# Practice Final Exam Solutions
# MAT 125 Spring 2008

Answer each question in the space provided and on the back of the sheets. Write full **solutions**, not just answers: unless otherwise marked, **answers without justification will get little or no partial credit**. Cross out anything the grader should ignore and circle or box the final answer. Do **NOT** round answers.

No books, notes, or calculators!

1. Compute the following limits by distinguishing between "$\lim f(x) = \infty$", "$\lim f(x) = -\infty$", and "limit does not exist even allowing for infinite values".

   (a) $\lim_{x\to -1} x^3 + 7x^2 - 1$

   *Solution*: Since any polynomial is continuous,

   $$\lim_{x\to -1} x^3 + 7x^2 - 1 = (-1)^3 + 7(-1)^2 - 1 = 5.$$

   (b) $\lim_{x\to\infty} x\tan\frac{1}{x}$

   *Solution*: This is expression of the type "$\infty \times 0$". Since $t = \frac{1}{x}$ goes to 0 as $x \to \infty$, we have

   $$\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{t\to 0}\frac{\tan t}{t},$$

   so that L'Hospital's rule is applicable. Namely, since $(\tan t)' = \sec^2 t$,

   $$\lim_{t\to 0}\frac{\tan t}{t} = \lim_{t\to 0}\frac{\sec^2 t}{1} = 1.$$

   Note: this problem can be also solved by writing

   $$x\tan\frac{1}{x} = \frac{\tan\frac{1}{x}}{\frac{1}{x}}$$

   and using L'Hospital's rule as $x \to \infty$.

   (c) $\lim_{x\to 5}\frac{x^2 - 4x - 5}{x - 5}$

   *Solution*: We can't just substitute $x = 5$, as it will give denominator zero. The numerator also becomes zero. However, factoring the numerator works:

   $$\lim_{x\to 5}\frac{x^2 - 4x - 5}{x - 5} = \lim_{x\to 5}\frac{(x+1)(x-5)}{x-5} = \lim_{x\to 5}(x+1) = 6.$$

   Note: this problem can also be solved by using L'Hospital's rule.

(d) $\lim_{x\to 0} x^4 \cos\frac{\pi}{x}$

*Solution*: Since $-1 \le \cos\frac{\pi}{x} \le 1$, we see that

$$-x^4 \le x^4 \cos\frac{\pi}{x} \le x^4$$

Since $\lim_{x\to 0} x^4 = 0$, by Squeeze Theorem, $\lim_{x\to 0} x^4 \cos\frac{\pi}{x} = 0$.

(e) $\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$

*Solution*: Using L'Hospital rule three times, we have

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3} = \lim_{x\to 0} \frac{e^x - 1 - x}{3x^2} = \lim_{x\to 0} \frac{e^x - 1}{6x} = \lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}.$$

(f) $\lim_{x\to\infty} \frac{x^3 + 1001x + 77}{x^3 - x^2 + 99}$

*Solution*:

$$\lim_{x\to\infty} \frac{x^3 + 1001x + 77}{x^3 - x^2 + 99} = \lim_{x\to\infty} \frac{1 + \frac{1001}{x^2} + \frac{77}{x^3}}{1 - \frac{1}{x^2} + \frac{99}{x^3}} = \frac{1}{1} = 1.$$

(g) $\lim_{x\to\pi/2} \frac{\cos x}{2x - \pi}$

*Solution*: Direct substituiton $x = \pi/2$ gives $\frac{0}{0}$ which is meaningless. Thus, we can use L'Hospital's rule, which gives

$$\lim_{x\to\pi/2} \frac{\cos x}{2x - \pi} = \lim_{x\to\pi/2} \frac{-\sin x}{2} = -\frac{1}{2}$$

(h) $\lim_{x\to\infty} (xe^{1/x} - x)$

*Solution*: We have, after introducing the variable $t = \frac{1}{x}$ and using L'Hospital rule,

$$\lim_{x\to\infty} (xe^{1/x} - x) = \lim_{x\to\infty} x(e^{1/x} - 1) = \lim_{t\to 0} \frac{e^t - 1}{t}$$
$$= \lim_{t\to 0} \frac{e^t}{1} = 1.$$

Note: this problem can be also solved by writing

$$\lim_{x\to\infty} (xe^{1/x} - x) = \lim_{x\to\infty} \frac{e^{1/x} - 1}{\frac{1}{x}}$$

and using L'Hospital's rule as $x \to \infty$.

