epsilon(\tilde{g}, z) \equiv \exp[-\phi(\tilde{g}, z)/h]$ is defined on $U_{\tilde{G}} \times \tilde{\mathfrak{M}}$ and is an analytic function of z for fixed g. From (3.12) it follows that for $\tilde{g}_1, \tilde{g}_2, \tilde{g}_1\tilde{g}_2 \in U_{\tilde{G}}$ and $z \in \mathfrak{M}_0$

$$\varepsilon(\tilde{g}_1\tilde{g}_2, z) = \varepsilon(\tilde{g}_1, g_2 z)\, \varepsilon(\tilde{g}_2, z) \quad (g_2 = \pi(\tilde{g}_2)). \tag{3.13}$$

Now recall that the representation (3.11) can be extended to a unitary representation of the whole group $\tilde{G}$. In particular, $\hat{T}_{\tilde{g}_1}\hat{T}_{\tilde{g}_2} = \hat{T}_{\tilde{g}_1\tilde{g}_2}$ with $\tilde{g}_1, \tilde{g}_2 \in U_{\tilde{G}}$. By combining these identities with (3.13), we find that $\epsilon(\tilde{g}, z)$ has a unique extension to $U^2_{\tilde{G}} \times \tilde{\mathfrak{M}}$, which preserves analyticity with respect to z, the property (3.13), and the fact that the operators $\hat{T}_{\tilde{g}}$ have the form (3.11) with $\tilde{g} \in U^2_{\tilde{G}}$. By repeating this argument we find that for any integer $n > 0$ there is a single-valued extension of $\epsilon(\tilde{g}, z)$ to $U^n_{\tilde{G}} \times \tilde{\mathfrak{M}}$ which preserves analyticity with respect to z and property (3.13), and extends formula (3.11) to $U^n_{\tilde{G}}$. This concludes the proof of the theorem, since $\tilde{G} = \bigcup U^n_{\tilde{G}}$.

3. *Covariant symbols of the operators* $\hat{T}_g$. By combining (3.2) and (3.12), we find that $\alpha(g_1, g_2) = \beta(\tilde{g}_1\tilde{g}_2) - \beta(\tilde{g}_1) - \beta(\tilde{g}_2)$. In particular, $\alpha(g^{-1}, g) = -\beta(\tilde{g}) - \beta(\tilde{g}^{-1})$. Therefore it follows from (3.7) and (3.11) that

$$(\hat{T}_{\tilde{g}}\Phi_{\bar{v}}, \Phi_{\bar{v}}) = (\Phi_{\overline{gv}}, \Phi_{\bar{v}})\, e^{-\frac{1}{h}\overline{\varphi(\tilde{g},v)}}$$
$$= \Phi_{\overline{gv}}(v)\, e^{-\frac{1}{h}\overline{\varphi(\tilde{g},v)}} = L_h(v, \overline{gv})\, e^{-\frac{1}{h}\overline{\varphi(\tilde{g},v)}}, \quad g = \pi(\tilde{g}).$$

Hence we have the covariant symbol for the operator $\hat{T}_{\tilde{g}}$:

$$T_{\tilde{g}}(z, \bar{z}) = \frac{(\hat{T}_{\tilde{g}}\Phi_{\bar{z}}, \Phi_{\bar{z}})}{(\Phi_{\bar{z}}, \Phi_{\bar{z}})} = \frac{L_h(z, \overline{gz})}{L_h(z, \bar{z})}\, e^{-\frac{1}{h}\overline{\varphi(\tilde{g}, z)}}. \tag{3.14}$$

Let $\tilde{g}(t)$ be a one-parameter subgroup of $\tilde{G}$. Represent the operator $\hat{T}_{\tilde{g}(t)}$ in the form $\hat{T}_{\tilde{g}(t)} = \exp((t/h)\hat{\mathfrak{L}})$. The covariant symbol $\mathfrak{L}(z, \bar{z})$ for the Lie operator $\hat{\mathfrak{L}}$ can be calculated by setting $\tilde{g} = \tilde{g}(t)$ in (3.14) and then evaluating the derivative of (3.14) at $t = 0$. It follows from hypothesis A that $\mathfrak{L}(z, \bar{z})$ does not depend on h:

$$\mathcal{L}(z, \bar{z}) = h\frac{d}{dt} e^{\frac{1}{h}[\Phi(z,\overline{gz}) - \overline{\varphi(\tilde{g},z)} - \Phi(z,\bar{z})]}\Big|_{t=0} = \frac{d}{dt}[\Phi(z, \overline{gz}) - \overline{\varphi(\tilde{g}, z)}]\Big|_{t=0}.$$

Theorem 3.4. *1) The symbols for the Lie operators possess the quasiclassical property. 2) If $\tilde{g}(t)$ is a one-parameter subgroup of $\tilde{G}$, $\mathfrak{L}(z, \bar{z})$ is the covariant symbol for the corresponding Lie operator, and $g(t) = \pi(\tilde{g}(t))$, then*

$$\frac{d}{dt}\tau_{g(t)}A\,|_{t=0} = \frac{d}{dt}A\,(g^{-1}(t)\,z,\ \overline{g^{-1}(t)\,z})\,|_{t=0} = \frac{1}{i}\,[\mathcal{L}, A], \tag{3.15}$$

where $[\mathfrak{L}, A]$ is the Poisson bracket in $A(\mathfrak{M})$.

Proof. In view of the connection between the representations τ_g and $T_{\tilde{g}}$ we have

$$\frac{d}{dt}\tau_{g(t)}A\,|_{t=0} = \frac{1}{h}\,(\mathcal{L} * A - A * \mathcal{L}). \tag{3.16}$$

Both functions $\mathfrak{L}$ and A do not depend on h. Therefore as $h \to 0$ the right-hand side of (3.16) is equal to $[\mathfrak{L}, A]/i + o(1)$, where $[\mathfrak{L}, A]$ is the Poisson bracket.(11) Since the left-hand side of (3.16) does not depend on h, we must have $o(1) = 0$. The theorem is proved.

§4. Quantizations in C^n(12)

1. *Wick quantization.* This version of the quantization of a mechanics with plane phase space is the simplest example of the two constructions in §2. In suitable coordinates the Kähler potential has the form

$$\Phi\,(z, \bar{z}) = \sum z_k \bar{z}_k. \tag{4.1}$$

The distinguished point is the origin.

The space F_h consists of entire functions which are square summable with respect to the measure

$$(f, f) = \frac{1}{h^n}\int |f(z)|^2 e^{-\frac{1}{h}\Phi(z,\bar{z})}\,d\mu\,(z, \bar{z}), \tag{4.2}$$

where $d\mu = d\mu_L/\pi^n$, $d\mu_L$ being ordinary Lebesgue measure on C^n. In this case F_h is called a Fock space.(13) One orthonormal basis in F_h consists of the vectors

$$f_k(z) = \prod \frac{z_s^{k_s}}{\sqrt{k_s!\,h^{k_s}}}, \quad k = k_1, \ldots, k_n. \tag{4.3}$$

It follows from (4.3) that

$$L_n(z, \bar{z}) = \sum f_k(z)\,\bar{f}_k(z) = e^{\frac{1}{h}\Sigma z_k \bar{z}_k} = e^{\frac{1}{h}\Phi(z,\bar{z})}.$$

(11) We omit the demonstration that the conditions of Theorem 2.2 are met.

(12) This section contains a short survey of the known results. For a more detailed exposition see [12], [13], [14] or [15]. Further references to the literature are given in these places.

(13) This was introduced by V. A. Fock in [16] and [17]. The scalar product he used was given not by the integral (3.2), but by a series in the Taylor coefficients of the function f. A description of the scalar product in F_h in the form (3.2) was first published in a paper by Bargmann [18]; however, it was well known well before the appearance of this article (see the report by R. A. Minlos, L. D. Faddeev and the author at the All-Union Mathematical Congress in 1961 [19].)

Consequently hypothesis A is valid. The correctness of hypotheses B and C is clear. The specialization of formulas (2.22), (2.23) and (2.24) is well known in the theory of Wick quantization.

In F_h consider the operators $\hat{a}_k$ and $\hat{a}_k^*$ (the "annihilation" and "creation" operators)

$$(\hat{a}_k f)(z) = h\frac{\partial f}{\partial z_k}, \quad (\hat{a}_k^* f)(z) = z_k f(z). \tag{4.4}$$

The expression of an operator $\hat{A}$ by means of $\hat{a}_k$ and $\hat{a}_k^*$ in the special form

$$\hat{A} = \sum A_{mn} (\hat{a}^*)^m \hat{a}^n$$

(m, n are multi-indices) is called its *Wick normal form.*

The term "Wick quantization" is related to the fact that the covariant symbol for the operator $\hat{A}$ is a product of functions in Wick normal form

$$A(z, \bar{z}) = \sum A_{mn} \bar{z}^m z^n. \tag{4.5}$$

If the operator $\hat{A}$ can be expressed in reverse (anti-Wick) normal form

$$\hat{A} = \sum \mathring{A}_{mn} \hat{a}^m (\hat{a}^*)^n,$$

then it can be associated with the product

$$\mathring{A}(z, \bar{z}) = \sum \mathring{A}_{mn} z^m \bar{z}^n.$$

The function $\mathring{A}$ is the contravariant symbol for the operator $\hat{A}$.

In this case the covariant and contravariant symbols for operators are also called Wick and anti-Wick.

The connection between the Wick and anti-Wick symbols, as well as the law of multiplication in the algebra of Wick symbols, according to the general theory, is given by the operator T_h. In the present case the kernel of this operator has the form

$$G_h = \frac{1}{h^n} e^{-\frac{1}{h}\Sigma(z_i - v_i)(\bar{z}_i - \bar{v}_i)}, \tag{4.6}$$

i.e. is the Poisson kernel. Consequently the operator T_h can be represented in the form

$$T_h = e^{h\Delta}, \tag{4.7}$$

where Δ is the Laplace operator.

2. *Weyl quantization.* This version of quantization does not make use of the complex structure, and therefore it is more natural to consider it as a quantization on R^{2n} rather than on C^n. As usual, we denote the coordinates in R^{2n} by $p = (p_1, \cdots, p_n)$, $q = (q_1, \cdots, q_n)$. Let $\phi(\alpha, \beta) d\alpha d\beta$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$, be a complex-valued measure of bounded variation in R^{2n}, and set

$$\mathcal{A}(p, q) = \int e^{i(p\alpha + q\beta)} \varphi(\alpha, \beta)\, d\alpha\, d\beta$$
$$\left(p\alpha = \sum p_k \alpha_k, \quad q\beta = \sum q_k \beta_k\right). \tag{4.8}$$

We associate a function $\mathcal{A}(p, q)$ with a bounded operator $\hat{A}$ in some Hilbert space H in the following manner. Let G denote the Heisenberg-Weyl group and let $\mathfrak{G}$ be the Lie algebra of this group. Let T_g be an irreducible unitary representation of this group in the space H. Let ξ_k, η_k, ζ be the basis in $\mathfrak{G}$ having the standard commutation relations $[\xi_k, \eta_j] = \delta_{kj}\zeta$ and $[\xi_k, \xi_j] = [\eta_k, \eta_j] = [\xi_k, \zeta] = [\eta_k, \zeta] = 0$. The representation T_g associates skew-Hermitian operators $\hat{\xi}_k, \hat{\eta}_k$ and $\hat{\zeta}$ with the elements ξ_k, η_k and ζ in H. Let $\hat{p}_k, \hat{q}_k$ and $\hat{z}$ denote the corresponding Hermitian operators $\hat{p}_k = -i\hat{\xi}_k$, $\hat{q}_k = -i\hat{\eta}_k$ and $\hat{z} = -i\hat{\zeta}$. Because of irreducibility we have $\hat{z} = hI$, where I is the identity operator.(14) Therefore among the $\hat{p}_k$ and $\hat{q}_k$ we have the relations

$$[\hat{p}_k, \hat{q}_j] = \delta_{kj}\frac{h}{i}I, \quad [\hat{p}_k, \hat{p}_j] = [\hat{q}_k, \hat{q}_j] = 0. \tag{4.9}$$

Let

$$\hat{A} = \int e^{i(\hat{p}\alpha + \hat{q}\beta)} \varphi(\alpha, \beta)\, d\alpha\, d\beta. \tag{4.10}$$

The association of the function (4.8) to the operator (4.10) was first proposed by Weyl [1] and is called Weyl quantization. The function $\mathcal{A}(p, q)$ is called the Weyl symbol for the operator $\hat{A}$. The operators (4.10) form a subalgebra of the algebra of bounded operators for which $\|\hat{A}\| \le \int |\phi|\, d\alpha\, d\beta$.

