MAT 126 Calculus B Fall 2005
Practice Midterm II Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Evaluate the following definite integrals

(a)

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx$$

Solution. Substitution $u = 2x+1$ gives $du = 2dx$ and $x = 0$ corresponds to $u = 1$ and $x = 13$ — to $u = 27$. Thus by the substitution rule,

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx = \int_1^{27} \frac{1}{u^{\frac{2}{3}}} du = 3u^{\frac{1}{3}}\Big|_1^{27}$$
$$= 3(27)^{\frac{1}{3}} - 3 = 3 \cdot 3 - 3 = 6.$$

(b)

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$$

Solution. Substitution $u = \sin x$ gives $du = \cos x dx$ and $x = 0$ corresponds to $u = 0$ and $x = \pi/2$ — to $u = 1$. Thus by the substitution rule,

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx = \int_0^1 e^u du = e^u\big|_0^1 = e - 1.$$

(c)

$$\int_0^1 x^4(1+x^5)^{20} dx$$

Solution. Substitution $u = 1 + x^5$ gives $du = 5x^4 dx$ and $x = 0$ corresponds to $u = 1$ and $x = 1$ — to $u = 2$. Thus by the substitution rule,

$$\int_0^1 x^4(1+x^5)^{20} dx = \frac{1}{5}\int_1^2 u^{20} du = \frac{u^{21}}{5 \cdot 21}\Big|_1^2 = \frac{2^{21}-1}{105}.$$

(d)

$$\int_0^1 \tan^{-1} x \, dx$$

Solution. We use integration by parts with $u = \tan^{-1} x$ and $dv = dx$. We have $du = \dfrac{dx}{1+x^2}$ and $v = x$, so that using $\tan^{-1}(1) = \dfrac{\pi}{4}$, we get

$$\int_0^1 \tan^{-1} x\, dx = \int_0^1 udv = uv|_0^1 - \int_0^1 vdu$$
$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx.$$

To evaluate the remaining integral, we use the substitution $u = 1 + x^2$, so that $du = 2xdx$ and $x = 0$ corresponds to $u = 1$ and $x = 1$ — to $u = 2$. Thus

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2}\int_1^2 \frac{du}{u} = \left.\frac{\ln u}{2}\right|_1^2 = \frac{\ln 2}{2}.$$

Therefore, we get

$$\int_0^1 \tan^{-1} x\, dx = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

(e)

$$\int_0^{\frac{\pi}{2}} \cos^5 t\, dt.$$

Solution. This is a trigonometric integral, so we use the main trigonometric identity and represent

$$\cos^5 t = \cos^4 t \cdot \cos t = (1 - \sin^2 t)^2 \cos t.$$

Now the substitution $u = \sin t$ gives $du = \cos t\, dt$, and $t = 0$ corresponds to $u = 0$ and $t = \dfrac{\pi}{2}$ — to $u = 1$. We have

$$\int_0^{\frac{\pi}{2}} \cos^5 t\, dt = \int_0^{\frac{\pi}{2}} (1 - \sin^2 t)^2 \cos t\, dt$$
$$= \int_0^1 (1 - u^2)^2 du = \int_0^1 (1 - 2u^2 + u^4) du$$
$$= \left.\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right)\right|_0^1 = \frac{8}{15}.$$

(f)

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

Solution. Substitution $u = \sin^{-1} x$ gives $du = \frac{dx}{\sqrt{1-x^2}}$, and $x = 0$ corresponds corresponds to $u = 0$ and $x = \frac{1}{2}$ — to $u = \frac{\pi}{6}$, so that

$$\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_0^{\frac{\pi}{6}} u\, du = \frac{\pi^2}{72}.$$

(g)

$$\int_{-\pi}^{\pi} \frac{x^2 \sin^3 x}{1+x^6}\, dx$$

Solution. This is the integral over symmetric interval $[-\pi, \pi]$, and the function

$$f(x) = \frac{x^2 \sin^3 x}{1+x^6}$$

is odd, $f(-x) = -f(x)$, so that

$$\int_{-\pi}^{\pi} \frac{x^2 \sin^3 x}{1+x^6}\, dx = 0$$

2. Evaluate the following indefinite integrals

(a)

$$\int x^3 e^{x^4} dx$$

Solution. Setting $u = x^4$, we get $du = 4x^3 dx$, so by the substitution rule,

$$\int x^3 e^{x^4} dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C.$$

(b)

$$\int t e^t dt$$

Solution. We use integration by parts with $u = t$ and $dv = e^t dt$. We have $du = dt$ and $v = e^t$, so that

$$\int t e^t dt = \int u dv = uv - \int v du = te^t - \int e^t dt = te^t - e^t + C.$$

