Practice Midterm II Solutions
MAT 125 Spring 2008

Answer each question in the space provided and write full **solutions**, not just answers. Unless otherwise marked, **answers without justification will get little or no partial credit**. Cross out anything the grader should ignore and circle or box the final answer. Do **NOT** round answers.

No books, notes, or calculators!

1. Explain (without using a graphing calculator!) why the equation
$$x^5 - 3x + 1 = 0$$
must have a solution with $0 < x < 1$.

   Solution: Function $f(x) = x^5 - 3x + 1$ is continuous and $f(0) = 1 > 0$, $f(1) = -1 < 0$. Thus by the Intermediate Value Theorem, there is x between 0 and 1, such that $f(x) = 0$.

2. Compute the following limits. Distinguish between "limit is equal to ∞", "limit is equal to $-\infty$" and "the limit doesn't exist even allowing for infinite values":

   (a) $\displaystyle\lim_{x\to\infty} \frac{x^3 + 2x + 1}{x^3 - 15x}$

   Solution:
$$\lim_{x\to\infty} \frac{x^3 + 2x + 1}{x^3 - 15x} = \lim_{x\to\infty} \frac{1 + \frac{2}{x^2} + \frac{1}{x^3}}{1 - \frac{15}{x^2}} = \frac{1}{1} = 1$$

   (b) $\displaystyle\lim_{x\to 2-} \frac{x^2 - 2x - 3}{x^2 - 5x + 6}$

   Solution:

   As $x \to 2-$,
$$x^2 - 2x - 3 \to 2^2 - 4 - 3 = -3$$
$$x^2 - 5x + 6 \to 4 - 10 + 6 = 0$$
   Thus, we have limit of the form $\frac{\text{negative number}}{0}$. This gives either ∞ or $-\infty$, depending on whether the denominator is positive or negative. To find this, we factor $x^2 - 5x + 6 = (x - 2)(x - 3)$, so as $x \to 2-$, $x - 3 \to -1$ and $x - 2 \to 0-$; thus, $x^2 - 5x + 6 \to 0+$. So we have limit of the form $\frac{\text{negative number}}{0+} = -\infty$

   (c) $\displaystyle\lim_{x\to 3+} \frac{x^2 - 2x - 3}{x^2 - 5x + 6}$

   Solution: Straightforward computation gives $\frac{0}{0}$ which is useless. We try to factor:
$$\frac{x^2 - 2x - 3}{x^2 - 5x + 6} = \frac{(x + 1)(x - 3)}{(x - 2)(x - 3)} = \frac{x + 1}{x - 2}$$

Thus,

$$\lim_{x\to 3+} \frac{x^2-2x-3}{x^2-5x+6} = \lim_{x\to 3+} \frac{x+1}{x-2} = \frac{4}{1} = 4$$

(d) $\lim\limits_{x\to\infty} \dfrac{1}{e^{(x^2)}+1}$

Solution: As $x \to \infty$, $x^2 \to \infty$, so $e^{x^2} \to \infty$. Thus, $\frac{1}{e^{x^2}+1} \to 0$.

3. Calculate derivatives of the following functions:

(a) $3(x+\sqrt{x})$

Solution:

$$(3(x+\sqrt{x}))' = 3(x+x^{1/2})' = 3(1+\tfrac{1}{2}x^{-1/2})$$

(b) $xe^x - 17x$

Solution:

$$\begin{aligned}(xe^x - 17x)' &= (xe^x)' - 17 = x'e^x + x(e^x)' - 17\\ &= e^x + x(e^x) - 17 = e^x(1+x) - 17\end{aligned}$$

(c) $\frac{2x}{x+1}$

Solution: By quotient rule,

$$\left(\frac{2x}{x+1}\right)' = \frac{(2x)'(x+1)-(x+1)'2x}{(x+1)^2} = \frac{2(x+1)-2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

(d) $\frac{1+\sqrt{x}}{1-\sqrt{x}}$

Solution: By quotient rule,

$$\begin{aligned}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)' &= \frac{(1+\sqrt{x})'(1-\sqrt{x})-(1+\sqrt{x})(1-\sqrt{x})'}{(1-\sqrt{x})^2}\\ &= \frac{\frac{1}{2\sqrt{x}}(1-\sqrt{x})+(1+\sqrt{x})\frac{1}{2\sqrt{x}}}{(1-\sqrt{x})^2} = \frac{1}{\sqrt{x}(1-\sqrt{x})^2}\end{aligned}$$

4. Let $f(x) = \left|2 - \frac{1}{x}\right|$.

(a) Sketch the graph of f and identify the asymptotes.

Solution: The graph is obtained by plotting the graph of $2 - \frac{1}{x}$ and then flipping the part of the graph which is below the x axis. Since $\lim_{x\to\infty} \left|2 - \frac{1}{x}\right| = \left|\lim_{x\to\infty}(2 - \frac{1}{x})\right| = |2| = 2$, and same for $x \to -\infty$, the horizontal asymptote is $y = 2$.

Vertical asymptote is $x = 0$: everywhere else this function is continuous (see part b), and as $x \to 0$, $2 - \frac{1}{x} \to \pm\infty$, so $\left|2 - \frac{1}{x}\right| \to \infty$.

(b) Find all values of x for which f is not continuous.

Solution: f is continuous everywhere where it is defined, i.e. everywhere except $x = 0$. So it is not continuos when $\boxed{x = 0}$.

(c) Find all values of x for which f is not differentiable (you do not have to calculate the derivative).

