8.6

4.

$$f(x)=\frac{x}{1-x}=x\cdot\frac{1}{1-x}$$
$$=x\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}x^{n+1}=\sum_{n=1}^{\infty}x^n$$

The series is convergent if $|x|<1$. Therefore, the interval of convergence is (-1, 1).

6.

$$f(x)=\frac{1}{1+9x^2}=\frac{1}{1-(-9x^2)}=\sum_{n=0}^{\infty}(-9x^2)^n=\sum_{n=0}^{\infty}(-1)^n 9^n x^{2n}$$

The series is convergent if $|-9x^2|<1 \iff x^2<\frac{1}{9} \iff |x|<\frac{1}{3}$

Therefore, the interval of convergence is $-\frac{1}{3}<x<\frac{1}{3}$

8.

$$f(x)=\frac{1+x^2}{1-x^2}=-1+\frac{2}{1-x^2}=-1+2\sum_{n=0}^{\infty}x^{2n}$$

The series is convergent if $x^2<1 \iff |x|<1$. Therefore, the interval of convergence is (-1, 1).

10.

$$f(x)=\frac{x}{4x+1}=x\cdot\frac{1}{4x+1}=x\cdot\frac{1}{1-(-4x)}$$
$$=x\sum_{n=0}^{\infty}(-4x)^n=\sum_{n=0}^{\infty}(-4)^n x^{n+1}$$

$$Let\ \ a_n=(-4)^n x^{n+1},\quad then\ \ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{4^{n+1}x^{n+2}}{4^n x^{n+1}}\right|=4|x|$$

The series is convergent if $|4x| < 1$, Therefore, the interval of convergence is $(-\frac{1}{4}, \frac{1}{4})$

16.

$$f(x) = \arctan(\frac{x}{3})$$

$$= \int \frac{1}{1+\left(x/3\right)^2} \cdot \frac{1}{3} dx = \int \frac{1}{3} \cdot \frac{1}{1-(-)\left(x/3\right)^2} dx = \int \frac{1}{3} \sum_{n=0}^{\infty} \left(-\left(\frac{x}{3}\right)^2 \right)^n dx$$

$$= \int \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^{2n}} dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{3^{2n+1}} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)}$$

Let $a_n = \frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)}$ *then*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x^{2n+3}|}{3^{2n+3}(2n+3)} \cdot \frac{3^{2n+1}(2n+1)}{|x^{2n+1}|} = \lim_{n\to\infty} \frac{x^2(2n+1)}{9(2n+3)} = \frac{x^2}{9}$$

The series is convergent if $\left|\frac{x^2}{9}\right| < 1 \Leftrightarrow |x^2| < 9 \Leftrightarrow |x| < 3$. Therefore, the radius of convergence is 3.

22.

$$\int \frac{x}{1+x^5} dx = \int x \cdot \frac{1}{1-(-x^5)} dx$$

$$= \int x \sum_{n=0}^{\infty} (-x^5)^n dx = \int \sum_{n=0}^{\infty} (-1)^n x^{5n+1} dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^{5n+1} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+2}}{5n+2}$$

24.

$$\int \tan^{-1}(x^2)dx$$
$$= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{2n+1} dx$$
$$= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{4n+2}}{2n+1} dx$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$$

Remark.

$$\tan^{-1} y = \int \frac{1}{1+y^2} dy = \int \sum_{n=0}^{\infty} (-y^2)^n dy = \sum_{n=0}^{\infty} \int (-1)^n y^{2n} dy$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1} \quad ------(*)$$

If you put $y = x^2$ in (*), then you can get the equation $\tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{2n+1}$

26.

$$\int_0^{0.4} \ln(1+x^4)dx$$
$$= \int_0^{0.4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{4(n+1)}}{n+1} dx = \int_0^{0.4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{n+1} dx$$
$$= \sum_{n=0}^{\infty} \int_0^{0.4} (-1)^n \frac{x^{4n+4}}{n+1} dx$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(n+1)(4n+5)} \Bigg]_0^{0.4}$$
$$= \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \cdots \cdots \right]_0^{0.4}$$

If we stop adding after three terms, then the error is smaller than the absolute value of 4^{th} term (because the series is an alternating series).

$$\frac{0.4^{17}}{68} \approx 2.52645 \times 10^{-9}$$

So we have $\int_0^{0.4} \ln(1+x^4)dx \approx \frac{0.4^5}{5} - \frac{0.4^9}{18} + \frac{0.4^{13}}{39} \approx 0.0020336$