MIDTERM I (MAT 127)

Instructions: Please do each of the following problems in the spaces provided. Show some work or give an explanation for each of your answers; answers for which no work is shown, or for which no explanation is given, will receive no credit. Please do not use any calculators during the exam.

(1) (20 points total) Which of the following differential equations does the function $y = xe^x$ solve.

(a) (5 points) $y' = y$.
Solution: $y' = e^x + xe^x \neq xe^x$. So xe^x is not a solution.

(b) (5 points) $y' = e^x$.
Solution: $y' = e^x + xe^x \neq e^x$. So xe^x is not a solution.

(c) (5 points) $y^{(2)} = 2y' - y$.
Solution:

$$y^{(2)} = (e^x + xe^x)' = 2e^x + xe^x$$

and

$$2y' - y = 2(e^x + xe^x) - xe^x = 2e^x + xe^x \quad .$$

So xe^x is a solution.

(d) (5 points) $y' = e^x + y$.
Solution: $y' = e^x + xe^x = e^x + y$. So xe^x is a solution.

(2) (15 points total) Consider the Initial Value Problem

$$y' = x + 2xy$$
$$y(0) = 1 \quad .$$

Use Euler's Method, with step size 1, to estimate the value $y(3)$.
Solution: We first estimate the value $y(1)$, then we estimate the value $y(2)$, and finally we estimate the value $y(3)$. Let $s_{a,b}$ denote the slope of the direction field for $y' = x + 2xy$ at the point (a, b) in the (x, y)-plane. Then:

$$y(1) \sim y(0) + s_{0,1} \times \triangle x \quad = \quad 1 + 0 \times 1 \quad = \quad 1$$

and

$$y(2) \sim 1 + s_{1,1} \times \triangle x \quad = \quad 1 + 3 \times 1 \quad = \quad 4$$

and

$$y(3) \sim 4 + s_{2,4} \times \triangle x \quad = \quad 4 + 18 \times 1 \quad = \quad 22 \quad .$$

(3) (30 points total) Indicate which of the following differential equations are seperable. Find all solutions to each of the seperable differential equations.

(a) (10 points) $y' = e^{2x-y}$.
Solution: $e^{2x-y} = e^{2x}e^{-y}$; so this is a seperable equation.

$$
\begin{aligned}
y' = e^{2x}e^{-y} \quad &\rightarrow \\
dy/dx = e^{2x}e^{-y} \quad &\rightarrow \\
e^{y}(dy/dx) = e^{2x} \quad &\rightarrow \\
e^{y}dy = e^{2x}dx \quad &\rightarrow \\
\int e^{y}dy = \int e^{2x}dx \quad &\rightarrow \\
e^{y} = e^{2x}/2 + c \quad &\rightarrow \\
y = ln(e^{2x}/2 + c) \quad &.
\end{aligned}
$$

(b) (10 points) $y' = x^2 y$.
Solution: This is seperable.

$$
\begin{aligned}
y' = x^2 y \quad &\rightarrow \\
dy/dx = x^2 y \quad &\rightarrow \\
y^{-1}dy = x^2 dx \quad &\rightarrow \\
\int y^{-1}dy = \int x^2 dx \quad &\rightarrow \\
ln(y) = x^3/3 + c \quad &\rightarrow \\
y = e^{x^3/3+c} \quad &.
\end{aligned}
$$

(c) (5 points) $y' = 2x + y$.
Solution: This is not seperable.

(d) (5 points) $y^3 = x^2 y$.
Solution: This is not seperable because it is not a first order differential equation.

(4) (20 points total) A bacteria culture grows at a rate proportional to its size.

(a) (5 points) Express the preceeding sentence as a differential equation.
Solution: $y' = ky$.

(b) (5 points) Solve the differential equation of part (a).

$$
\begin{aligned}
y' = ky \quad &\rightarrow \\
dy/dt = ky \quad &\rightarrow \\
y^{-1}dy/dt = k \quad &\rightarrow \\
y^{-1}dy = kdt \quad &\rightarrow \\
\int y^{-1}dy = \int kdt \quad &\rightarrow \\
ln(y) = kt + c \quad &\rightarrow
\end{aligned}
$$

$$y = e^{kt+c} = e^c e^{kt} = e^c (e^k)^t \quad .$$

(c) (10 points) Suppose that bacteria culture starts with are 500 bacteria, and after 1 hour there are 700 bacteria. Find a mathematical expression for the number of bacteria in the culture after 8 hours have elapsed.

Solution: Since 500 is the initial value, we have

$$500 = y(0) = e^c e^{k0} = e^c \qquad \rightarrow$$
$$e^c = 500 \quad .$$

Moreover we have that

$$700 = y(1) = 500e^k \qquad \rightarrow$$
$$700/500 = e^k \qquad \rightarrow$$
$$e^k = 7/5 \quad .$$

Substituting these values for e^c and e^k into part (b), we get that

$$y(t) = 500(7/5)^t \qquad \rightarrow$$
$$y(8) = 500(7/5)^8 \quad .$$

(5) (15 points total) A tank contains 500 L of pure water. A solution of water and salt (which contains .05 kg of salt per liter of solution) is added to the tank at the rate of 10L/min. The solution in the tank is kept thoroughly mixed, and is drained off at the rate of 10L/min.

Let y denotes the the number of kilograms of salt in the tank at any given time t (time is in minutes). Write down an Initial Value Problem which y must solve.

Solution: The differential equation is gotten as follows:

$$y' = (rate \quad in) - (rate \quad out) \qquad \rightarrow$$
$$y' = 10 \times .05 - 10 \times (y/500) \qquad \rightarrow$$
$$y' = .5 - y/50 = .5 - .02y \quad .$$

Since the initial valule is 0, the desired Initial Value Problem is

$$y' = .5 - .02y$$
$$y(0) = 0 \quad .$$