(c)

$$\int x^2 \cos x\, dx$$

Solution. We use integration by parts with $u = x^2$ and and $dv = \cos x dx$. We have $du = 2x dx$ and $v = \sin x$, so that

$$\int x^2 \cos x\, dx = \int u dv = uv - \int v du = x^2 \sin x - 2 \int x \sin x\, dx.$$

For the remaining integral we again use integration by parts with $u = 2x$ and $dv = \sin x dx$, so that $du = 2dx$ and $v = -\cos x$. We have

$$\begin{aligned} 2\int x \sin x\, dx = \int u dv = uv - \int v du = -2x\cos x + 2\int \cos x\, dx \\ = -2x\cos x + 2\sin x + C, \end{aligned}$$

so that

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x\cos x - 2\sin x + C.$$

(Double-check the answer by differentiating!)

(d)

$$\int \cos(\sqrt{x}) dx$$

Solution. First, we use the substitution rule with $t = \sqrt{x}$, so that $dt = \frac{1}{2\sqrt{x}}\, dx$, or $dx = 2\sqrt{x}\, dt = 2t\, dt$. We get

$$\int \cos(\sqrt{x}) dx = 2\int t\cos t dt.$$

To evaluate this integral, we use integration by parts with $u = 2t$ and $dv = \cos t dt$. We have $du = 2dt$ and $v = \sin t$, so that

$$\begin{aligned} 2\int t\cos t dt = \int u dv = uv - \int v du \\ = 2t\sin t - 2\int \sin t dt = 2t\sin t + 2\cos t + C. \end{aligned}$$

Finally, remembering that $t = \sqrt{x}$, we get

$$\int \cos(\sqrt{x}) dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

3. Evaluate the following indefinite integrals

(a)

$$\int \frac{1}{x^2} \ln x dx$$

Solution. Here we use integration by parts with $u = \ln x$ and $dv = \frac{1}{x^2} dx$, so that

$$du = \frac{1}{x} dx \quad \text{and} \quad v = -\frac{1}{x}.$$

(Note that substitution rule with $u = \ln x$ does not simplify the integral since in the denominator we have x^2; if it was x, then the substitution rule would work.) Thus we have

$$\int \frac{1}{x^2} \ln x dx = \int u dv = uv - \int v du$$
$$= -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

(b)

$$\int \frac{1}{x} (\ln x)^2 dx$$

Solution. Here we use the substitution rule with $u = \ln x$ and $du = \frac{1}{x} dx$ (since we have x in the denominator). Therefore,

$$\int \frac{1}{x} (\ln x)^2 dx = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} (\ln x)^3 + C.$$

(c)

$$\int x^{\frac{3}{2}} \ln x \, dx$$

Solution. Here we use integration by parts with $u = \ln x$ and $dv = x^{\frac{3}{2}} dx$, so that

$$du = \frac{1}{x} dx \quad \text{and} \quad v = \frac{2}{5} x^{5/2}.$$

Thus we have

$$\int x^{\frac{3}{2}} \ln x \, dx = \int u dv = uv - \int v du$$
$$= \frac{2}{5} x^{5/2} \ln x - \frac{2}{5} \int x^{3/2} dx = \frac{2}{5} x^{5/2} \ln x - \frac{4}{25} x^{5/2} + C.$$

4. Evaluate the following indefinite integrals

(a)

$$\int \frac{2x^2}{x^2+1} dx$$

Solution. We have

$$\frac{2x^2}{x^2+1} = 2 - \frac{2}{x^2+1}$$

(either by doing the long division, or by writing $2x^2 = 2x^2 + 2 - 2 = 2(x^2+1) - 2$, and dividing both terms by

x^2+1). Thus

$$\int \frac{2x^2}{x^2+1}dx = \int \left(2 - \frac{2}{x^2+1}\right) dx = 2x - 2\tan^{-1} x + C.$$

(b)

$$\int \frac{2}{x^2-1}dx$$

Solution. Using partial fractions, we write

$$\frac{2}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

This decomposition is valid if and only if for all x

$$2 = A(x+1) + B(x-1).$$

Setting $x = 1$ we get $A = 1$, and setting $x = -1$ we get $B = -1$. (Another way to solve for A and B is to rewrite the above equation as

$$2 = (A+B)x + (A-B),$$

which is equivalent to the following system of linear algebraic equations for A and B:

$$A + B = 0,$$
$$A - B = 2.$$

The solution of this system is, as before, $A = 1$, $B = -1$.) Therefore we have

$$\int \frac{2}{x^2-1}dx = \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$
$$= \ln|x-1| - \ln|x+1| + C = \ln\left|\frac{x-1}{x+1}\right| + C.$$