Let us consider the realization of the space H in the form $L^2(R^n)$ and try to express the operators $\hat{p}_k$ and $\hat{q}_k$ by the formulas

$$(\hat{p}_k f)(x) = \frac{h}{i}\frac{\partial}{\partial x_k} f(x), \quad (\hat{q}_k f)(x) = x_k f(x). \tag{4.11}$$

In this realization of the algebra $\mathfrak{G}$ the operator (4.10) is described by means of a kernel $K(x, y)$ which is associated with its Weyl symbol by the formulas

$$K(x, y) = \frac{1}{(2\pi h)^n}\int \mathcal{A}\left(p, \frac{x+y}{2}\right) e^{\frac{ip}{h}(x-y)}\, dp,$$
$$\mathcal{A}(p,q) = \int K\left(q - \frac{\xi}{2}, q + \frac{\xi}{2}\right) e^{-\frac{ip\xi}{h}}\, d\xi. \tag{4.12}$$

The multiplication law in the algebra of Weyl symbols follows from (4.12): if $\hat{A} = \hat{A}_1\hat{A}_2$, then the corresponding symbols are connected by the relation(15)

$$\mathcal{A}(p,q) = \left(\frac{2}{h}\right)^n \int \mathcal{A}_1(p^{(1)}, q^{(1)})\, \mathcal{A}_2(p^{(2)}, q^{(2)})\, e^{\frac{2i}{h}\int_\Delta \omega}\, dp^{(1)} dq^{(1)} dp^{(2)} dq^{(2)}, \tag{4.13}$$

where Δ is the triangle in R^{2n} with vertices at the points $(p^{(1)}, q^{(1)})$, $(p^{(2)}, q^{(2)})$ and (p, q), while $\omega = \sum dp_i \wedge dq_i$. The integral $\int_\Delta \omega$ is easily calculated:

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(14) By von Neumann's famous theorem [20], the representation T_g is defined by the number h up to unitary equivalence.

(15) In the preceding works concerning formula (4.13), the integral $\int_\Delta \omega$ is replaced by the explicit expression (4.14).

$$\int_{\Delta} \omega = \sum \begin{vmatrix} 1 & 1 & 1 \\ q_k^{(1)} & q_k^{(2)} & q_k \\ p_k^{(1)} & p_k^{(2)} & p_k \end{vmatrix}. \tag{4.14}$$

For the sake of completeness we give the relation between the Wick and Weyl quantizations. For this purpose consider a representation of the algebra $\mathfrak{G}$ in the space F_h. Let

$$\hat{p}_k = \frac{1}{i\sqrt{2}} (\hat{a}_k - \hat{a}_k^*), \quad \hat{q}_k = \frac{1}{\sqrt{2}} (\hat{a}_k + \hat{a}_k^*), \tag{4.15}$$

where $\hat{a}_k$ and $\hat{a}_k^*$ are defined by (4.4).

The operator $\hat{A}$ now acts in F_h. Being bounded, it has the Wick symbol $A(z, \bar{z})$. We see that

$$A(z, \bar{z}) = (T_{\frac{h}{2}} \mathcal{A})(z, \bar{z}), \tag{4.16}$$

where T_h is the integral operator with kernel (4.6). In applying formulas (4.16) we assume that $v = (q + ip)/\sqrt{2}$ and $\bar{v} = (q - ip)/\sqrt{2}$. By combining (4.16) and (4.7) we find

$$\mathcal{A}(p, q) = (T_{\frac{h}{2}} \overset{\circ}{A})(p, q)$$

(now $z = (q + ip)/\sqrt{2}$ and $\bar{z} = (q - ip)/\sqrt{2}$).

3. *Projective representations of the group of parallel translations in the space* F_h. The group of parallel translations acts transitively on C^n. According to the general theory developed in §3, if $A(z, \bar{z})$ is the covariant symbol for the operator $\hat{A}$, then $A(z - \xi, z - \bar{\xi})$ is the covariant symbol for the operator $\hat{T}_\xi^{(h)} \hat{\Delta} (\hat{T}_\xi^{(h)})^{-1}$, where $\hat{T}_\xi^{(h)}$ is the operator in F_h defined by

$$(\hat{T}_\xi^{(h)} f)(z) = f(z - \xi) e^{\frac{1}{h}(z\bar{\xi} - \frac{1}{2}\xi\bar{\xi})} \tag{4.17}$$

The relation among the $\hat{T}_\xi^{(h)}$ follows from (4.17):

$$\hat{T}_\xi^{(h)} \hat{T}_\eta^{(h)} = e^{\frac{1}{h}(\xi\bar{\eta} - \bar{\xi}\eta)} \hat{T}_\eta^{(h)} \hat{T}_\xi^{(h)}. \tag{4.18}$$

The relation (4.18) shows that the operators $\hat{T}_\xi^{(h)}$ form a projective representation of the group H and, at the same time, form a linear representation of the Heisenberg-Weyl group G.([16]) From von Neumann's theorem [20] it follows that the representations $\hat{T}_\xi^{(h)}$ are inequivalent for different h and that every irreducible unitary representation of G is equivalent to one of the $\hat{T}_\xi^{(h)}$.

§5. Quantization in homogeneous bounded regions

1. *Construction of the quantization.* Let Ω be a bounded homogeneous region in C^n, and let $K(z, \bar{z})$ be a Bergman kernel. Ω is a Kähler manifold with respect to the metric ds^2, whose potential is $\Phi(z, \bar{z}) = \ln K(z, \bar{z})$. By following the scheme described in §2,

([16]) The group G is a one-dimensional central extension of the group H.

we consider the space F_h of analytic functions in Ω with the scalar product

$$(f, f) = c(h) \int |f(z)|^2 K^{-1/h}(z, \bar{z})\, d\mu(z, \bar{z}), \tag{5.1}$$

where

$$d\mu(z, \bar{z}) = \det \frac{\partial^2 \ln K}{\partial z^i \partial \bar{z}^k} \cdot \frac{d\mu_L(z, \bar{z})}{\pi^n},$$

and $d\mu_L(z, \bar{z})$ is Lebesgue measure on Ω. Consider the function $L_h(z, \bar{v})$ defined by (2.16) and the complete system $\Phi_{\bar{v}}(z) = L_h(z, \bar{v})$. We form the algebra A_h of covariant symbols of bounded operators in F_h and the algebra $\mathfrak{A}$, which consists of the functions $f(h|z, \bar{z})$, $0 < h \leq 1$, which for fixed h are elements of A_h and continuous in h, z and $\bar{z}$.

Theorem 5.1. *The algebra* $\mathfrak{A}$ *is a special quantization with the weak correspondence principle.*

In view of the fact that the set E in this case consists of segments, according to Lemma 2.2 and the remark made after it, it is sufficient to verify hypotheses A, B and C.

The proof of Theorem 5.1 is based on the following assertions.

Theorem 5.2. *For homogeneous bounded regions* Ω:

1) $\det \|\partial^2 \ln K/\partial z^i \partial \bar{z}^k\| = \lambda K(z, \bar{z})$;

2) $L_h(z, \bar{z}) = \mu K^{1/h}(z, \bar{z})$, *where* $\lambda = \lambda(\Omega)$ *and* $\mu = \mu(\Omega)$ *are constants which depend only on* Ω;

3) *for* $0 < h \leq 1$ *the spaces* F_h *are not empty*: $F_h \supset H$, *where* H *is the space of analytic functions on* Ω *which are Lebesgue square summable.*

The second assertion of the theorem shows that hypothesis A is valid.

Proof of Theorem 5.2. Let H denote the Hilbert space of analytic functions on Ω with Lebesgue square summable modulus, let G be a group of transitive actions on Ω, and let $\partial(gz)/\partial z$ be the analytic Jacobian of the transformation $z \to gz$: $\partial(gz)/\partial z = \det \|\partial v^i/\partial z^k\|$, where z^k are the coordinates of the point z and v^i are the coordinates of the point $v = gz$.

It follows immediately from the definition of the kernel function that[17]

$$K(z, \bar{z}) = K(gz, \overline{gz}) \left| \frac{\partial (gz)}{\partial z} \right|^2. \tag{5.2}$$

Now let $\rho(z, \bar{z}) = \det \|\partial^2 \ln K(z, \bar{z})/\partial z^i \partial \bar{z}^i\|$. In turn, it follows from (5.2) that $\rho(z, \bar{z})$ satisfies the similar identity:

$$\rho(z, \bar{z}) = \rho(gz, \overline{gz}) \left| \frac{\partial (gz)}{\partial z} \right|^2. \tag{5.3}$$

Therefore the function $\lambda = \rho(z, \bar{z})/K(z, \bar{z})$ is invariant with respect to G and, because

(17) If $\phi_n(z)$ is an orthonormal basis in H, then $\chi_n(z) = \phi_n(gz)\partial(gz)/\partial z$ has the same property. Therefore

$$K(z, \bar{z}) = \sum |\chi_n(z)|^2 = \sum |\varphi_n(gz)|^2 \left| \frac{\partial (gz)}{\partial z} \right|^2 = K(gz, \overline{gz}) \left| \frac{\partial (gz)}{\partial z} \right|^2.$$

of transitivity, is a constant. This proves the first assertion. The third is an immediate consequence. Indeed, by the first assertion of Theorem 5.2 the measure $d\mu$ in (5.1) is equal to

$$d\mu(z, \bar{z}) = \lambda\pi^{-n}K(z, \bar{z})\, d\mu_L(z, \bar{z}). \tag{5.4}$$

Let H denote the space of analytic functions over Ω which are Lebesgue square summable. Consider in H an orthonormal basis which includes the function $f_0(z) =$ const. By using this basis we see that $K(z, \bar{z}) \geq |f_0|^2$. Therefore, for $0 < h \leq 1$

$$K^{-\frac{1}{h}+1}(z, \bar{z}) \leqslant |f_0|^{1-\frac{1}{h}} = \text{const}$$

and it follows from (5.4) that $H \subset F_h$. We note that $H = F_1$. We turn to the second assertion.

Lemma 5.1. *There exists a neighborhood U of the identity of the group of motions G, for which $|\partial(gz)/\partial z - 1| < \frac{1}{2}$ for all $z \in \Omega$ and $g \in U$.*

Proof. First of all, consider the fact that the action of the group G can be extended by continuity to the closure $\bar{\Omega}$ of the region Ω.(18) We define

$$j(g, z) = \lim_{z_n \to z} \frac{\partial(gz_n)}{\partial z_n} \quad \text{for} \quad z \in \bar{\Omega}.$$

It is clear that the function $j(g, z)$ is continuous in the variables $g \in G$ and $z \in \bar{\Omega}$. Note that $j(e, z) = 1$ for all $z \in \bar{\Omega}$. Now suppose that there is no neighborhood U having the required properties. If this is the case, then there is a sequence $g_n \to e$ and a sequence $z_n \in \Omega$ such that

$$\left| \frac{\partial(g_n z_n)}{\partial z_n} - 1 \right| \geqslant \frac{1}{2}. \tag{5.5}$$

By compactness, with no loss of generality it can be assumed that $z_n \to z \in \bar{\Omega}$. By taking the limit in (5.5) as $n \to \infty$, we find that $|j(e, z) - 1| \geq \frac{1}{2}$, which contradicts the equation $j(e, z) = 1$. The lemma is proved.

Now let $\sigma_n(z)$ be an orthonormal system of functions in the space F_h, and take $g \in U$. By introducing a cut along the negative real axis in the w-plane we define a single-valued branch of the function $w^{1/h}$ by the condition $1^{1/h} = 1$. Keeping this branch in mind, we consider the function

$$\chi_n(z) = \sigma_n(gz) \left[\frac{\partial(gz)}{\partial z}\right]^{1/h},$$

which, in view of the definition of the neighborhood U, is a single-valued analytic function on Ω. It follows from (5.2) that the transformation

$$f(z) \to f(gz) \left[\frac{\partial(gz)}{\partial z}\right]^{\frac{1}{h}}$$

(18) Actually, the action of G extends in a continuous fashion even to some complex manifold which contains Ω [21].

is a unitary operator in F_h. Therefore the functions $\chi_n(z)$ form an orthonormal basis in F_h. Consequently

$$L_h(z, \bar{z}) = \sum \chi_n(z) \overline{\chi_n(z)} = \sum \sigma_n(gz) \overline{\sigma_n(gz)} \left| \frac{\partial (gz)}{\partial z} \right|^{\frac{2}{h}} = L_h(gz, \overline{gz}) \left| \frac{\partial (gz)}{\partial z} \right|^{2/h}. \quad (5.6)$$

Since the last identity is valid for all $g \in U$, it is automatically true for all $g \in G$. On the other hand, it follows from (5.2) that the function $K^{1/h}(z, \bar{z})$ satisfies a similar identity. Therefore $\mu = L_h K^{-1/h}$ is a function invariant with respect to G, so that $\mu = \text{const}$. Theorem 5.2 is completely proved.