2. Compute the derivatives of the following functions

   (a) $f(x) = x^3 - 12x^2 + x + 137\pi$
   *Solution*: $f'(x) = 3x^2 - 24x + 1$.

   (b) $f(x) = (2x+1)\sin x$
   *Solution*: $f'(x) = (2x+1)'\sin x + (2x+1)(\sin x)' = 2\sin x + (2x+1)\cos x$.

   (c) $g(s) = \sqrt{1+e^{2s}}$
   *Solution*: By chain rule, using $u = 1 + e^{2s}$:

$$\frac{dg}{ds} = \frac{dg}{du}\frac{du}{ds} = \frac{d(\sqrt{u})}{du}\frac{d(1+e^{2s})}{ds} = \frac{1}{2\sqrt{u}}2e^{2s} = \frac{e^{2s}}{\sqrt{1+e^{2s}}}$$

   (d) $h(t) = \dfrac{1+e^t}{1-e^t}$
   *Solution*: By quotient rule,

$$\begin{aligned} h'(t) &= \frac{(1+e^t)'(1-e^t) - (1+e^t)(1-e^t)'}{(1-e^t)^2} = \frac{e^t(1-e^t) - (1+e^t)(-e^t)}{(1-e^t)^2} \\ &= \frac{e^t - (e^t)^2 + e^t + (e^t)^2}{(1-e^t)^2} = \frac{2e^t}{(1-e^t)^2} \end{aligned}$$

   (e) $f(x) = (2x+2)^{100}$
   *Solution*: By chain rule,

$$f'(x) = 100(2x+2)^{99}(2x+2)' = 200(2x+2)^{99}$$

   (f) $g(x) = x^{\sin x}$
   *Solution*: We will use logarithmic derivative:

$$\begin{aligned} (\ln g(x))' &= (\ln x^{\sin x})' = (\sin x \ln x)' = (\sin x)' \ln x + (\sin x)(\ln x)' \\ &= \cos x \, \ln x + \sin x \, \frac{1}{x} \end{aligned}$$

   Thus, using $(\ln g)' = \dfrac{g'}{g}$, we get

$$g'(x) = g(x)(\ln g(x))' = x^{\sin x}\left(\cos x \, \ln x + \frac{\sin x}{x}\right).$$

3. Let $f(x) = xe^{-x^2}$.

   (a) Find asymptotes of $f(x)$ (hint: $f(x) = \frac{x}{e^{x^2}}$)
   *Solution*: This function is continuous everywhere, so there are no vertical asymptotes. To find horizontal asymptotes, we need to compute $\lim_{x\to\pm\infty} f(x)$. Writing $f(x) = \frac{x}{e^{x^2}}$, we see that as $x \to \infty$, both numerator and denominator have limit $\infty$. Thus, we can not use quotient rule (it would give $\frac{\infty}{\infty}$, which is meaningless); however, we can use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0$$

