MAT 126 Calculus B Fall 2005
Practice Final Exam — Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. You **do not need** to simplify numerical answers or write their approximate values. This practice exam contains more problems then the actual test to give you more practice.

1. Evaluate the following definite integrals:

(a)

$$\int_1^9 \ln \sqrt{x}\, dx$$

$$\int_1^9 \ln \sqrt{x}\, dx = \frac{1}{2}\int_1^9 \ln x\, dx = \frac{1}{2}\left(x\ln x - x\right)\Big|_1^9$$
$$= \frac{1}{2}(9\ln 9 - 9 - (\ln 1 - 1)) = 9\ln 3 - 4,$$

where we have used that the antiderivative of $\ln x$ is $x\ln x - x$ (see Section 5.6, Example 2).

(b)

$$\int_0^2 \frac{x}{1+2x^2}\, dx$$

Using the substitution $u = 1 + 2x^2$ we get $du = 4xdx$, so that $xdx = \frac{1}{4}du$, and the limits of integration $x = 0$ and $x = 2$ correspond to $u = 1$ and $u = 9$. We get

$$\int_0^2 \frac{x}{1+2x^2}\, dx = \frac{1}{4}\int_1^9 \frac{du}{u}$$
$$= \frac{1}{4}\ln u\Big|_1^9 = \frac{1}{2}\ln 3.$$

(c)

$$\int_1^e \frac{(\ln x)^3}{x}\, dx$$

The substitution $u = \ln x$ gives $du = \frac{dx}{x}$ and limits of integration $x = 1$ and $x = e$ correspond to $u = 0$ and $u = 1$. We have

$$\int_1^e \frac{(\ln x)^3}{x}\, dx = \int_0^1 u^3 du = \frac{1}{4}\, u^4\Big|_0^1 = \frac{1}{4}.$$

(d)

$$\int_{-1}^{1} x^2 \sin(x^5)\, dx$$

The function $f(x) = x^2 \sin(x^5)$ is odd, $f(-x) = -f(x)$, so using the property of symmetric functions (see Section 5.5), we get

$$\int_{-1}^{1} x^2 \sin(x^5)\, dx = 0.$$

(e)

$$\int_{0}^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

Using the substitution $u = \sin^{-1} x$, we get $du = \dfrac{dx}{\sqrt{1-x^2}}$, and the limits of integration $x = 0$ and $x = 1/2$ correspond to $u = 0$ and $u = \pi/6$. We get

$$\int_{0}^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_{0}^{\pi/6} u\, du = \frac{1}{2}\, u^2\Big|_0^{\pi/6} = \frac{\pi^2}{72}.$$

(f)

$$\int_{1}^{4} \sqrt{t} \ln t\, dt$$

Here we use integration by parts with $u = \ln t$ and $dv = \sqrt{t}$. We have $du = \frac{dt}{t}$ and $v = \frac{2}{3}t^{3/2}$, so that

$$\begin{aligned}\int_{1}^{4} \sqrt{t} \ln t\, dt &= \int_{1}^{4} u\, dv = uv|_1^4 - \int_{1}^{4} vdu \\ &= \frac{2}{3}(16 \ln 2) - \frac{2}{3}\int_{1}^{4} \sqrt{t}\, dt \\ &= \frac{32}{3} \ln 2 - \frac{4}{9}\, t^{3/2}\Big|_1^4 = \frac{32}{3} \ln 2 - \frac{28}{9}.\end{aligned}$$

(g)

$$\int_{0}^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$$

We use the substitution $u = 1 + 2x$, so that $du = 2dx$ and the limits of integration $x = 0$ and $x = 13$ correspond to $u = 1$ and $u = 27$. We get

$$\int_{0}^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \frac{1}{2}\int_{1}^{27} u^{-\frac{2}{3}}\, du = \frac{1}{2}\, 3u^{\frac{1}{3}}\Big|_1^{27} = 3.$$

(h)

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$$

This is a trigonometric integral. Writing

$$\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x,$$

we recognize the substitution $u = \cos x$. We have $du = -\sin x dx$, and the limits of integration $x = 0$ and $x = \frac{\pi}{2}$ correspond to $u = 1$ and $u = 0$. We get

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = -\int_1^0 (1 - u^2) du = \int_0^1 (1 - u^2) du = \left(u - \frac{u^3}{3} \right) \Big|_0^1 = \frac{2}{3}.$$

2. Evaluate the following indefinite integrals:

(a)

$$\int x^2 e^x \, dx$$

This is Example 3 in Section 5.6.