This establishes the validity of hypothesis A. That of B and C is obvious for $0 < h \leq 1$. This concludes the proof of Theorem 5.1.

Remarks. 1) In the case where Ω is a circular region, the point $z = 0$ is the distinguished point for the potential $\Phi = \ln(K(z, \bar{z})/K(0, 0))$. In this case there exists in Ω a complete orthonormal system consisting of homogeneous polynomials of nonnegative integer degree. By using this system we find that $K(z, 0) = K(0, \bar{z}) = K(0, 0)$,(19) which is equivalent to the fact that the point 0 is distinguished.

2) The space F_h is certainly not empty if $0 < h \leq 1$. In case $h > 1$, the integral (5.1) can begin to diverge. In this case it is natural to try to understand matters by means of analytic continuation in h. For the case where Ω is a symmetric region, this possibility has been studied in detail. It appears that for the case where Ω is the complex sphere, the space F_h can be constructed for any $h > 0$. In all other cases the permissible values of h are bounded by a constant $c(\Omega)$: for $h > c(\Omega)$ the scalar product defined by means of analytic continuation is not positive definite.

3) A projective representation $\hat{U}_g$ of the group G acts in the space F_h. In a sufficiently small neighborhood of the identity the operator $\hat{U}_g = \hat{T}_g$ is defined by the general formula (3.4); the function $\psi(g, z)$ is defined in (5.2). By taking the logarithm of (5.2), we find that

$$\Phi(z, \bar{z}) = \Phi(gz, \overline{gz}) + \ln j(g, z) + \ln \overline{j(g, z)},$$

where $j(g, z)$ is the analytic Jacobian. Consequently

$$(\hat{T}_g f)(z) = f(g^{-1}z)\, j^{1/h}(g^{-1}, z). \quad (5.7)$$

§6. Quantization on a cylinder and torus

1. *General remarks.* The two-dimensional cylinder and torus are Kähler manifolds. Their complex structure and metric can be defined in the usual way by unfolding them onto a plane. We denote real coordinates on the cylinder and torus by p and q, where in the case of the cylinder p runs over the real axis and q is a cyclic coordinate with period 2π, while in the case of the torus both coordinates are cyclic with period 2π. In either case we set $z = q + ip$. There is a global potential

(19) $= \mu(\Omega)^{-1/2}$, where $\mu(\Omega)$ is the Lebesgue volume of Ω.

$$\Phi(z, \bar{z}) = -\frac{1}{2}(z-\bar{z})^2 = 2p^2 \tag{6.1}$$

for the metric on the cylinder. In the case of the torus, a global potential exists on the set $\widetilde{\mathfrak{M}}$ obtained from the torus by removing the circle $q = \text{const}$. Equation (6.1) can be considered as the potential on $\widetilde{\mathfrak{M}}$. In neither case is hypothesis A valid, and therefore it is not clear that by the method in §2 a quantization can be constructed using the algebra A_h which admits the correspondence principle. In this connection we shall construct quantizations on the cylinder and torus along the lines of Weyl quantization in the plane.

2. *Cylinder.* Formula (4.13), which provides a law of multiplication in the algebra A_h for the case of Weyl quantization in the plane, cannot be exactly carried over to the cylinder. The problem is that in the plane there is a unique triangle whose vertices lie at three given points, whereas on the cylinder there exist many such geodesic triangles. In this connection we introduce a definition.

A set $\mathfrak{M}_h$ of points on the cylinder $\mathfrak{M}$ is called admissible if when the vertices of the triangle Δ belong to $\mathfrak{M}_h$ the function

$$\exp\frac{2i}{h}\int_{\Delta}\omega \tag{6.2}$$

does not depend on the choice of the triangle.

We begin with a description of admissible sets. Note that

$$\int_{\Delta}\omega = q(p_2 - p_1) + q_2(p_1 - p) + q_1(p - p_2). \tag{6.3}$$

By taking into consideration the fact that q is a cyclic variable of period 2π, we find that for the function (6.2) to be independent of the sides of the triangle, but to depend only on its vertices, it is necessary and sufficient that the differences $p_2 - p_1$, $p_1 - p$ and $p - p_2$ take on values of the form $nh/2$ with integer n.

Thus those sets $\mathfrak{M}_h$ are admissible which consist of circles on $\mathfrak{M}$ which are parallel to the base and are a distance $hn/2$ apart.

Corresponding to this we modify formula (4.13):

$$\mathcal{A}(p, q) = \frac{h}{2}\sum_{n_1, n_2}\int \mathcal{A}_1(p_1, q_1)\,\mathcal{A}_2(p_2, q_2)\, e^{\frac{2i}{h}\begin{vmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p \\ q_1 & q_2 & q \end{vmatrix}} dq_1\, dq_1,$$
$$p_1 = n_1\frac{h}{2}, \quad p_2 = n_2\frac{h}{2}, \quad p = n\frac{h}{2}. \tag{6.4}$$

Going over to the Fourier transform, we obtain

$$\mathcal{A}(p, q) = \sum_{\beta}\int_0^{\frac{4\pi}{h}} e^{i(\alpha p + \beta q)}\varphi(\alpha, \beta)\, d\alpha, \quad \beta = 0, \pm 1, \ldots \tag{6.5}$$

(the function $\phi(\alpha, \beta)$ is periodic in α with period $4\pi/h$).

A law of composition for ϕ follows from (6.4):

$$\varphi(\alpha, \beta) = \sum_{\beta'} \int_0^{\frac{4\pi}{h}} \varphi_1(\alpha - \alpha', \beta - \beta') \varphi_2(\alpha', \beta') e^{\frac{ih}{2}\begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix}} d\alpha'. \tag{6.6}$$

Let $\|\phi\| = \Sigma_\beta \int |\phi(\alpha, \beta)| d\alpha$ It follows from (6.6) that $\|\phi\| \le \|\phi_1\| \, \|\phi_2\|$. Consequently the set of functions ϕ such that $\|\phi\| < \infty$ is closed relative to the composition (6.6), and thus the set of functions $\mathfrak{A}$ of the form (6.5) with $\|\phi\| < \infty$ is closed relative to the composition (6.4). We denote the latter set by A_h.

We shall establish the associativity of the composition (6.4). In $L^2(0, 2\pi)$ consider an integral operator with kernel $K(x, y)$, which is 2π-periodic in x and y. To it we associate a function $\mathfrak{A}(p, q)$ by a formula similar to (4.12):

$$\mathcal{A}(p, q) = 2\int_0^{2\pi} K(q - \xi, q + \xi) e^{-\frac{2ip\xi}{h}} d\xi, \quad p = \frac{nh}{2}, \quad n = 0, \pm 1, \ldots. \tag{6.7}$$

Inverting (6.7), we have

$$K(x, y) = \frac{1}{4\pi} \sum \mathcal{A}\left(p, \frac{x+y}{2}\right) e^{\frac{ip}{h}(x-y)}. \tag{6.8}$$

The operator product $\hat{A} = \hat{A}_1 \hat{A}_2$ in $L^2(0, 2\pi)$ corresponds to the composition of their kernels $K(x, y) = \int K_1(x, s) K_2(s, y) ds$. Going over to the functions $\mathfrak{A}(p, q)$ by formulas (6.7) and (6.8), we find, after an obvious transformation, that the composition (6.4) is generated by the operator product. Thus the functions $\mathfrak{A}(p, q)$ of the form (6.5) with $\|\phi\| < \infty$ and with the multiplication law (6.4) form an associative algebra. Formulas (6.7) and (6.8) describe a linear representation of the algebra A_h in $L^2(0, 2\pi)$.

We construct the quantization $\mathfrak{A}$ out of those functions $\mathfrak{A}(h|p, q)$, $0 < h < \infty$, for which $\mathfrak{A}(h|p, q) \in A_h$ for fixed h.

The correspondence principle (in its weak form) follows from the following considerations. Let the functions $\mathfrak{A}_1(h|p, q)$ and $\mathfrak{A}_2(h|p, q)$ be defined for all $(p, q) \in \mathfrak{M}$ and $0 < h < \infty$ and be continuously differentiable in all variables. In this case the right-hand side of (6.4), after multiplication by $h/2$, is a defining sum for the integral (4.13) (for $n = 1$), extended over $\mathfrak{M}$. Consequently (6.4) and (4.13) have a common asymptote as $h \to 0$.

Thus the correspondence principle for our quantization on a cylinder follows from the correspondence principle for the Weyl quantization on the plane. We shall not dwell on this point.

3. *The torus.* As a starting point for the construction of a quantization, we again consider formula (4.13). The same arguments as in the case of the cylinder show that, on an admissible manifold $\mathfrak{M}_h$, p_2, p_1 and p can take on only the discrete values $hn/2$. In view of symmetry with respect to p and q, it follows that q_1, q_2 and q also take on only these discrete values. Since, on the other hand, p and q are cyclic coordinates, the number of distinct values accepted by p and q is finite and is given by the relation

$Nh/2 = 2\pi$, with N an integer. Thus we see that h can take on only the discrete set of values[20]

$$h = \frac{4\pi}{N}. \tag{6.9}$$

The lattice $\mathfrak{M}_h$: $(p, q) = (m, n)h/2$ on the torus is an admissible set.

We modify equation (4.13) in the following way:

$$\mathcal{A}(p, q) = \left(\frac{h}{2}\right)^3 \sum_{p_i, q_i \in \mathfrak{M}_h} \mathcal{A}_1(p_1, q_1)\, \mathcal{A}_1(p_2, q_2)\, e^{\frac{2i}{h}\begin{vmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p \\ q_1 & q_2 & q \end{vmatrix}}. \tag{6.10}$$

Let K_h denote the lattice on the circle $0 < x \leq 2\pi$ consisting of the points $x = nh/2$, and let $L^2(K_h)$ be the Hilbert space of functions on K_h with the scalar product

$$(f, g) = \frac{h}{2} \sum_{x \in K_h} f(x)\, \overline{g(x)}.$$

It is clear that $\dim L^2(K_h) = N = 4\pi/h$. Every operator in the space $L^2(K_h)$ is defined by a kernel $K(x, y)$, $x, y \in K_h$. We assign to each operator in $L^2(K_h)$ a function $\mathfrak{A}(p, q)$ on $\mathfrak{M}_h$ by the formula, which is similar to (6.7),

$$\mathcal{A}(p, q) = h \sum K(q - \xi, q + \xi)\, e^{\frac{2ip}{h}\xi}. \tag{6.11}$$

The inverse of (6.11) coincides in form with (6.8). The operator product $\hat{A} = \hat{A}_1 \hat{A}_2$ in $L^2(K_h)$ corresponds to the composition of kernels $K(x, y) = (h/2) \Sigma K_1(x, s) K_2(s, y)$. Thus by means of (6.11) and (6.8) we can go over to the functions $\mathfrak{A}(p, q)$. An obvious calculation shows that the composition of the functions $\mathfrak{A}(p, q)$, which comes about in this manner, is identical with (6.10).

In this way (6.10) defines an associative algebra. The algebra $\mathfrak{A}$ is defined just as for the cylinder,[21] and the validity of the correspondence principle is established exactly as in that case.

In concluding we point out that all the formulas which relate to the quantization in the case of the torus are obtained from the similar formulas for the Weyl quantization of the plane by replacing the integrals by their defining sums with increment $h/2$

§7. Questions of uniqueness

In this section we examine, on the basis of some general considerations, the uniqueness of the Wick and Weyl quantizations in C^n.

1. *Additional definitions.* Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be quantizations of the same classical mechanics. We call $\mathfrak{A}_1$ a *subquantization* of $\mathfrak{A}_2$ ($\mathfrak{A}_1 \subset \mathfrak{A}_2$) if an admissible homomorphism $\psi: \mathfrak{A}_1 \to \mathfrak{A}_2$ exists.