   since $\lim_{x\to\infty} 2xe^{x^2} = \infty$. Similar computation gives

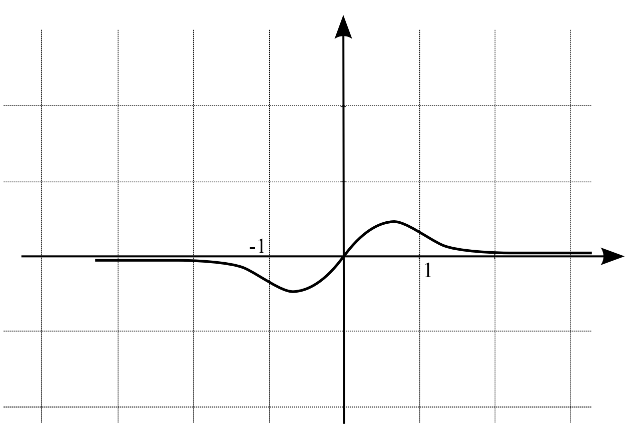
$$\lim_{x\to-\infty} f(x) = 0$$

   Thus, the horizontal asymptote is $y = 0$.

   (b) Compute the derivative of $f(x)$
   *Solution*: $f'(x) = (x)'e^{-x^2} + x(e^{-x^2})' = e^{-x^2} + x(-2xe^{-x^2}) = (1 - 2x^2)e^{-x^2}$

   (c) On which intervals is $f(x)$ increasing? decreasing?
   *Solution*: $f(x)$ is increasing when $f'(x) > 0$, i.e. $(1-2x^2)e^{-x^2} > 0$. Since $e^{-x^2} > 0$, it is equivalent to $1 - 2x^2 > 0$, i.e. $1 < 2x^2$, or $x^2 < 1/2$. Solutions of this last inequality are $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. So $f(x)$ is increasing on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
   Same argument shows that $f(x)$ decreasing on $(-\infty, -\frac{1}{\sqrt{2}})$ and on $(\frac{1}{\sqrt{2}}, \infty)$

   (d) Sketch a graph of $f(x)$ using the results of the previous parts and the fact that $f(0) = 0$.



4. Let $f(x) = -2x^3 + 6x^2 - 3$.

   (a) Compute $f'$, $f''$.
   *Solution*:

$$f'(x) = -6x^2 + 12x$$
$$f''(x) = -12x + 12$$

   (b) On which intervals is $f(x)$ increasing/decreasing?
   *Solution*: $f(x)$ is increasing when $f'(x) > 0$:

$$-6x^2 + 12x > 0$$
$$-6x(x-2) > 0$$

   Since the graph of $-6x^2 + 12x$ is a parabola witht he branches going down, this expression is positive between the roots, i.e. for $0 < x < 2$. Thus, $f'(x) > 0$ on the interval $(0, 2)$, and $f(x)$ is increasing on $(0, 2)$.

   Similar argument shows that $f'(x) < 0$ on $(-\infty, 0)$ and on $(2, \infty)$; thus, on these intervals $f(x)$ is decreasing.

   (c) On which intervals is $f(x)$ concave up/down?
   *Solution*: $f(x)$ is concave up when $f''(x) > 0$, i.e. $-12x + 12 > 0$, or $1 - x > 0$, $x < 1$. Threfore, $f(x)$ is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$.

   (d) Find all critical points of $f(x)$. Which of them are local maximums? local minimums? neither? Justify your answer.
   *Solution*: Critical points are where $f''(x) = 0$, i.e.

$$-6x^2 + 12x = 0$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$

   So the critical points are $x = 0$, $x = 2$.

   Since $f(x)$ is decreasing for $x < 0$ and increasing for $0 < x < 2$, by first derivative test, $x = 0$ is a local minimum. Similarly, since $f(x)$ is increasing for $0 < x < 2$ and decreasing for $x > 2$, $x = 2$ is a local maximum.

5. It is known that the polynomial $f(x) = x^3 - x - 1$ has a unique real root. Between which two whole numbers does this root lie? Justify your answer.