(b)

$$\int \frac{2x^3 + 1}{x^2 + 1} \, dx$$

Doing long division, or simplifying as follows:

$$\frac{2x^3 + 1}{x^2 + 1} = \frac{(2x^3 + 2x) + 1 - 2x}{x^2 + 1} = 2x + \frac{1 - 2x}{x^2 + 1},$$

we get

$$\begin{aligned} \int \frac{2x^3 + 1}{x^2 + 1} \, dx &= \int \left(2x + \frac{1 - 2x}{x^2 + 1} \right) dx \\ &= x^2 + \int \frac{1}{x^2 + 1} \, dx - 2 \int \frac{x}{x^2 + 1} \, dx \\ &= x^2 + \tan^{-1} x - \ln(x^2 + 1) + C, \end{aligned}$$

where in the last integral we have used the substitution $u = x^2 + 1$.

(c)

$$\int \frac{\tan^{-1} x}{1 + x^2} \, dx$$

Using the substitution $u = \tan^{-1} x$, we get $du = \dfrac{dx}{1 + x^2}$, so that

$$\int \frac{\tan^{-1} x}{1 + x^2} \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\tan^{-1} x)^2 + C.$$

(d)

$$\int \sin^{-1} x \, dx$$

Using integration by parts with $u = \sin^{-1} x$ and $dv = dx$ we get

$$du = \frac{dx}{\sqrt{1-x^2}} \quad \text{and} \quad v = x,$$

so that

$$\begin{aligned} \int \sin^{-1} x \, dx = \int u \, dv = uv - \int v \, du \\ = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx. \end{aligned}$$

To compute the last integral, use the substitution $u = 1 - x^2$, so that $du = -2xdx$ and

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

Thus, finally,

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

(e)

$$\int \frac{x-1}{x^2+3x+2} \, dx$$

Using partial fractions,

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$

where A and B are such that

$$x - 1 = A(x+2) + B(x+1)$$

holds for all x. Setting $x = -1$, we get $A = -2$, and setting $x = -2$, we get $B = 3$. Thus

$$\begin{aligned} \int \frac{x-1}{x^2+3x+2} \, dx &= \int \left(-\frac{2}{x+1} + \frac{3}{x+2} \right) dx \\ &= -\log(x+1)^2 + \log|x+2|^3 + C. \end{aligned}$$

(f)

$$\int t^2 \cos(1-t^3) \, dt$$

Using the substitution $u = 1 - t^3$ we get $du = -3t^2 dt$ and

$$\int t^2 \cos(1-t^3)\, dt = -\frac{1}{3}\int \cos u\, du = -\frac{1}{3}\sin u + C = -\frac{1}{3}\sin(1-t^3) + C.$$

(g)

$$\int e^x \sqrt[3]{1+e^x}\, dx$$

Using the substitution $u = 1 + e^x$ we get $du = e^x dx$ and

$$\int e^x \sqrt[3]{1+e^x}\, dx = \int u^{\frac{1}{3}}\, du = \frac{3}{4}u^{\frac{4}{3}} + C = \frac{3}{4}(1+e^x)^{\frac{4}{3}} + C.$$

(h)

$$\int \cos^5 x\, dx$$

This is a trigonometric integral. Writing

$$\cos^5 x = \cos^4 \cos x = (1-\sin^2 x)^2 \cos x,$$

we recognize the substitution $u = \sin x$. We have $du = \cos x dx$ and

$$\int \cos^5 x\, dx = \int (1-\sin^2 x)^2 \cos x\, dx = \int (1-u^2)^2\, du = \int (1 - 2u^2 + u^4) du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

3. (a) Write a formula for $\cos^2 x$ in terms of $\sin^2 x$.