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(20) It will be shown elsewhere that in the case of a compact symmetric space with a semicircle group of actions the situation is similar.

(21) The only difference is that h takes on values of the form (6.9), and not the half line $(0, \infty)$.

The quantization $\mathfrak{A}$ is called *maximal* if $\mathfrak{A} \subset \mathfrak{A}_1$ implies that $\mathfrak{A} = \mathfrak{A}_1$.

Let $\mathfrak{A}$ be a special quantization of the mechanics $(\mathfrak{M}, \omega)$ which is natural with respect to some category $\mathcal{K}$ to which the group G of motions of the manifold $\mathfrak{M}$ belongs.

As has been repeatedly noted, in this case the translations generate automorphisms of the algebra A_h by the formula

$$(\tau_g f)(x) = f(g^{-1}x). \tag{7.1}$$

The quantization $\mathfrak{A}$ will be called *effective* if there is no natural isomorphism between algebras A_{h_1} and A_{h_2} for $h_1 \neq h_2$ (i.e. none which commutes with all τ_g).

A quantization is called *irreducible* if the algebras A_h have faithful irreducible representations as bounded operators in a Hilbert space.

A quantization is called a *w^*-quantization* if the algebras A_h are w^*-algebras.

In particular, in the case of an irreducible w^*-quantization the algebras A_h are isomorphic to the complete algebras of bounded operators in a Hilbert space.

2. *General considerations.* Consider the set $\widetilde{M}$ of $*$-algebras A *consisting of functions on a homogeneous manifold $\mathfrak{M}$ which admits a group G of motions and which has* the properties:

i) The algebra A is isomorphic to an algebra of bounded operators on a Hilbert space.

ii) The translations $(\tau_g f)(x) = f(g^{-1}x)$ are isomorphisms of the algebras A.

iii) The identity of A is the function $f_0(x) \equiv 1$.

The algebras A_1 and $A_2 \in \widetilde{M}$ *are called naturally isomorphic if there is an isomorphism between them which commutes with the automorphisms τ_g.*

We denote the set of classes of pairwise naturally isomorphic algebras A by M.

Next, let $\widetilde{T}$ *denote the set of all irreducible projective representations of the* group G, and let T denote the set of classes of unitarily equivalent projective representations of G. We shall construct a monomorphic mapping $M \to T$.

Fix an algebra $A \in \widetilde{M}$. Let A be isomorphic to an algebra of bounded operators L in the Hilbert space $\mathcal{H}$, and let ϕ denote the isomorphism $A \to L$. Let $\sigma_g = \phi\tau_g\phi^{-1}$ be an automorphism of L. Since all the isomorphisms of L are inner, there is a bounded *operator $\hat{U}_g$, defined up to a constant factor λ, which generates σ_g:*

$$\sigma_g \hat{f} = \hat{U}_g \hat{f} \hat{U}_g^{-1}. \tag{7.2}$$

In view of the fact that σ_g^* is an automorphism, in particular it takes Hermitian operators into Hermitian operators. It therefore follows that $\hat{U}_g$ differs by only a constant factor from a unitary operator. We can therefore consider $\hat{U}_g$ to be unitary with $|\lambda| = 1$.

The operators $\hat{U}_g$ form a unitary projective representation of the group G.

We shall show that the $\hat{U}_g$ are irreducible. Let $\hat{f}_0 \in L$ commute with $\hat{U}_g$ and $f_0(x) = \phi^{-1}(\hat{f}_0)$. By (7.2),

$$f_0(g^{-1}x) = (\tau_g f_0)(x) = \varphi^{-1}(\sigma_g \hat{f}_0) = \varphi^{-1}(\hat{U}_g \hat{f}_0 U_g) = \varphi^{-1}(\hat{f}_0) = f_0(x).$$

Because of the transitivity of the action of G on $\mathfrak{M}$, it follows that $f_0(x) = f_0 = \text{const}$. Because of iii), this means that $\hat{f}_0 = f_0 I$, where I is the identity operator in $\mathcal{H}$. Thus to each algebra $A \in \widetilde{M}$ we have assigned an irreducible projective representation $\hat{U}_g$ of the group G.

Theorem 7.1. *The algebras A_1 and A_2 are naturally isomorphic if and only if the corresponding representations $\hat{U}_g^{(1)}$ and $\hat{U}_g^{(2)}$ are unitarily equivalent.*

Proof. We shall supply the index $i = 1, 2$ to objects which relate to the algebras A_i. Suppose that the representations $\hat{U}_g^{(i)}$ are unitarily equivalent, and let $V : \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism of the Hilbert spaces which underlies this equivalence: $V\hat{U}_g^{(1)} = \hat{U}_g^{(2)} V$. By means of V we construct the isomorphisms $L_1 \to L_2$ and $A_1 \to A_2$:

$$\hat{f} \to V\hat{f}V^{-1}, \quad f(x) \to \varphi_2^{-1}(V\varphi_1(f)V^{-1}) = \psi(f). \tag{7.3}$$

The isomorphism $f \to \psi(f)$ in (7.3) is natural:

$$\psi(\tau_g^{(1)} f) = \varphi_2^{-1}(V\hat{U}_g^{(1)}\varphi_1(f)(\hat{U}_g^{(1)})^{-1}V^{-1}) = \varphi_2^{-1}(\hat{U}_g^{(2)}V\varphi_1(f)V^{-1}(\hat{U}_g^{(2)})^{-1}) = \tau_g^{(2)}\psi(f).$$

Conversely, let the algebras A_1 and A_2 be naturally isomorphic, and let $\psi : A_1 \to A_2$ be the isomorphism. In this case $\chi = \phi_2\psi\phi_1^{-1}$ is an isomorphism between the algebras L_1 and L_2. Since the L_i are complete operator algebras, an isomorphism $V : \mathcal{H}_1 \to \mathcal{H}_2$ exists which generates $\chi : \chi\hat{f} = V\hat{f}V^{-1}$, $\hat{f} \in L_1$. The uniqueness of ψ implies the identity $\psi\tau_g^{(1)} = \tau_g^{(2)}\psi$. It follows from this that $\chi\sigma_g^{(1)} = \sigma_g^{(2)}\chi$, where $\sigma_g^{(i)}$ are automorphisms of the L_i of the form (7.2). In more detail,

$$V\hat{U}_g^{(1)}\hat{f}(\hat{U}_g^{(1)})^{-1}V^{-1} = \hat{U}_g^{(2)}V\hat{f}V^{-1}(\hat{U}_g^{(2)})^{-1}. \tag{7.4}$$

From this we have

$$V^{-1}(\hat{U}_g^{(2)})^{-1}V\hat{U}_g^{(1)}\hat{f} = \hat{f}\,V^{-1}(\hat{U}_g^{(2)})^{-1}V\hat{U}_g^{(1)}. \tag{7.5}$$

Since (7.5) is correct for any $\hat{f} \in L_1$, we have $V^{-1}(\hat{U}_g^{(2)})^{-1}V\hat{U}_g^{(1)} = \lambda I$, so that $\hat{U}_g^{(1)} = \lambda V^{-1}\hat{U}_g^{(2)}V$. This equation implies the unitary equivalence of the projective representations $\hat{U}_g^{(1)}$ and $\hat{U}_g^{(2)}$. The theorem is proved.

3. *Uniqueness of Wick quantization.* In this subsection we shall consider C^n as a uniform space with a group of actions G_1 which is composed of parallel translations and unitary transformations. We denote the first of these subgroups by H and the second by U. We adjoin to the algebras A_h, $0 < h < \infty$, which enter into the construction of the Wick quantization, the algebra C of complex numbers. For convenience we shall assume that $C = A_\infty$. The quantization formed by means of the algebras A_h, $0 < h < \infty$, will be called the extended Wick quantization.

Theorem 7.2. *All irreducible, effective, maximal w^*-quantizations in C^n which are natural with respect to the group G_1 are naturally equivalent. One such quantization is the extended Wick quantization. In this way all these quantizations are naturally equivalent to the extended Wick quantization.*

Proof. To each algebra which arises in the extended Wick quantization we assign an irreducible unitary projective representation $\hat{U}_g^{(h)}$ of the group G_1, as was done in subsection 2. By Theorem 7.1, to prove Theorem 7.2 it is sufficient to verify that the representations $\hat{U}_g^{(h)}$, up to unitary equivalence, exhaust all the irreducible projective representations of G_1. Every projective representation of G_1 is a linear representation of some central extension G_1 with respect to a center which is no more than one-dimensional. By means of standard homological algebraic methods it is easily established that: 1) each such *extension* is a *semidirect product* $\widetilde{G}_1 = U \times \widetilde{H}$, where U is a unitary group and $\widetilde{H}$ is an extension of the group of parallel translations of H; 2) if the group $\widetilde{H}$ is not commutative, then it is the Heisenberg-Weyl group.

The group H acts transitively on C^n. Consequently the restriction of the representation $\hat{U}_g^{(h)}$ to H is irreducible. Hence it immediately follows that if for some h_0 the group $\widetilde{H}$ is commutative, then the space F_{h_0} is one-dimensional, and consequently $A_{h_0} = C$.[22]

In the case where the group $\widetilde{H}$ is the Heisenberg-Weyl group, each of its irreducible representations, up to projective equivalence, can be uniquely completed to an irreducible representation of the group $\widetilde{G}_1$.[23] Now recall that according to §4.3 the representations $\hat{U}_g^{(h)}$, $0 < h < \infty$, generated by the Wick quantization, up to equivalence, exhaust all the irreducible unitary representations of $\widetilde{H}$. Theorem 7.2 is completely proved.

Remark. Let B_h be an algebra which is naturally isomorphic to the algebra A_h of Wick symbols. Since the isomorphism $\psi: B_h \to A_h$ is a linear correspondence, it must have the form

$$A(z, \bar{z}) = \int \mathcal{K}(z, \bar{z} \mid u, \bar{u}) B(u, \bar{u})\, d\mu(u, \bar{u}), \tag{7.6}$$

where $B \in B_h$ and $A = \psi B \in A_h$.

The requirement that it be natural leads to the fact that

$$\mathcal{K}(gz, g\bar{z} \mid gu, g\bar{u}) = K(z, \bar{z} \mid u, \bar{u}) \text{ for } g \in G_1.$$

Therefore we have

$$\mathcal{K}(z, \bar{z} \mid u, \bar{u}) = K(|z-u|^2) \quad (|z|^2 = \sum |z|^2). \tag{7.7}$$

Formula (4.16), which relates the Wick and Weyl quantizations, is a particular case of (7.6) and (7.7).

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[22] It is natural to consider the unit representation ϵ_g as $\hat{U}_g^{(h_0)}$. Note that $\epsilon_g = \lim_{h\to\infty} \hat{U}_g^{(h)}$ in the topology of the representation space (see [3]). This circumstance is the basis for the notation $A_\infty = C$.

[23] Let $\hat{T}_g$ and $\hat{L}_g$ be irreducible unitary representations of $\widetilde{G}_1$ whose restrictions to $\widetilde{H}$ coincide: $\hat{T}_\xi = \hat{L}_\xi$ for $\xi \in \widetilde{H}$. Since $\widetilde{H}$ is a normal divisor of $\widetilde{G}_1$, we have

$$\hat{T}_g \hat{T}_\xi \hat{T}_g^{-1} = \hat{T}_{g\xi g^{-1}}, \quad \hat{L}_g \hat{T}_\xi \hat{L}_g^{-1} = \hat{L}_{g\xi g^{-1}} = \hat{T}_{g\xi g^{-1}},$$

whence $\hat{L}_g^{-1}\hat{T}_g\hat{T}_\xi = \hat{T}_\xi \hat{L}_g^{-1}\hat{T}_g$. Because of the irreducibility of T_ξ it follows that $\hat{T}_g = \lambda \hat{L}_g$, i.e. $\hat{T}_g$ and $\hat{L}_g$ are projectively equivalent.

4. *Uniqueness of the Weyl quantization.* Let G_2 denote the group of all linear nonhomogeneous canonical transformations in R^{2n}, i.e. the group of all linear nonhomogeneous transformations which leave the form $\omega = \Sigma\, dp_i \wedge dq_i$ invariant. In the complex coordinates $z_k = (q_k + ip_k)/\sqrt{2}$ we have $\omega = (1/i)\, \Sigma\, dz_k \wedge d\bar{z}_k$, so that the group G_1 considered above, which preserves the metric $ds^2 = \Sigma\, dz_k d\bar{z}_k$, is a subgroup of G_2.