   *Solution*: Computing the values of $f(x)$ for several whole values of $x$, we get

   $f(-2) = -7$

   $f(-1) = -1$

   $f(0) = -1$

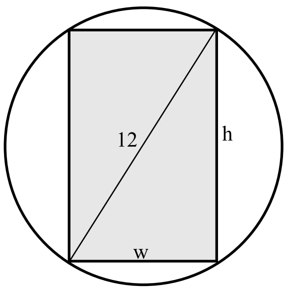
$f(1) = -1$

$f(2) = 5$

Thus, we see that $f(x)$ changes sign on the intrerval $[1, 2]$. SInce any polynomial is continuous, by Intermediate Value Theorem $f(x)$ must have a root somewhere on this interval. Thus, the root is between 1 and 2.

6. It is known that for a rectangular beam of fixed length, its strength is proportional to $w \cdot h^2$, where $w$ is the width and $h$ is the height of the beam's cross-section.

   Find the dimensions of the strongest beam that can be cut from a 12" diameter log (thus, the cross-section must be a rectangle with diagonal 12").

   *Solution*: The dimensions of the beam are width $w$ and height $h$. They must satisfy the conditions $h \geq 0$, $w \geq 0$. In addition, since the diagonal of the cross-section must be 12 inches, Pythagorean theorem gives $h^2 + w^2 = 12^2 = 144$. Thus, we need to find the maximum of the function $wh^2$, where $h, w$ are real numbers subject to the above conditions.

   Let us rewrite everything in terms of $w$. Then $h = \sqrt{144 - w^2}$; restrictions $h \geq 0$, $w \geq 0$ give $0 \leq w \leq 12$, and the strength is given by

   $$s(w) = w(\sqrt{144 - w^2})^2 = w(144 - w^2) = -w^3 + 144w$$

   So we need to find the maximum of this function on the interval $[0, 12]$.

   $f'(w) = -3w^2 + 144$, so critical points are when

   $$\begin{aligned} &-3w^2 + 144 = 0 \\ &144 = 3w^2 \\ &w^2 = 48 \\ &w = \pm\sqrt{48} = \pm\sqrt{16 \cdot 3} = \pm 4\sqrt{3} \end{aligned}$$

   Thus, on $[0, 12]$ there is a unique critical point, $w = 4\sqrt{3}$.

   To find the maximum, we compare the values of the function at the critical point and the endpoints:

   $$\begin{aligned} &f(0) = 0(144 - 0^2) = 0 \\ &f(12) = 12(144 - 12^2) = 0 \\ &f(4\sqrt{3}) = 4\sqrt{3}(144 - (4\sqrt{3})^2) = 4\sqrt{3}(144 - 48) = 4\sqrt{3} \cdot 96 \end{aligned}$$

Clearly, the largest value is $f(4\sqrt{3})$; thus, this is the maximum. So the best width is $4\sqrt{3}$, and the corresponding height is $h = \sqrt{144 - w^2} = \sqrt{96} = 4\sqrt{6}$.

7. The curve defined by the equation

$$y^2(y^2 - 4) = x^2(x^2 - 5)$$

is known as the "devil's curve". Use implicit differentiation to find the equation of the tangent line to the curve at the point $(0; -2)$.

*Solution*: Rewriting the equation in the form

$$y^4 - 4y^2 = x^4 - 5x^2$$

and taking derivative of both sides, we get $y'(4y^3 - 8y) = 4x^3 - 10x$, so

$$y' = \frac{4x^3 - 10x}{4y^3 - 8y}$$

Subsituting $x = 0, y = -2$, we get $y' = 0$, so the tangent line is horizaontal and the equation of the tangent line is $y = -2$.

8. Find the most general function $f(x)$ satisfying

(a) $f''(x) = \cos x$
*Solution*: We have $f'(x) = \sin x + C_1$, so that

$$f(x) = -\cos x + C_1 x + C_2,$$

where $C_1$ and $C_2$ are arbitrary constants.

(b) $f'(x) = \dfrac{x^2 + x + 1}{x}$
*Solution*: Since $f'(x) = x + 1 + \dfrac{1}{x}$,

$$f(x) = \frac{x^2}{2} + x + \ln|x| + C,$$

where $C$ is an arbitrary constant.