Solution: First of all, f is not differentiable for $x = 0$, since f is not defined at this point. Next, looking at the graph we see that it has a corner at $x = \frac{1}{2}$ (when $2 - \frac{1}{x} = 0$); thus, at this point f is not differentiable. It can be verified by direct computation of derivative as limit:

$$\begin{aligned} f'(0.5) &= \lim_{h\to 0} \frac{f(0.5+h) - f(0.5)}{h} \\ &= \lim_{h\to 0} \left(\left|2 - \frac{1}{0.5+h}\right| - \left|2 - \frac{1}{0.5}\right| \right) h^{-1} \\ &= \lim_{h\to 0} \left(\left|\frac{1+2h-1}{0.5+h}\right| - 0 \right) h^{-1} = \lim_{h\to 0} \left|\frac{2h}{0.5+h}\right| h^{-1} \end{aligned}$$

For $h > 0$ and small, $\frac{h}{0.5+h} > 0$, so $\left|\frac{h}{0.5+h}\right| = \frac{h}{0.5+h}$ and

$$\lim_{h\to 0^+} \left|\frac{h}{0.5+h}\right| h^{-1} = \lim_{h\to 0^+} \frac{2h}{0.5+h} h^{-1} = \frac{2}{0.5+0} = 4$$

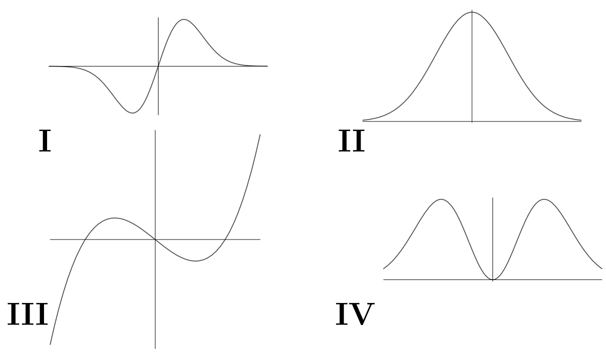
Similarly, for $h < 0$

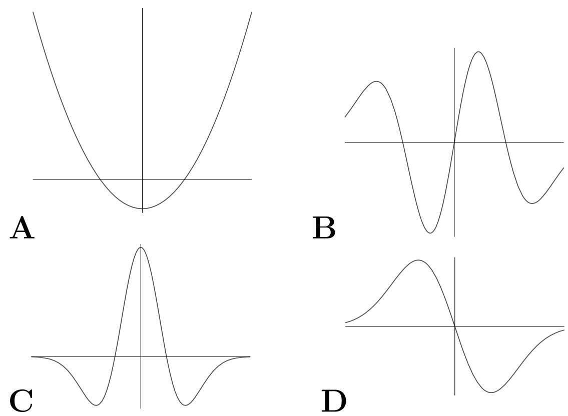
$$\lim_{h\to 0^-} \left|\frac{2h}{0.5+h}\right| h^{-1} = -\lim_{h\to 0^-} \frac{2h}{0.5+h} h^{-1} = -\frac{2}{0.5+0} = -4.$$

Since these limits do not match, $f'(0.5)$ does not exist.

Thus, the answer is $\boxed{x = 0 \text{ and } 0.5}$.

5. Match the graphs of functions **I–IV** below with the graphs of their derivatives **A–D**. (Justification is not required.)





Solution: Looking at graph I, we see that $f_I'(0) > 0$; the only function among A–D which satisfies this is C, so

I–C

Similarly, by looking at points where f increases/decreases/has derivative zero, we get the remaining ones:

II–D

III–A

IV–B

6. Let $f(x) = x^3 - 3x^2 - 9x + 7$.

   (a) Calculate f'

   Solution: $\boxed{f'(x) = 3x^2 - 6x - 9}$

   (b) Calculate f''

   Solution: $\boxed{f''(x) = 6x - 6}$

   (c) On which intervals does f increase? decrease?

   Solution: f increases when $f'(x) > 0$, which gives

$$\begin{gathered} 3x^2 - 6x - 9 > 0 \\ x^2 - 2x - 3 > 0 \\ (x-3)(x+1) > 0 \\ \boxed{x < -1 \text{ or } x > 3} \end{gathered}$$

   Similarly, f decreases when $f'(x) < 0$, which gives $\boxed{x \in (-1, 3)}$.

   (d) On which intervals is f concave up?

   Solution: f is concave up when $f''(x) > 0$, i.e. when $6x - 6 > 0, 6(x-1) > 0, \boxed{x > 1}$.

7. Find all tangent lines to the graph of $f(x) = 1/x$ which have slope $m = -1/4$; write equations of each of these tangent lines.

   Solution: The slope of tangent line to the graph of f at $x = a$ is $f'(a)$. Thus, to find for which a we have slope $-1/4$, we have to solve

$$\begin{aligned} f'(a) &= -1/4 \\ -\frac{1}{a^2} &= -1/4 \\ a^2 &= 4 \\ a &= \pm 2 \end{aligned}$$

   Now let us find the equation of the tangent line for each of these a, using general formula: $y = f(a) + f'(a)(x - a)$.

   For $a = 2$:

$$y = \frac{1}{2} + \left(-\frac{1}{4}\right) \cdot (x-2) = \frac{1}{2} + \frac{1}{2} - \frac{x}{4} = \boxed{1 - \frac{x}{4}}$$

   For $a = -2$:

$$y = \frac{1}{-2} + \left(-\frac{1}{4}\right) \cdot (x+2) = -\frac{1}{2} - \frac{1}{2} - \frac{x}{4} = \boxed{-1 - \frac{x}{4}}$$