We adjoin to the algebras A_h, $0 < h < \infty$, which compose the Weyl quantization, the algebra $A_\infty = C$, and call the quantization constructed by means of the algebras A_h, $0 < h \leq \infty$, the extended Weyl quantization.

Theorem 7.3. *The extended Weyl quantization is the unique maximal, irreducible and effective w^*-quantization which is natural with respect to the group G_2.*(24)

In order not to obscure simple ideas with complicated details, we shall restrict ourselves to giving a heuristic proof of this theorem.

First step. By repeating in full detail the proof of Theorem 7.2, we see that the extended Weyl quantization is, up to natural equivalence, the unique quantization having the properties enumerated in the conditions of Theorem 7.3.

Second step. Let B_h be the algebra of Weyl symbols in A_h. Write the isomorphism $\psi : B_h \to A_h$ in the form (7.6). The requirement that it be natural with respect to G_2 leads to the fact that $\mathcal{K}(gx|gy) = \mathcal{K}(x|y)$, where $x = (z, \bar{z})$, $y = (v, \bar{v})$ and $g \in G_2$.

Thus $\mathcal{K}(x|y)$ is an invariant pair of points. However, in R^{2n} there is no pair of points invariant under G_2. Consequently $\mathcal{K}(x|y) = \delta(x - y)$, where $\delta(x)$ is the Dirac δ-function, and $A = B$.(25)

§8. Concluding remarks

1. *Nonexistence of a universal quantization.* Let $\mathcal{G}$ denote the group of all one-to-one transformations of the space R^{2n} which leave invariant the form $\omega = \Sigma\, dp_i \wedge dq_i$, i.e. the total group, including nonlinear canonical transformations. A quantization which is natural with respect to the group $\mathcal{G}$ will be called universal.

Theorem 8.1. *There exists no irreducible universal w^*-quantization.*

Before proving this theorem, we note that it provides a negative solution to the question of the existence of a quantization which is natural with respect to the category of all morphisms of classical mechanics.

Proof of Theorem 8.1. Let Q be a universal quantization. With no loss of generality we can assume that it is maximal. It must therefore coincide with the Weyl quantization. In fact, since the group of all linear canonical transformations is a subgroup of $\mathcal{G}$, the quantization Q satisfies the conditions of Theorem 7.3. Consequently the law of multiplication in the algebras A_h is given by (4.13). Thus for a universal quantization Q it is necessary that

(24) A similar result in another context is proved in [22].

(25) The basis for a rigorous proof of Theorem 7.3 is the irreducibility of the representation of G_2 in $L^2(R^{2n})$ defined by the formula $(T_g f)(x) = f(g^{-1}x)$.

$$F(gx_1, gx_2, gx_3) = F(x_1, x_2, x_3), \tag{8.1}$$

where

$$F(x_1, x_2, x_3) = \int\limits_{\Delta(x_1, x_2, x_3)} \omega,$$

and $\Delta(x_1, x_2, x_3)$ is a triangle with vertices x_1, x_2, x_3. If the transformation $g \in \mathfrak{G}$ does not take straight lines into straight lines, i.e. is not affine, then (8.1) is impossible (since the sides of $\Delta(x_1, x_2, x_3)$ are straight line segments).

The theorem is proved. Its proof can be summarized by saying that the group $\mathfrak{G}$ does not leave a triple of points invariant.

2. *Remarks concerning terminology.* Let $\mathfrak{L}(\mathfrak{M})$ denote the Lie algebra with respect to the Poisson bracket which consists of infinitely differentiable functions on a symplectic manifold $\mathfrak{M}$. The term "quantization" is sometimes applied to a linear representation of this algebra. To me this usage seems incorrect because of the fact that the quantization used in physics cannot in any of its mathematical interpretations be considered as a linear representation of the algebra $\mathfrak{L}(\mathfrak{M})$. In this connection we shall prove the following theorem.

Let $\mathfrak{L}_0$ denote the Lie algebra with respect to the ordinary Poisson bracket which consists of polynomials in two variables $\phi(p, q)$.

Theorem 8.2. *There exists no representation T_ϕ of the algebra $\mathfrak{L}_0$ in $L^2(R^1)$ with the following properties:*

1) *A Schwartz space S forms a part of the region of definition of all the T_ϕ and is invariant with respect to all the T_ϕ.*

2) $$T_p f = i\hat{p}f = i\,\frac{h}{i}\,\frac{df}{dx}, \quad T_q f = i\hat{q}f = ixf. \tag{8.2}$$

Proof. We shall break the proof up into several steps.

1) Let T_ϕ be a representation with the property (8.2). We shall show that the operator $T_{p^m q^n}$ can be represented as a polynomial in the operators $\hat{p}$ and $\hat{q}$ whose degree is no higher than m with respect to $\hat{p}$ and no higher than n with respect to $\hat{q}$. But first we note that from the equations

$$0 = T_{[q, q^n]} = [T_q, T_{q^n}] = [i\hat{q}, T_{q^n}]$$

it follows that $T_{q^n} = f_n(\hat{q})$. Similarly, $T_{p^n} = g_n(\hat{p})$. From the invariance of the space S under T_{p^n}, T_{q^n} and the Fourier transformation it follows that the functions of a real variable $f_n(x)$ and $g_n(x)$ are infinitely differentiable

Next, if $f(x)$ is an infinitely differentiable function, then

$$[p, f(q)] = f'(q), \quad [\hat{p}, f(\hat{q})] = \frac{h}{i} f'(\hat{q}) \tag{8.3}$$

(in the first equation we have the Poisson bracket; in the second, the commutator).

By applying (8.3) $n+1$ times, we find that

$$0 = T_{[p,\ldots,[p,[p,q^n]]]} = i^{n+1}[\hat{p}, \ \ldots \ , \ [\hat{p}, f_n(\hat{q})]] = h^{n+1} f_n^{(n+1)}(\hat{q}),$$

where $f_n^{(n+1)}(x) = d^{n+1}f_n(x)/dx^{n+1}$. Consequently $f_n(x)$ is a polynomial of degree no higher than n. $g_n(x)$ has the same property. Now note that

$$p^m q^n = \frac{1}{(m+1)(n+1)}[p^{m+1}, \ q^{n+1}].$$

Consequently

$$T_{p^m q^n} = \frac{1}{(m+1)(n+1)}[q_{m+1}(\hat{p}), \ f_{n+1}(\hat{q})].$$

By making use of the commutation relation $[\hat{p}, \hat{q}] = hI/i$, we find from this, after an obvious transformation, that the operator $T_{p^m q^n}$ can be written as a polynomial in $\hat{p}$ and $\hat{q}$ of degree no higher than m in $\hat{p}$ and n in $\hat{q}$.

2) Let $\mathcal{A}_\phi(p, q)$ denote the Weyl symbol(26) of the operator T_ϕ. We shall show that(27)

$$\mathcal{A}_\varphi(p, q) = ih\varphi\left(\frac{p}{h}, \ \frac{q}{h}\right) + c, \quad c = c(\varphi) = \text{const.} \tag{8.4}$$

We make use of the following general formula [15]. Let $\hat{f}_1$ and $\hat{f}_2$ be any operators, and let $f_1(p, q)$ and $f_2(p, q)$ be their Weyl symbols. In addition, let $\hat{g} = [\hat{f}_1, \hat{f}_2]$ and $g(p, q)$ be the Weyl symbol for $\hat{g}$. Then

$$g(p, q) = \frac{h}{i}[f_1, f_2] + \sum_1^\infty \left(\frac{h}{2}\right)^{2n+1} \sum_{k+l=2n+1} \frac{i^{k-l}}{k!l!}\left(\frac{\partial^{k+l} f_1}{\partial q^k \partial p^l}\frac{\partial^{k+l} f_2}{\partial p^k \partial q^l} - \frac{\partial^{k+l} f_1}{\partial p^k \partial q^l}\frac{\partial^{k+l} f_2}{\partial p^l \partial q^k}\right), \tag{8.5}$$

where

$$[f_1, f_2] = \frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial q} - \frac{\partial f_1}{\partial q}\frac{\partial f_2}{\partial p}$$

is the Poisson bracket.

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(26) The Weyl symbol for a differential operator with polynomial coefficients is obtained from the general formula (4.8), where $\phi(\alpha, \beta)$ is a linear combination of δ-functions and their derivatives. For these operators one can provide an independent definition of the Weyl symbol as follows. Let A and B be any noncommuting operators. The symmetrized product $(A^m B^n)$ of these operators is the coefficient of $(m+n)\alpha^m \beta^n/m!n!$ in the expansion

$$(\alpha A + \beta B)^N = \sum_{m+n=N} \frac{N!}{m!n!}\alpha^m \beta^n (A^m B^n).$$

(For example, $(AB) = \frac{1}{2}(AB + BA)$.) By using the relation $[\hat{p}, \hat{q}] = Ih/2$, we can write every operator which is a polynomial in $\hat{p}$ and $\hat{q}$ in symmetrized form: $A = \Sigma a_{mn}(\hat{p}^m \hat{q}^n)$. The Weyl symbol for the operator $\hat{A}$ in terms of the coefficients is a_{mn}: $\mathcal{A}(p, q) = \Sigma a_{mn} p^m q^n$.

(27) It will also be shown that the constant c in (8.4) is equal to 0. However, this stronger version of (8.4) is not needed here.

For the following it is essential that if f_1 is a polynomial of no higher than second degree and f_2 is an arbitrary polynomial, then all the terms in (8.5) beyond the first are equal to zero.

By the conditions of the theorem, formula (8.4) with the constant $c = 0$ is valid with $\phi = p$ and $\phi = q$. Let us find $\mathfrak{A}_{q^2}(p, q)$. By the above, $\mathfrak{A}_{q^2}(p, q) = f_2(q)$ is a polynomial of at most second degree. Applying (8.5), we find

$$2iq = \mathcal{A}_{[p,q^2]}(p, q) = h\,[p, f_2(q)] = hf_2'(q).$$

Hence $f_2(q) = iq^2/h + c_1$. Similarly, $g_2(p) = ip^2/h + c_2$.

Next, we have $pq = \frac{1}{4}[p^2, q^2]$. Therefore

$$\mathcal{A}_{pq} = \frac{1}{4}\mathcal{A}_{[p^2,q^2]}(pq) = \frac{1}{4}\frac{h}{i}[g_2(p),\ f_2(q)] = \frac{i}{h}pq.$$

Thus (8.4) is valid if ϕ is a polynomial of no higher than second degree.

We now turn to the general case. Let ϕ be a polynomial of at most second degree and let f be any polynomial. By (8.5),

$$\mathcal{A}_{[\varphi,f]}(p, q) = \frac{h}{i}[\mathcal{A}_\varphi, \mathcal{A}_f]. \tag{8.6}$$

By setting $\phi = p$ and $\phi = q$ in (8.6), we obtain

$$\mathcal{A}_{\frac{\partial f}{\partial q}} = h\frac{\partial \mathcal{A}_f}{\partial q}, \quad \mathcal{A}_{\frac{\partial f}{\partial p}} = h\frac{\partial \mathcal{A}_f}{\partial p}. \tag{8.7}$$

Let $\mathfrak{B}_f(p, q) = \mathfrak{A}_f(ph, qh)$. It follows from (8.7) that

$$\mathfrak{B}_{\frac{\partial f}{\partial q}} = \frac{\partial \mathfrak{B}_f}{\partial q}, \quad \mathfrak{B}_{\frac{\partial f}{\partial p}} = \frac{\partial \mathfrak{B}_f}{\partial p}. \tag{8.8}$$

Let L denote the linear operator $f \to \mathfrak{B}_f$. As was noted above, $\mathfrak{A}_f$, and consequently also $\mathfrak{B}_f$, is a polynomial whose degree in p and q does not exceed the degree of f. Let $\mathfrak{L}_n$ denote the space of polynomials of degree at most n in p and q. The space $\mathfrak{L}_n$ is invariant with respect to L; and, in addition, it follows from (8.8) that L commutes with $\partial/\partial p$ and $\partial/\partial q$. Therefore

$$L = \sum_0^\infty L_n, \quad L_n = \sum_{k+l=n} a_{kl}\frac{\partial^n}{\partial p^k \partial q^l}, \quad a_{kl} = \text{const}. \tag{8.9}$$

(The series in the first equation of (8.9) is a formal series. It is meaningful only insofar as L can be only applied to polynomials.)