$$\cos^2 x = 1 - \sin^2 x.$$

(b) Evaluate

$$\int \cos^3 x \sin^2 x\, dx$$

$$\int \cos^3 x \sin^2 x\, dx = \int (1-\sin^2 x)\sin^2 x \cos x\, dx,$$

and using the substitution $u = \sin x$, we get $du = \cos x\, dx$ and

$$\int \cos^3 x \sin^2 x\, dx = \int u^2(1-u^2) du = \int (u^2 - u^4) du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

4. Let

$$f(x) = \int_2^{\sqrt{x}} \frac{\sin t}{t} dt + x^2$$

(a) Find $f'(x)$.
Using the chain rule with $u = \sqrt{x}$ and the Fundamental Theorem of Calculus, we get

$$\frac{df}{dx}(x) = \frac{d}{du}\left(\int_2^u \frac{\sin t}{t}\, dt\right) \frac{du}{dx}\bigg|_{u=\sqrt{x}} + 2x$$
$$= \frac{\sin\sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + 2x = \frac{\sin\sqrt{x}}{2x} + 2x.$$

(b) Evaluate $f(4)$.

$$f(4) = \int_2^{\sqrt{4}} \frac{\sin t}{t} dt + 4^2 = \int_2^2 \frac{\sin t}{t}\, dt + 16 = 16.$$

5. Find a function f and a number a such that for x,

$$1 + \int_a^x t f(t)\, dt = x^3$$

Setting in the equation $x = a$, we get $1 = a^3$, so that $a = 1$. Differentiating both sides of the equation with respect to x and using the Fundamental Theorem of Calculus, we get

$$x f(x) = 3x^2,$$

so that $f(x) = 3x$.

6. (a) Let

$$I = \int_0^4 e^{x^2} dx$$

For any value of n list the numbers L_n, R_n, M_n, T_n and I in increasing order.
The function $f(x) = e^{x^2}$ is increasing and concave upward on the real line (check it using the second derivative test). From the graph (sketch it!) we get

$$L_n < M_n < I < T_n < R_n.$$

(Here we used the analog of Fig. 5 on p. 419 (sketch it!) to get the relation $M_n < I < T_n$).

(b) Repeat part (a) for

$$I = \int_0^{\sqrt{2}/2} e^{-x^2} dx$$

The function $f(x) = e^{-x^2}$ is decreasing and concave downward when $0 \le x \le \sqrt{2}/2$ (check it using the second derivative test) . From the graph and analog of Fig. 5 (sketch them!) we get

$$R_n < T_n < I < M_n < L_n.$$

7. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a)

$$\int_0^\infty e^{-x}\,dx$$

The integral $\int_0^t e^{-x}\,dx$ exists for every number $t \ge 0$ and

$$\begin{aligned}\lim_{t\to\infty}\int_0^t e^{-x}\,dx &= \lim_{t\to\infty}\left(-e^{-x}\right)\Big|_0^t \\ &= \lim_{t\to\infty}(1 - e^{-t}) = 1.\end{aligned}$$

The improper integral of Type 1 is convergent and

$$\int_0^\infty e^{-x}\,dx = 1.$$

(b)

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

The integral $\int_t^1 \frac{1}{\sqrt{x}}\,dx$ exists for every number $t > 0$ and

$$\begin{aligned}\lim_{t\to 0}\int_t^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{t\to 0}\left(2\sqrt{x}\right)\Big|_t^1 \\ &= 2\lim_{t\to 0}(1 - \sqrt{t}) = 2.\end{aligned}$$

The improper integral of Type 2 is convergent and

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

(c)

$$\int_0^3 \frac{1}{x\sqrt{x}}\,dx$$

The integral $\int_t^3 \frac{1}{x\sqrt{x}}\,dx$ exists for every number $t>0$ and

$$\begin{aligned}\lim_{t\to 0}\int_t^3 \frac{1}{x\sqrt{x}}\,dx &= \lim_{t\to 0}\left(-2x^{-1/2}\right)\Big|_t^3\\ &= -\frac{2}{\sqrt{3}} + 2\lim_{t\to 0}\frac{1}{\sqrt{t}}.\end{aligned}$$

The last limit is ∞ — does not exist as a finite number, so that improper integral of Type 2 is divergent.