(c)

$$\int \frac{2x}{x^2+1}dx$$

Solution. To evaluate this integral, we use the substitution $u = 1 + x^2$, so that $du = 2xdx$ (compare with the last integral in problem **2** (d)). We have

$$\int \frac{2x}{x^2+1}dx = \int \frac{du}{u} = \ln|u| + C = \ln(1+x^2) + C.$$

(d)

$$\int \frac{4x+7}{2x^2+7x-15}\,dx$$

Solution. We have (either by guessing or using the formula for the roots of quadratic polynomial)

$$2x^2+7x-15 = (2x-3)(x+5)$$

so that we use method of partial fractions,

$$\frac{4x+7}{2x^2+7x-15} = \frac{A}{2x-3} + \frac{B}{x+5} = \frac{A(x+5)+B(2x-3)}{(2x-3)(x+5)}.$$

Clearing the denominators, we get the equation

$$4x+7 = A(x+5)+B(2x-3)$$

and setting x=-5 we get $-13 = -13B$, so $B = 1$; setting $x = \frac{3}{2}$ we get $13 = 13A/2$, so that $A = 2$. As the result,

$$\int \frac{4x+7}{2x^2+7x-15}\,dx = \int \left(\frac{2}{2x-3} + \frac{1}{x+5}\right) dx$$
$$= \ln|2x-3| + \ln|x+5| + C,$$

where for the first term we used the substitution $u = 2x-3$.

5. (a) Write a formula for $\tan x$ in terms of $\sin x$ and $\cos x$.
Solution.

$$\tan x = \frac{\sin x}{\cos x}.$$

(b) Evaluate

$$\int \tan x\,dx$$

Solution. Using part (a) we have

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx,$$

which suggests the substitution $u = \cos x$. We have $du = -\sin x dx$, so that

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

6. Evaluate

(a)

$$\int \sqrt{16-x^2}\,dx$$

Solution. This integral is simplified with the help of the substitution $x = 4\sin\theta$ (see Section 5.7). We have $16 -$

$x^2 = 16(1 - \sin^2\theta) = 16\cos^2\theta$, so that $\sqrt{16 - x^2} = 4\cos\theta$. Since $dx = 4\cos\theta d\theta$, we have

$$\int \sqrt{16 - x^2}\, dx = 16 \int \cos^2\theta d\theta.$$

To evaluate the last integral, we use the half-angle formula

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}.$$

Thus

$$\begin{aligned}\int \sqrt{16 - x^2}\, dx &= 16 \int \cos^2\theta d\theta = 8 \int (1 + \cos 2\theta) d\theta \\ &= 8\theta + 4\sin 2\theta + C \\ &= 8\theta + 8\sin\theta\cos\theta + C \\ &= 8\sin^{-1}\frac{x}{4} + 2x\cos\left(\sin^{-1}\frac{x}{4}\right) + C \\ &= 8\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{16 - x^2} + C.\end{aligned}$$

Here in the last three lines we have used the double angle formula for $\sin 2\theta$, the equation $\theta = \sin^{-1}\frac{x}{4}$, and the formula $\cos\theta = \sqrt{1 - \sin^2\theta}$, which follows from the fundamental trigonometric identity.

(b)

$$\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx$$

Solution. This integral is simplified with the help of the substitution $x = 2\tan\theta$ (see Section 5.7). We have, using the fundamental trigonometric identity,

$$x^2 + 4 = 4\tan^2\theta + 4 = \frac{4}{\cos^2\theta},$$

so that $\frac{1}{\sqrt{x^2 + 4}} = \frac{1}{2}\cos\theta$. Since $dx = \frac{2}{\cos^2\theta}\, d\theta$, we get

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx &= \int \frac{1}{2}\cos\theta \frac{\cos^2\theta}{4\sin^2\theta} \frac{2}{\cos^2\theta}\, d\theta \\ &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = -\frac{1}{4\sin\theta} + C \\ &= -\frac{1}{4\sin(\tan^{-1}\frac{x}{2})} + C,\end{aligned}$$

where in the last line we have used that $\theta = \tan^{-1}(\frac{x}{2})$.