We make use of (8.6) for the case where ϕ is a homogeneous second degree polynomial:

$$\mathfrak{B}_{[\varphi,f]}(p, q) = \mathcal{A}_{[\varphi,f]}(ph, qh) = \frac{h}{i}[\mathcal{A}_\varphi, \mathcal{A}_f](ph, qh) = [\varphi, \mathcal{A}_f](ph, qh)$$
$$= [\varphi, \mathfrak{B}_f](p, q).$$

Consequently L commutes with the operation of taking the Poisson bracket with ϕ, i.e. with the operators

$$p\frac{\partial}{\partial q},\quad q\frac{\partial}{\partial p},\quad p\frac{\partial}{\partial p},\quad q\frac{\partial}{\partial q}. \tag{8.10}$$

Note that commutation with the operators (8.10) does not change the order of a homogeneous differential operator with constant coefficients. Consequently each operator L_n commutes with the operators (8.10). In particular,

$$0=\left[L_n,\ p\frac{\partial}{\partial p}\right]=\sum_{k+l=n} ka_{kl}\frac{\partial^n}{\partial p^k\partial q^l},$$

$$0=\left[L_n,\ q\frac{\partial}{\partial q}\right]=\sum_{k+l=n} la_{kl}\frac{\partial^n}{\partial p^k\partial q^l}.$$

Hence $a_{kl}=0$ for $k\neq 0,\ l\neq 0$, i.e. $L_n=0$ for $n\neq 0$. Thus $L=L_0$, $\mathcal{B}_f(p,q)=L_0f(p,q)$ and $\mathcal{A}_f(p,q)=L_0f(p/h,q/h)$. From the condition $\mathcal{A}_p(p,q)=ip$ we find that $L_0=ih$. This proves (8.4).

3) According to (8.4),

$$\mathcal{A}_{p^4}(p,q)=\frac{i}{h^3}p^4+c_1,\quad \mathcal{A}_{q^4}(p,q)=\frac{i}{h^3}q^4+c_2,$$

$$\mathcal{A}_{[p^4,q^4]}=16\,\mathcal{A}_{p^3q^3}=16\frac{i}{h^5}p^3q^3+c_3.$$

Applying (8.5), we find

$$\begin{gathered}16\frac{i}{h^5}p^3q^3+c_3=\mathcal{A}_{[p^4,q^4]}=\frac{h}{i}[\mathcal{A}_{p^4},\mathcal{A}_{q^4}]\\ +\left(\frac{h}{2}\right)^3 2\frac{i^{-3}}{3!}\frac{\partial^3\mathcal{A}_{p^4}}{\partial p^3}\frac{\partial^3\mathcal{A}_{q^4}}{\partial q^3}=16\frac{i}{h^5}p^3q^3+\frac{i}{h^3 4\cdot 3!}(4!)^2pq.\end{gathered} \tag{8.11}$$

Equation (8.11) is inconsistent. Therefore the theorem is proved.

Appendix

1. *The asymptotic behavior of some integrals.* Let $\phi(t,\bar{t})$ be 4 times and $u(t,\bar{t})$ 3 times continuously differentiable in the region $D\subset C^n$ defined by the conditions $|t_i|<\epsilon$. In addition, suppose that the function $\phi(t,\bar{t})$ has a local maximum at $t=0$ and that $\det\|\partial^2\phi/\partial t_i\partial\bar{t}_k\|\neq 0$.

Let L_t be the operator $L_t=\sum t_i\partial/\partial z_i$, and let $\bar{L}_t$ be the operator $\sum\bar{t}_i\partial/\partial\bar{z}_i$. Let R be the function

$$R=\sum_{p+q=5} t_{i_1}\dots t_{i_p}\bar{t}_{j_1}\dots t_{j_q}\widetilde{R}_{i_1,\dots,i_p|j_1,\dots,j_q}(t,\bar{t}),$$

where $\widetilde{R}_{i_1,\dots,i_p|j_1,\dots,j_q}(t,\bar{t})$ is a function continuous in the closure of D. From this point on it will be implicitly assumed for simplicity that

$$L_t^p\bar{L}_t^q\varphi=L_t^p\bar{L}_t^q\varphi\,|_{z=\bar{z}=0}\,.$$

Finally, let

$$g(t,\bar t)=\det\left\|\frac{\partial^2\varphi}{\partial t_i\partial\bar t_k}\right\|,\qquad dtd\bar t=\prod\frac{dt^k\wedge d\bar t^k}{2\pi i}.$$

Consider the integral

$$\mathcal{I}_h(u)=h^{-n}\int u(t,\bar t)\,g(t,\bar t)\,e^{-\frac{1}{h}Q(t,\bar t)}\,dtd\bar t,$$

where

$$Q(t,\bar t)=L_t\bar L_t\varphi+\frac{1}{2}(L_t^2\bar L_t+\bar L_t^2L_t)\varphi+\frac{1}{4}L_t^2\bar L_t^2\varphi+\frac{1}{6}(L_t^3\bar L_t+\bar L_t^3L_t)\varphi+R(t,\bar t). \tag{A.1}$$

Lemma 1. *For ϵ sufficiently small the integral $\mathcal{I}_h(u)$ has as $h\to 0$ the asymptotic behavior*

$$\mathcal{I}_h(u)=u(0,0)+h\left(\Delta u+\frac{3}{2}u\Delta\ln g\right)\Big|_{t=\bar t=0}+o(h),$$

where Δ is the Laplace-Beltrami operator for the metric

$$ds^2=\sum\frac{\partial^2\varphi}{\partial t_i\partial\bar t_k}dt_id\bar t_k.$$

Proof. Let

$$\psi(t,\bar t)=e^{-\frac{1}{h}Q'(t,\bar t)}-1,$$

$$Q'=\frac{1}{2}(L_t^2\bar L_t+\bar L_t^2L_t)\varphi+\frac{1}{4}L_t^2\bar L_t^2\varphi+\frac{1}{6}(L_t^3\bar L_t+\bar L_t^3L_t)\varphi+R, \tag{A.2}$$

and expand ψ in powers of h^{-1}:

$$\psi=\psi_0+\widetilde R,$$

$$\psi_0=-\frac{1}{h}\Big[\frac{1}{2}(L_t^2\bar L_t+\bar L_t^2L_t)\varphi+\frac{1}{4}L_t^2\bar L_t^2\varphi+\frac{1}{6}(L_t^3\bar L_t+\bar L_t^3L_t)\varphi\Big]+\frac{1}{4h^2}L_t^2\bar L_t\varphi\cdot\bar L_t^2L_t\varphi. \tag{A.3}$$

Corresponding to (A.2) and (A.3) the integral $\mathcal{I}_h(u)$ can be represented in the form $\mathcal{I}_h(u)=\mathcal{I}_h^0(u)+\widetilde{\mathcal{I}}_h(u)$, where

$$\mathcal{I}_h^0(u)=\frac{1}{h^n}\int(1+\psi_0)\,u\,g e^{-\frac{1}{h}L_t\bar L_t\varphi}\,dtd\bar t, \tag{A.4}$$

$$\widetilde{\mathcal{I}}_h(u)=\frac{1}{h^n}\int\widetilde R u g e^{-\frac{1}{h}L_t\bar L_t\varphi}\,dtd\bar t. \tag{A.5}$$

The standard saddle-point methods show that $\widetilde{\mathcal{I}}_h(u)=o(h)$. Therefore it is sufficient for our purposes to limit ourselves to considering the integral $\mathcal{I}_h^0(u)$.

In the neighborhood of the point $t = \bar{t} = 0$ we have

$$u(t, \bar{t})\, g(t, \bar{t}) = u(0, 0)\, g(0, 0) + L_t(ug) + \bar{L}_t(ug) + L_t\bar{L}_t(ug) + T(t, \bar{t}), \quad (A.6)$$

where

$$T(t, \bar{t}) = \sum_{p+q=3} t_{i_1} \ldots t_{i_p} \bar{t}_{j_1} \ldots \bar{t}_{j_q} \tilde{T}_{i_1,\ldots,i_p|j_1,\ldots,j_q}$$

and $\tilde{T}_{i_1,\ldots,i_p|j_1,\ldots,j_q}$ is a continuous function.

Corresponding to (A.6), the integral $\mathfrak{I}_h^0(u)$ breaks up into the sum of two terms which it is convenient to discuss separately. We begin with the first:

$$\mathfrak{I}_h^{01}(u) = u(0, 0)\, g(0, 0) \frac{1}{h^n} \int (1 + \psi_0)\, e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\, dt d\bar{t}. \quad (A.7)$$

We insert ψ_0 from (A.3) into (A.7) and investigate the resulting integrals.

$$1) \int L_t^2 \bar{L}_t \varphi e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\, dt d\bar{t} = \int L_t \bar{L}_t^2 \varphi e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\, dt d\bar{t} = 0. \quad (A.8)$$

Indeed, by making the change of variables $t \to \theta t$, $\bar{t} \to \bar{\theta}\bar{t}$, $|\theta| = 1$, we find that the first of these integrals is multiplied by θ and the second by $\bar{\theta}$. But since, on the other hand, they do not depend on θ, they both must vanish. In the same way we establish the equation

$$\int L_t^3 \bar{L}_t \varphi e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\, dt d\bar{t} = \int L_t \bar{L}_t^3 \varphi e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\, dt d\bar{t} = 0.$$

2) Consider the following formal identity:

$$\int (L_t^2 \bar{L}_t^2 \varphi)(z, \bar{z})\, e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\, dt dt = \left(\frac{\partial}{\partial \mu}\right)^2 \left(\frac{\partial}{\partial \bar{\mu}}\right)^2 \int e^{-\frac{1}{h} L_t \bar{L}_t \varphi + \mu L_t + \bar{\mu} \bar{L}_t}\ \varphi(z, \bar{z})\, dt d\bar{t}\, \Big|_{\mu=\bar{\mu}=0}, \quad (A.9)$$

where $\exp[-h^{-1} L_t \bar{L}_t \phi + \mu L_t + \bar{\mu} \bar{L}_t]$ denotes the operator which acts on a function $\phi(z, \bar{z})$ by the formula

$$e^{-\frac{1}{h} L_t \bar{L}_t \varphi + \mu L_t + \bar{\mu} \bar{L}_t}\ \varphi(z, \bar{z}) = e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\ \varphi(z + \bar{\mu} t, \bar{z} + \mu t).$$

By completing the square in the exponent on the right-hand side of (A.9) and integrating with respect to t and $\bar{t}$, we find that

$$\int (L_t^2 \bar{L}_t^2 \varphi)(z, \bar{z})\, e^{-\frac{1}{h} L_t \bar{L}_t \varphi}\, dt d\bar{t} = h^n g^{-1} \frac{\partial^4}{\partial \mu^2 \partial \bar{\mu}^2} e^{h\mu\bar{\mu} \frac{\partial}{\partial \bar{z}} \Phi \frac{\partial}{\partial z}} \varphi(z, \bar{z}) \Big|_{\mu=\bar{\mu}=0} = 2g^{-1} h^{n+2} \left(\frac{\partial}{\partial \bar{z}} \Phi \frac{\partial}{\partial z}\right)^2 \varphi(z, \bar{z}), \quad (A.10)$$

where

$$\frac{\partial}{\partial \bar z}\Phi\frac{\partial}{\partial z}=\sum\frac{\partial}{\partial \bar z_i}\tilde\varphi_{ik}\frac{\partial}{\partial z_k},$$

and $\Phi = \|\tilde\phi_{ik}\|$ is the matrix inverse to $\|\partial^2\phi/\partial z_i\partial\bar z_k\|$.