(d)

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

Since

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx,$$

we must evaluate both integrals separately. Using the substitution $u = x^2$ (one could also use $u = -x^2$), we get $du = 2xdx$ and

$$\begin{aligned}\int_0^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx &= \frac{1}{2}\lim_{t\to\infty}\int_0^{t^2} e^{-u}du\\ &= \frac{1}{2}\lim_{t\to\infty}\left(-e^{-u}\right)\Big|_0^{t^2}\\ &= \frac{1}{2}\lim_{t\to\infty}(1-e^{-t^2}) = \frac{1}{2}.\end{aligned}$$

Using this result and the fact that xe^{-x^2} is an odd function we get

$$\begin{aligned}\int_{-\infty}^{0} xe^{-x^2}\,dx &= \lim_{t\to -\infty}\int_t^0 xe^{-x^2}\,dx\\ &= -\lim_{-t\to\infty}\int_0^{-t} xe^{-x^2}\,dx = -\frac{1}{2}.\end{aligned}$$

Thus the improper integral of Type 1 is convergent and

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.$$

(e)

$$\int_0^1 \frac{1}{4y-1}\,dy$$

The integrand is $f(y) = \frac{1}{4y-1}$, and it is discontinuous (blows up) at $y = \frac{1}{4}$. Thus

$$\int_0^1 \frac{1}{4y-1}\,dy = \int_0^{\frac{1}{4}} \frac{1}{4y-1}\,dy + \int_{\frac{1}{4}}^1 \frac{1}{4y-1}\,dy,$$

and we need to investigate both improper integrals of Type 2. For the first integral we have, using the substitution $u = 4y - 1, du = 4dy$,

$$\int_0^{\frac{1}{4}} \frac{1}{4y-1}\,dy = \lim_{t\to\frac{1}{4}} \int_0^t \frac{1}{4y-1}\,dy = \frac{1}{4}\lim_{t\to\frac{1}{4}} \int_{-1}^{4t-1} \frac{1}{u}\,du$$
$$\frac{1}{4}\lim_{t\to\frac{1}{4}} \ln|u|\Big|_{-1}^{4t-1} = \frac{1}{4}\lim_{t\to\frac{1}{4}}(\ln|4t-1| - \ln|-1|)$$
$$= \frac{1}{4}\lim_{t\to\frac{1}{4}} \ln|4t-1| = -\infty,$$

since $\ln 0 = -\infty$. Thus the first improper integral is divergent, so that the integral in question is also divergent.

8. Find the area of the region bounded by the curves:

(a) $y = x^2$ and $y = x^4$.
The curves intersect at the points $x = -1, 0, 1$ and the top and bottom boundaries of the enclosed region are $y = x^2$ and $y = x^4$ (sketch the graph!). We have

$$A = \int_{-1}^1 (x^2 - x^4)dx = 2\int_0^1 (x^2 - x^4)dx$$
$$= 2\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{4}{15}.$$

(b) $x + y^2 = 2$ and $x + y = 0$.
The curves intersect at the points with coordinates $(-2, 2)$ and $(1, -1)$, and the top and bott om boundaries of the enclosed region are $x = 2 - y^2$ and $x = -y$, where we are using y as an independent variable (sketch the graph!). We have

$$A = \int_{-1}^2 (2 - y^2 - (-y))dy = \int_{-1}^2 (2 - y^2 + y)dy$$
$$= \left(2y - \frac{y^3}{3} + \frac{y^2}{2}\right)\Big|_{-1}^2 = 4\frac{1}{2}.$$

9. (a) Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y^2 = x$ about the x-axis.
The curves $y = x^2$ and $y = \sqrt{x}$ (we solved the second equation for y, which is assumed to be positive) intersect at $x = 0$ and $x = 1$. The region has the curve $y = \sqrt{x}$ as the top boundary and the curve $y = x^2$ as the bottom boundary (sketch the graph!). A cross-section is a washer with the inner radius x^2 and the outer radius $\sqrt{x}$. The cross-sectional area is $A(x) = \pi(x - x^4)$, and the volume of the solid of revolution is

$$\begin{aligned} V &= \int_0^1 A(x)dx = \pi \int_0^1 (x - x^4)dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Bigg|_0^1 = \frac{3\pi}{10}. \end{aligned}$$

