

MAT 126 Calculus B Spring 2005 Practice Midterm I — Solutions

Name: _____

I.D.: _____ Section number: _____

Please answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. You **do not need** to simplify numerical answers or write their approximate values: if the answer you got is $\sqrt{2}$ you should not replace it by 1.414. Pay close attention to the distinction between sequences of numbers and series (which are sums). The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. (a) (10 points) Estimate the area under the graph of $f(x) = 16 - x^2$ from $x = 0$ to $x = 4$ using four rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate and underestimate or an overestimate?

Solution.

$$R_4 = f(1) + f(2) + f(3) + f(4) = 15 + 12 + 7 = 34$$

The function $f(x)$ is concave downward, so that $R_4 < A$, i.e., it is an underestimate (sketch the graph!)

- (b) (10 points) Repeat part (a) using left endpoints.

Solution.

$$L_4 = f(0) + f(1) + f(2) + f(3) = 50$$

It is an overestimate, $L_4 > A$ (sketch the graph!)

2. (a) (10 points) Evaluate integral by interpreting it as area

$$\int_{-5}^5 \sqrt{25 - x^2} dx$$

Solution. It is the upper half of the circle of radius 5 centered at the origin, so

$$\int_{-5}^5 \sqrt{25 - x^2} dx = \frac{1}{2} \pi 5^2 = 12.5\pi$$

(b) (5 points) Determine a region whose area is equal to

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{4n} \tan \frac{i\pi}{4n}$$

Do not evaluate the limit.

Solution. It is the right endpoint sum for the function $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$. Thus

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{4n} \tan \frac{i\pi}{4n} = \int_0^{\pi/4} \tan x dx$$

3. Given two functions $f(x)$ and $g(x)$ which satisfy

$$\begin{aligned} \int_0^3 f(x) dx &= 5, & \int_0^5 f(x) dx &= 7, \\ \int_3^5 g(x) dx &= 1, & \int_0^5 g(x) dx &= 9, \end{aligned}$$

find

(a) (5 points)

$$\int_3^5 (3f(x) - g(x)) dx$$

Solution. We have

$$\int_3^5 f(x) dx = \int_0^5 f(x) dx - \int_0^3 f(x) dx = 7 - 5 = 2$$

so that

$$\int_3^5 (3f(x) - g(x)) dx = 3 \int_3^5 f(x) dx - \int_3^5 g(x) dx = 6 - 1 = 5$$

(b) (5 points)

$$\int_0^3 (f(x) + 2g(x)) dx$$

Solution. Similarly,

$$\int_0^3 g(x) dx = 9 - 1 = 8$$

and

$$\int_0^3 (f(x) + 2g(x)) dx = 5 + 2 \times 8 = 21$$

4. (5 points) Express the limit as a definite integral on the given interval $[0, 4]$:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{e^{x_i}}{1+x_i} \Delta x$$

Solution.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{e^{x_i}}{1+x_i} \Delta x = \int_0^4 \frac{e^x}{1+x} dx$$

5. Evaluate the following indefinite integrals

- (a) (5 points)

$$\int (3 \cos x - 4 \sin x) dx$$

Solution.

$$\int (3 \cos x - 4 \sin x) dx = 3 \sin x + 4 \cos x + C$$

- (b) (10 points)

$$\int \frac{\cos x}{1 - \cos^2 x} dx$$

Solution. Using the fundamental trigonometric identity,

$$\int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C$$

6. Evaluate the following definite integrals

- (a) (5 points)

$$\int_1^2 x^{-2} dx$$

Solution.

$$\int_1^2 x^{-2} dx = \left. \frac{x^{-1}}{-1} \right|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

- (b)

$$\int_1^8 \frac{x-1}{\sqrt[3]{x^2}} dx$$

Solution.

$$\begin{aligned} \int_1^8 \frac{x-1}{\sqrt[3]{x^2}} dx &= \int_1^8 (x^{1/3} - x^{-2/3}) dx \\ &= \left(\frac{3}{4} x^{4/3} - 3x^{1/3} \right) \Big|_1^8 = \left(\frac{3}{4} \times 8^{4/3} - 3 \times 8^{1/3} \right) - \left(\frac{3}{4} - 3 \right) = 8.25 \end{aligned}$$

where we have used that $8 = 2^3$.

(c) (5 points)

$$\int_1^{27} \frac{1}{9t} dt$$

Solution.

$$\int_1^{27} \frac{1}{9t} dt = \frac{1}{9} \ln t \Big|_1^{27} = \frac{1}{9} \ln 27 = \frac{1}{3} \ln 3$$

, where we have used that $\ln 1 = 0$ and $27 = 3^3$.

(d) (5 points)

$$\int_{\ln 3}^{\ln 6} 5e^x dx$$

Solution.

$$\int_{\ln 3}^{\ln 6} 5e^x dx = 5e^x \Big|_{\ln 3}^{\ln 6} = 5(e^{\ln 6} - e^{\ln 3}) = 5(6 - 3) = 15$$

(e) (10 points)

$$\int_{\pi/3}^{\pi/2} \csc x \cot x dx$$

Solution.

$$\begin{aligned} \int_{\pi/3}^{\pi/2} \csc x \cot x dx &= \int_{\pi/3}^{\pi/2} \frac{\cos x}{\sin^2 x} dx \\ &= \left(-\frac{1}{\sin x} \right) \Big|_{\pi/3}^{\pi/2} = -1 + \frac{2}{\sqrt{3}}. \end{aligned}$$

7. Let

$$F(x) = \int_{x^3}^8 e^{-t^2} dt + \sqrt{x}$$

(a) Find $F'(x)$

Solution. We have, using the chain rule with $u = x^3$ and the FTC,

$$\begin{aligned} \frac{dF}{dx} &= \frac{d}{du} \left(- \int_8^u e^{-t^2} dt \right) \frac{du}{dx} + \frac{d}{dx} \sqrt{x} \\ &= -3x^2 e^{-x^6} + \frac{1}{2\sqrt{x}}. \end{aligned}$$

(b) Find $F(2)$

Solution.

$$F(2) = \int_8^8 e^{-t^2} dt + \sqrt{2} = \sqrt{2}.$$

8. It is given that the function $f(x)$ for all $x > 0$ satisfies the equation

$$3 + \int_a^x \frac{f(t)}{t} dt = \sqrt[3]{x}$$

with some number a . Find the function $f(x)$ and the number a .

Solution. Write

$$\int_a^x \frac{f(t)}{t} dt = \sqrt[3]{x} - 3$$

and differentiate both sides of this equation with respect to x . Using the FTC, we get

$$\frac{f(x)}{x} = \frac{1}{3}x^{-2/3},$$

and solving for $f(x)$ we get

$$f(x) = \frac{1}{3}\sqrt[3]{x}.$$

Now again consider the equation

$$3 + \int_a^x \frac{f(t)}{t} dt = \sqrt[3]{x}$$

and set $x = a$. We get

$$3 = \sqrt[3]{a},$$

so that $a = 3^3 = 27$.