3) Note that

$$L_t^2\bar L_t\varphi\cdot L_t\bar L_t^2\varphi$$

$$=\left(\sum t_i\frac{\partial}{\partial v_i}\right)^2\left(\sum \bar t_i\frac{\partial}{\partial \bar v_i}\right)\left(\sum t_i\frac{\partial}{\partial w_i}\right)\left(\sum \bar t_i\frac{\partial}{\partial \bar w_i}\right)^2\varphi(v,\bar v)\varphi(w,\bar w)\Big|_{\substack{v=w=0\\ \bar v=\bar w=0}},$$

and for simplicity introduce the notation

$$L_t^{(1)}=\sum t_i\frac{\partial}{\partial v_i},\quad L_t^{(2)}=\sum t_i\frac{\partial}{\partial w_i},$$
$$\bar L_t^{(1)}=\sum \bar t_i\frac{\partial}{\partial \bar v_i},\quad \bar L_t^{(2)}=\sum \bar t_i\frac{\partial}{\partial \bar w_i}. \tag{A.11}$$

By a transformation similar to the preceding one we get

$$\int L_t^{(1)}(\bar L_t^{(1)})^2(L_t^{(2)})^2(\bar L_t^2)\varphi(v,\bar v)\varphi(w,\bar w)e^{-\frac{1}{h}L_t\bar L_t\varphi}\,dtd\bar t$$
$$=\frac{\partial^6}{\partial\mu^2\partial\bar\mu\partial\sigma\partial\bar\sigma^2}\int e^{-\frac{1}{h}L_t\bar L_t\varphi+\mu\bar L_t^{(1)}+\bar\mu L_t^{(1)}+\sigma\bar L_t^{(2)}+\bar\sigma\bar L_t^{(2)}}$$
$$\times\varphi(v,\bar v)\varphi(w,\bar w)\prod dtd\bar t\Big|_{\mu=\sigma=0} \tag{A.12}$$
$$=h^n g^{-1}\frac{\partial^6}{\partial\mu^2\partial\bar\mu\partial\sigma\partial\bar\sigma^2}e^{h\left(\mu\frac{\partial}{\partial\bar v}+\sigma\frac{\partial}{\partial\bar w}\right)\Phi\left(\bar\mu\frac{\partial}{\partial v}+\bar\sigma\frac{\partial}{\partial w}\right)}\varphi(v,\bar v)\varphi(w,\bar w)\Big|_{\mu=\sigma=0}$$
$$=h^{n+3}g^{-1}\left[2\left(\frac{\partial}{\partial\bar v}\Phi\frac{\partial}{\partial w}\right)^2\left(\frac{\partial}{\partial\bar w}\Phi\frac{\partial}{\partial v}\right)+4\left(\frac{\partial}{\partial\bar v}\Phi\frac{\partial}{\partial w}\right)\left(\frac{\partial}{\partial\bar v}\Phi\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial\bar w}\Phi\frac{\partial}{\partial w}\right)\right]$$
$$\times\varphi(v,\bar v)\varphi(w,\bar w).$$

We extend the transformation of the integrals (A.10) and (A.12). We write for brevity

$$\varphi_{ik}=\frac{\partial^2\varphi}{\partial z_i\partial\bar z_k}\Big|_{z=\bar z=0}\quad\text{and}\quad\frac{\partial^{p+q}}{\partial\bar z^p\partial z^q}\varphi_{ik}=\frac{\partial^{p+q}}{\partial\bar z^p\partial z^q}\frac{\partial^2\varphi}{\partial z^i\partial z^k}\Big|_{z=\bar z=0}.$$

Using these abbreviations we find that the integral (A.10) for $z=\bar z=0$ is equal to

$$2g^{-1}h^{n+2}\sum\tilde\varphi_{ik}\tilde\varphi_{\alpha\beta}\frac{\partial^2\varphi_{ki}}{\partial\bar z_\alpha\partial z_\beta}.$$

Similarly, (A.12) with $v=\bar v=w=\bar w=0$ is equal to

$$2h^{n+3}g^{-1}\left[\frac{\partial\varphi_{ti}}{\partial\bar z_j}\frac{\partial\varphi_{ls}}{\partial z_k}+2\frac{\partial\varphi_{li}}{\partial\bar z_j}\frac{\partial\varphi_{ts}}{\partial z_k}\right]\tilde\varphi_{ik}\tilde\varphi_{jl}\tilde\varphi_{st}.$$

Thus we find the following final expression for (A.7):

$$\mathcal{J}_h^{01}(u) = u(0,\ 0)\left[1 + \frac{h}{2}\left\{\sum \tilde{\varphi}_{ik}\tilde{\varphi}_{\alpha\beta}\frac{\partial^2\varphi_{kl}}{\partial\bar{z}_\alpha\partial z_\beta} + \sum \tilde{\varphi}_{tk}\tilde{\varphi}_{jl}\tilde{\varphi}_{st}\left(\frac{\partial\varphi_{tl}}{\partial\bar{z}_j}\frac{\partial\varphi_{ts}}{\partial z_k} + 2\frac{\partial\varphi_{tl}}{\partial\bar{z}_j}\frac{\partial\varphi_{ts}}{\partial z_k}\right)\right\}\right]. \tag{A.13}$$

We turn our attention to the second integral:

$$\mathcal{J}^{02}(u) = \frac{1}{h^n}\int L_t(ug)(1+\psi_0)\,e^{-L_t\bar{L}_t\varphi}dtd\bar{t}. \tag{A.14}$$

First of all we find that

$$\int L_t(ug)\,e^{-\frac{1}{h}L_tL_t\varphi}\prod dtd\bar{t} = \int L_t(ug)(L_t^2\bar{L}_t^2\varphi)\,e^{-\frac{1}{h}L_t\bar{L}_t\varphi},$$

$$= \int L_t(ug)(L_t^2\bar{L}_t\varphi)(\bar{L}_t^2L_t\varphi)\,e^{-\frac{1}{h}L_t\bar{L}_t\varphi}\prod dtd\bar{t} \tag{A.15}$$

$$= \int L_t(ug)(L_t^2\bar{L}_t\varphi)\,e^{-\frac{1}{h}L_t\bar{L}_t\varphi}\prod dtd\bar{t} = 0.$$

Equation (A.15) is established by the same calculation as (A.8). Thus

$$\mathcal{J}_h^{02}(u) = -\frac{1}{2h^{n+1}}\int L_t(ug)(\bar{L}_t^2L_t\varphi)\,e^{-\frac{1}{h}L_t\bar{L}_t\varphi}dtd\bar{t}. \tag{A.16}$$

As above, we introduce operators $L_t^{(1)}$, $\bar{L}_t^{(1)}$, $L_t^{(2)}$ and $\bar{L}_t^{(2)}$ as in (A.11) and consider the auxiliary integral, which differs from (A.16) by the fact that the factor before the exponential has been replaced by

$$(L_t^{(1)}ug)(v,\bar{v})\,((\bar{L}_t^{(2)})^2L_t^{(2)}\varphi)(w,\bar{w}).$$

By using the argument by which we calculated the integral $\mathcal{J}^{01}$, we find that

$$\mathcal{J}_h^{02}(u) = -\frac{g^{-1}}{2h}\frac{\partial^4}{\partial\sigma^2\partial\bar{\mu}\partial\bar{\sigma}} \times e^{h\left(\mu\frac{\partial}{\partial\bar{v}}+\sigma\frac{\partial}{\partial\bar{w}}\right)\Phi\left(\bar{\mu}\left(\frac{\partial}{\partial v}+\bar{\sigma}\frac{\partial}{\partial w}\right)\right.}u(v,\bar{v})\,g(v,\bar{v})\,\varphi(w,\bar{w})\Big|_{\substack{\sigma=\mu=0\\ v=w=0}}.$$

Hence

$$\mathcal{J}_h^{02}(u) = -hg^{-1}\left(\frac{\partial}{\partial\bar{w}}\Phi\frac{\partial}{\partial w}\right)\left(\frac{\partial}{\partial\bar{w}}\Phi\frac{\partial}{\partial v}\right)u(v,\bar{v})\,g(v,\bar{v})\,\varphi(w,\bar{w})\Big|_{v=w=0}$$
$$= -hg^{-1}\tilde{\varphi}_{tk}\tilde{\varphi}_{jl}\frac{\partial\varphi_{kl}}{\partial\bar{z}_j}\frac{\partial}{\partial z_t}(u(z,\bar{z})\,g(z,\bar{z}))\Big|_{z=\bar{z}=0}. \tag{A.17}$$

The integral

$$\mathcal{J}_h^{03}(u) = \frac{1}{h^n}\int\bar{L}_t(ug)(1+\psi_0)\,e^{-\frac{1}{h}L_t\bar{L}_t\varphi}dtd\bar{t}.$$

is calculated similarly. It is equal to

$$\mathfrak{I}_h^{03}(u) = -hg^{-1}\,\tilde{\varphi}_{ik}\tilde{\varphi}_{jl}\frac{\partial\varphi_{kl}}{\partial z_l}\frac{\partial}{\partial \bar{z}_j}(ug)\Big|_{z=\bar{z}=0} \tag{A.18}$$

Turning to the integral

$$\mathfrak{I}_h^{04}(u) = \frac{1}{h^n}\int (L_t\bar{L}_t ug)(1+\psi_0)\,e^{-\frac{1}{h}L_t\bar{L}_t\varphi}dt d\bar{t}. \tag{A.19}$$

the usual considerations show that

$$\frac{1}{h^n}\int (L_t\bar{L}_t ug)\,\psi_0\, e^{-\frac{1}{h}L_t\bar{L}_t\varphi}dt d\bar{t} = o(h).$$

Therefore

$$\mathfrak{I}_h^{04}(u) = \frac{1}{h^n}\int (L_t\bar{L}_t ug)\,e^{-\frac{1}{h}L_t\bar{L}_t\varphi}dt d\bar{t} + o(h).$$

By using the method used above, we find from this that

$$\mathfrak{I}_h^{04}(u) = hg^{-1}\sum\tilde{\varphi}_{ik}\frac{\partial^2}{\partial z_i\partial\bar{z}_k}(ug)\Big|_{z=\bar{z}=0} + o(h). \tag{A.20}$$

Lastly, by an obvious calculation,

$$\mathfrak{I}_h^{05}(u) = h^{-n}\int Te^{-\frac{1}{h}L_t\bar{L}_t\varphi}\prod dt\,d\bar{t} = o(h).$$

Finally we obtain

$$\mathfrak{I}_h^{0}(u) = u(0,0) + hu_1(0,0) + o(h), \tag{A.21}$$

where

$$\begin{aligned} u_1 = u\Big\{&\frac{1}{2}\sum\tilde{\varphi}_{ik}\tilde{\varphi}_{\alpha\beta}\frac{\partial^2\varphi_{ki}}{\partial\bar{z}_\alpha\partial z_\beta} + \frac{1}{2}\sum\tilde{\varphi}_{ik}\tilde{\varphi}_{jl}\tilde{\varphi}_{st}\Big(\frac{\partial\varphi_{tl}}{\partial\bar{z}_j}\frac{\partial\varphi_{is}}{\partial z_k} \\ &+ 2\frac{\partial\varphi_{il}}{\partial\bar{z}_j}\frac{\partial\varphi_{ts}}{\partial z_k}\Big) - g^{-1}\sum\tilde{\varphi}_{ik}\tilde{\varphi}_{jl}\Big(\frac{\partial\varphi_{ki}}{\partial\bar{z}_j}\frac{\partial g}{\partial z_l} + \frac{\partial\varphi_{kl}}{\partial z_l}\frac{\partial g}{\partial\bar{z}_j}\Big) \\ &+ g^{-1}\sum\tilde{\varphi}_{ik}\frac{\partial^2 g}{\partial\bar{z}_i\partial z_k}\Big\} + \sum\frac{\partial u}{\partial z_l}\Big(-\sum\tilde{\varphi}_{ik}\tilde{\varphi}_{jl}\frac{\partial\varphi_{ki}}{\partial\bar{z}_j} \\ + \sum\tilde{\varphi}_{il}g^{-1}\frac{\partial g}{\partial\bar{z}_i}\Big\} &+ \sum\frac{\partial u}{\partial\bar{z}_j}\Big(-\sum\tilde{\varphi}_{ik}\tilde{\varphi}_{jl}\frac{\partial\varphi_{ki}}{\partial z_l} + \sum\tilde{\varphi}_{jk}g^{-1}\frac{\partial g}{\partial z_k}\Big) + \sum\tilde{\varphi}_{ik}\frac{\partial^2 u}{\partial\bar{z}_i\partial z_k}. \end{aligned} \tag{A.22}$$

We shall transform this result. For this purpose we use the easily proven identities

$$\frac{\partial\tilde{\varphi}_{ik}}{\partial x} = -\sum\tilde{\varphi}_{is}\frac{\partial\varphi_{st}}{\partial x}\tilde{\varphi}_{tk}, \tag{A.23}$$

$$g^{-1}\frac{\partial g}{\partial x} = \sum \tilde{\varphi}_{ks}\frac{\partial \varphi_{sk}}{\partial x}, \tag{A.24}$$

where x denotes either of the variables z_α or $\bar{z}_\alpha$. By (A.24) we find that

$$\sum \tilde{\varphi}_{ll} g^{-1}\frac{\partial g}{\partial \bar{z}_i} = \sum \tilde{\varphi}_{ll}\tilde{\varphi}_{ks}\frac{\partial \varphi_{sk}}{\partial \bar{z}_l},$$

whence we have the vanishing of the coefficient of $\partial u/\partial z_l$. The vanishing of the coefficient of $\partial u/\partial \bar{z}_j$ is shown similarly.