(b) Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \sec x$, $y = 1$, $x = -1$ and $x = 1$ about the x-axis.
The region has the horizontal line $y = \sec x$ as the top boundary, the curve $y = 1$ as the bottom boundary, and the lines $x = -1$ and $x = 1$ as the vertical boundaries (sketch the graph!). A cross-section is a washer with the inner radius 1 and the outer radius $\sec x$. The cross-sectional area is $A(x) = \pi(\sec^2 x - 1)$, and the volume of the solid of revolution is

$$\begin{aligned} V &= \int_{-1}^1 A(x)dx = \pi \int_{-1}^1 (\sec^2 x - 1)dx = 2\pi \int_0^1 (\sec^2 x - 1)dx \\ &= 2\pi \left(\tan x - x\right)\big|_0^1 = 2\pi(\tan 1 - 1). \end{aligned}$$

10. Find the length of the following curves:
(a) $y = x^{3/2}$, $0 \le x \le 2$.

$$L = \int_0^2 \sqrt{1 + (y')^2}\, dx = \int_0^2 \sqrt{1 + \tfrac{9}{4}x}\, dx.$$

Using the substitution $u = 1 + \frac{9}{4}x$, we get

$$\begin{aligned} L &= \frac{4}{9} \int_1^{11/2} \sqrt{u}\, du \\ &= \frac{4}{9} \cdot \frac{2}{3}\, u^{3/2}\big|_1^{11/2} = \frac{8}{27} \left(\left(\frac{11}{2} \right)^{3/2} - 1 \right). \end{aligned}$$

(b)

$$y = \frac{x^2}{4} - \frac{\ln x}{2}, \quad 1 \le x \le 2$$

$$L = L = \int_1^2 \sqrt{1+(y')^2}\,dx = \int_1^2 \sqrt{1+\frac{1}{4}\left(x-\frac{1}{x}\right)^2}\,dx.$$

We have, by simple algebra,

$$1+\frac{1}{4}\left(x-\frac{1}{x}\right)^2 = 1+\frac{1}{4}(x^2-2+x^{-2}) = \frac{1}{4}(x^2+x^{-2})+\frac{1}{2}$$
$$= \frac{1}{4}(x^2+2+x^{-2}) = \frac{1}{4}(x+x^{-1})^2.$$

Thus

$$L = \int_1^2 \sqrt{\frac{1}{4}(x+x^{-1})^2}dx = \frac{1}{2}\int_1^2 (x+x^{-1})dx = \frac{1}{2}\left(\frac{x^2}{2}+\ln x\right)\Big|_1^2 = \frac{3}{4}+\frac{\ln 2}{2}.$$

11. Find the average value f_{ave} of f on the given interval.

(a) $f(x) = x\sin(x^2)$ on $[0, \sqrt{\pi}]$.
We get, using the substitution $u = x^2$, $du = 2xdx$,

$$f_{ave} = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} x\sin(x^2)dx = \frac{1}{2\sqrt{\pi}}\int_0^{\pi} \sin u\,du$$
$$= \frac{1}{2\sqrt{\pi}}\left(-\cos u\right)\Big|_0^{\pi} = \frac{1}{2\sqrt{\pi}}(-\cos\pi+\cos 0) = \frac{1}{\sqrt{\pi}}.$$

(b) $f(x) = 4 - x^2$ on $[0, 3]$.

$$f_{ave} = \frac{1}{3}\int_0^3 (4-x^2)\,dx = \frac{1}{3}\left(4x-\frac{x^3}{3}\right)\Big|_0^3$$
$$= \frac{1}{3}(12-9) = 1.$$

(c) For f as in part (b) find the number c in $[0, 3]$ such that $f(c) = f_{ave}$.
Solving $4-c^2 = 1$ we get $c = \sqrt{3}$ as the only solution which belongs to the interval $[0, 3]$.