Let us turn to the coefficient of u. It follows from (A.24) that

$$\sum \tilde{\varphi}_{lk}\tilde{\varphi}_{jl}\left(\frac{\partial \varphi_{kl}}{\partial \bar{z}_j}\frac{\partial g}{\partial z_l} + \frac{\partial \varphi_{kl}}{\partial z_l}\frac{\partial g}{\partial \bar{z}_j}\right) = 2g^{-1}\sum \tilde{\varphi}_{jl}\frac{\partial g}{\partial \bar{z}_j}\frac{\partial g}{\partial z_l}.$$

By using this identity, and also the fact that $\partial\phi_{ti}/\partial\bar{z}_j = \partial\phi_{tj}/\partial\bar{z}_i$ (which is a consequence of the definition $\phi_{ti} = \partial^2\phi/\partial z_t\partial\bar{z}_i$), we see that the coefficient of u is equal to

$$\frac{1}{2}\sum \tilde{\varphi}_{lk}\left[\sum \tilde{\varphi}_{\alpha\beta}\frac{\partial^2\varphi_{kl}}{\partial\bar{z}_\alpha\partial z_\beta} + \sum \tilde{\varphi}_{jl}\tilde{\varphi}_{st}\frac{\partial \varphi_{tj}}{\partial \bar{z}_l}\frac{\partial \varphi_{ls}}{\partial z_k}\right]$$
$$+ g^{-2}\sum \tilde{\varphi}_{lk}\left(g\frac{\partial^2 g}{\partial\bar{z}_l\partial z_k} - \frac{\partial g}{\partial \bar{z}_l}\frac{\partial g}{\partial z_k}\right).$$

Next, combining (A.23) and (A.24), we find

$$g^{-1}\frac{\partial^2 g}{\partial\bar{z}_l\partial z_k} = \sum \tilde{\varphi}_{\alpha\beta}\frac{\partial^2\varphi_{\beta\alpha}}{\partial\bar{z}_l\partial z_k} + \sum \tilde{\varphi}_{jl}\tilde{\varphi}_{st}\left(\frac{\partial \varphi_{lj}}{\partial \bar{z}_l}\frac{\partial \varphi_{ts}}{\partial z_k} - \frac{\partial \varphi_{tj}}{\partial \bar{z}_l}\frac{\partial \varphi_{ls}}{\partial z_k}\right).$$

Hence we have

$$\sum \tilde{\varphi}_{lk}\left[\sum \tilde{\varphi}_{\alpha\beta}\frac{\partial^2\varphi_{kl}}{\partial\bar{z}_\alpha\partial z_\beta} + \sum \tilde{\varphi}_{jl}\tilde{\varphi}_{st}\frac{\partial \varphi_{tj}}{\partial \bar{z}_l}\frac{\partial \varphi_{ls}}{\partial z_k}\right]$$
$$= \sum \tilde{\varphi}_{lk}\left[g^{-1}\frac{\partial^2 g}{\partial\bar{z}_l\partial z_k} - \sum \tilde{\varphi}_{jl}\tilde{\varphi}_{st}\frac{\partial \varphi_{lj}}{\partial \bar{z}_i}\frac{\partial \varphi_{ts}}{\partial z_k}\right]$$
$$= g^{-2}\sum \tilde{\varphi}_{lk}\left(g\frac{\partial^2 g}{\partial\bar{z}_l\partial z_k} - \frac{\partial g}{\partial \bar{z}_i}\frac{\partial g}{\partial z_k}\right).$$

Now note that

$$\frac{\partial^2 \ln g}{\partial\bar{z}_l\partial z_k} = g^{-2}\left(g\frac{\partial^2 g}{\partial\bar{z}_l\partial z_k} - \frac{\partial g}{\partial \bar{z}_l}\frac{\partial g}{\partial z_k}\right).$$

Thus the coefficient of u is equal to

$$\frac{3}{2}\sum \tilde{\varphi}_{lk}\frac{\partial^2 \ln g}{\partial\bar{z}_l\partial z_k}.$$

It will be shown below (see Lemma 3) that the Laplace-Beltrami operator for the metric

$\phi_{ik} = \partial^2\phi/\partial z_i\partial\bar{z}_k$ when applied to functions is equal to

$$\Delta = \sum \tilde{\varphi}_{ik} \frac{\partial^2}{\partial \bar{z}_i \partial z_k}.$$

Therefore our result can be reformulated:

$$u_1 = \Delta u + \frac{3}{2} u \Delta \ln g.$$

The lemma is proved

Let $u(t, \bar{t})$ be a continuously differentiable function, let $\phi = \sum t_i \phi_{ik} \bar{t}_k$, $i, k = 1, \cdots, n$, be a positive definite quadratic form, and let R be a function of the form

$$R(t, \bar{t}) = \sum_{p+q=3} t_{i_1} \ldots t_{i_p} \bar{t}_{j_1} \ldots \bar{t}_{j_q} \tilde{R}_{i_1,\ldots,i_p|j_1,\ldots,j_q}(t, \bar{t}),$$

where the $\tilde{R}_{i_1,\cdots,i_p|j_1,\cdots,j_q}$ are continuous functions. We denote by $g(t, \bar{t})$ a continuously differentiable function such that $g(0, 0) = \det \|\phi_{ik}\|$.

Consider the integral

$$\mathcal{J}_h(u) = \frac{1}{h^n} \int_{|t_i|<\varepsilon} u(t, \bar{t}) g(t, \bar{t}) e^{-\frac{1}{h}[\varphi(t,\bar{t})+R(t,\bar{t})]} dt d\bar{t}. \qquad (A.25)$$

Lemma 2. *For sufficiently small* ϵ

$$|\mathcal{J}_h(u) - u(0, 0)| \leqslant c\sqrt{h} \max_{\substack{i \\ |t_i|<\varepsilon}} \left(\left| \frac{\partial u}{\partial t_i} \right|, \left| \frac{\partial u}{\partial \bar{t}_i} \right| \right) + u(0, 0) \cdot o(1), \qquad (A.26)$$

where c *and* $o(1)$ *do not depend on* u.

Proof. In the integral (A.25) make the change of variable $t \to t\sqrt{h}$, $\bar{t} \to \bar{t}\sqrt{h}$. We shall show that following this we can take the limit under the integral sign. We decompose ϕ into the sum $\phi = \phi_0 + \phi'$, where ϕ_0 and ϕ' are positive definite quadratic forms, with $\phi' > |R|$ for $|t_i| < \epsilon$ (it is this condition which determines ϵ). Thus

$$e^{-\frac{1}{h}(\varphi+R)} \leqslant e^{-\frac{1}{h}\varphi_0} \quad \text{and} \quad e^{-\varphi+R(t\sqrt{h},\, \bar{t}\sqrt{h})} \leqslant e^{-\varphi_0(t,\bar{t})}.$$

Consequently the integral $\mathcal{J}_h(u)$, after the substitution $t \to t\sqrt{h}$, $\bar{t} \to \bar{t}\sqrt{h}$, is boundedly convergent, so that we can take the limit under the integral sign, whereupon we find that

$$\lim_{h\to 0} \mathcal{J}_h(u) = u(0, 0).$$

In particular, if $u = \text{const}$, it follows that $\mathcal{J}_h(u) - u = u \cdot o(1)$, where $o(1)$ does not depend on u.

Now let $u(0, 0) = 0$ and $u(t, \bar{t}) = \sum t_i u_i + \bar{t}\tilde{u}_i$, where u_i and $\tilde{u}_i$ are bounded functions. A repetition of the above argument shows that

$$|\mathfrak{I}_h(u)| \leqslant \sqrt{h}\int \sum (|u_i(t\sqrt{h}, \bar{t}\sqrt{h})| + |\tilde{u}_i(t\sqrt{h}, \bar{t}\sqrt{h})|)|t_i|$$
$$\times g(t\sqrt{h}, \bar{t}\sqrt{h}) e^{-\varphi_0} \prod dt d\bar{t} \leqslant c\sqrt{h} \max_{i, |t_i|<\varepsilon} (|u_i(t, \bar{t})|, |\tilde{u}_i(t, \bar{t})|).$$

Now u_i and $\tilde{u}_i$ can be chosen so that

$$|u_i| \leqslant c \max_{|t_\alpha|<\varepsilon} \left|\frac{\partial u_i}{\partial t_i}\right|, \quad |\tilde{u}_i| \leqslant c \max_{|t_\alpha|<\varepsilon} \left|\frac{\partial u}{\partial \bar{t}_i}\right|.$$

Let $u_0(t, \bar{t}) = u(0, 0) = \text{const}$. By combining this result with the previous one, we find that

$$\mathfrak{I}_h(u) = \mathfrak{I}_h(u - u_0) + \mathfrak{I}_h(u_0) = \mathfrak{I}_h(u - u_0) + u_0 + u_0 \cdot o(1),$$

so that

$$|\mathfrak{I}_h(u) - u(0, 0)| \leqslant c\sqrt{h} \max_{i, |t_\alpha|<\varepsilon} \left(\left|\frac{\partial u}{\partial t_i}\right|, \left|\frac{\partial u}{\partial \bar{t}_i}\right|\right) + u(0, 0) \cdot o(1).$$

The lemma is proved.

2. *The Laplace-Beltrami operator on a Kähler manifold.* Let a metric

$$ds^2 = \sum \psi_{ik} dz_i d\bar{z}_k. \tag{A.27}$$

be given in some region $D \subset C^n$. A necessary and sufficient condition for (A.27) to be a Kähler metric, and also for the local existence of a function ψ such that $\psi_{ik} = \partial^2\psi/\partial z_i \partial\bar{z}_k$, is that the identities

$$\frac{\partial \psi_{ik}}{\partial z_\alpha} = \frac{\partial \psi_{\alpha k}}{\partial z_i},$$
$$\frac{\partial \psi_{ik}}{\partial \bar{z}_\alpha} = \frac{\partial \psi_{i\alpha}}{\partial \bar{z}_k} \tag{A.28}$$

be satisfied. We have the following general assertion.

Lemma 3. *The Laplace-Beltrami operator for the metric* (A.27) *when applied to functions*[28] *has the form*

$$\Delta = \sum \tilde{\psi}_{ik} \frac{\partial^2}{\partial \bar{z}_i \partial z_k},$$

where $\|\tilde{\psi}_{ik}\|$ *is the inverse of the matrix* $\|\psi_{ik}\|$.

Proof. By our general formula,

$$\Delta u = \frac{1}{2} \sum \left[g^{-1} \frac{\partial}{\partial \bar{z}_i} \tilde{\psi}_{ik} g \frac{\partial}{\partial z_k} u + g^{-1} \frac{\partial}{\partial z_k} \tilde{\psi}_{ik} g \frac{\partial}{\partial \bar{z}_i} u \right] =$$

(28) But not to tensor fields!

$$= \sum \widetilde{\psi}_{lk} \frac{\partial^2 u}{\partial \bar{z}_l \partial z_k} + \frac{1}{2g} \sum \left[\left(\frac{\partial}{\partial \bar{z}_l} g \widetilde{\psi}_{lk} \right) \frac{\partial u}{\partial z_k} + \left(\frac{\partial}{\partial z_k} g \widetilde{\psi}_{lk} \right) \frac{\partial u}{\partial \bar{z}_l} \right].$$

By (A.23) and (A.24)

$$g^{-1} \sum \frac{\partial}{\partial \bar{z}_l} g \widetilde{\psi}_{lk} = \sum \left(g^{-1} \frac{\partial g}{\partial \bar{z}_l} \widetilde{\psi}_{lk} + \frac{\partial \widetilde{\psi}_{lk}}{\partial \bar{z}_l} \right)$$

$$= \sum \widetilde{\psi}_{ls} \frac{\partial \psi_{sl}}{\partial \bar{z}_l} \widetilde{\psi}_{lk} - \sum \widetilde{\psi}_{ls} \frac{\partial \psi_{st}}{\partial \bar{z}_l} \widetilde{\psi}_{tk} = 0,$$

so that, according to (A.28), $\partial\psi_{sl}/\partial\bar{z}_i = \partial\psi_{si}/\partial\bar{z}_l$. In the same way one can show that the coefficient of $\partial u/\partial\bar{z}_i$ also vanishes.

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