MAT 127 PRACTICE FINAL

(1) Consider the initial value problem

$$y'' - y' + 3y = 0$$
$$y(0) = 1, y'(0) = -1 \quad .$$

Assuming the solution to this initial value problem has is the power series

$$y = \sum_{n=0}^{\infty} c_n x^n \quad ,$$

find all the coeffiecients c_n for $n \leq 6$. **Solution:** We have that

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n \quad ,$$

$$-y' = \sum_{n=0}^{\infty} -(n+1)c_{n+1}x^n \quad ,$$

$$3y = \sum_{n=0}^{\infty} 3c_n x^n \quad .$$

If we add these series together (adding term by term) we must get the power series $\sum_{n=0}^{\infty} 0x^n$ (this is what the differential equation states). Thus, for all $n \geq 0$ we have that

$$(1) \quad (n+2)(n+1)c_{n+2} - (n+1)c_{n+1} + 3c_n = 0 \quad .$$

On the other hand, we deduce from $y(0) = 1$ and from $y'(0) = -1$ that

$$(2) \quad c_0 = 1$$

and

$$(3) \quad c_1 = -1 \quad .$$

To compute c_2 we use properties (1)-(3) (with $n = 0$ in (1)) to get

$$2c_2 - (-1) + 3 = 0$$

from which we can solve for

$$(4) \quad c_2 = -2 \quad .$$

To compute c_3 we use properties (1),(3),(4) (with $n = 1$ in (1)) to get

$$6c_3 - 2(-2) + 3(-1) = 0$$

from which we can solve for

$$(5) \quad c_3 = -1/6 \quad .$$

To compute c_4 we use properties (1)(4)(5) (with $n = 2$ in (1)) .....

(2) Use the seperation of variables technique to solve the initial value problem

$$y' = yln(x)$$
$$y(1) = 2 \quad .$$

Solution: We have

$$dy/dx = yln(x)$$
$$dy/y = ln(x)dx$$
$$\int dy/y = \int ln(x)dx$$
$$ln(y) = xln(x) - x + c$$
$$y = e^{xln(x)-x+c} = e^c x^x e^{-x} \quad .$$

Now we can use the initial value $y(1) = 2$ to solve for e^c as follows:

$$2 = y(1) = e^c 1^1 e^{-1} = e^c e^{-1}$$
$$e^c = 2e \quad .$$

Thus

$$y = 2ex^x e^{-x} \quad .$$

(3)

(a) Use Euler's Method with step size 1 to estimate the value $y(3)$, where y denotes the solution to the initial value problem

$$y' = y + x^2$$
$$y(0) = 1 \quad .$$

Solution: We have

$$y(0) = 1$$
$$y(1) \approx 1 + (1 + 0)1 = 2$$
$$y(2) \approx 2 + (2 + 1)1 = 5$$
$$y(3) \approx 5 + (5 + 4)1 = 14 \quad .$$

(b) Sketch the direction field for the differential equation given in part (a).

(4) Determine whether or not each of the following sequences $\{a_n\}$ converges. If the sequence converges, then compute the limit.

(a) $a_n = 2 + (-2/\pi)^n$
Solution: $limit_{n\to\infty} a_n = 2$

(b) $a_n = (n^3 - n + 2)/(n^2 - 3n^3)$
Solution: $a_n = (1-(1/n^2)+(2/n^3))/((1/n)-3)$, so $limit_{n\to\infty} a_n = -1/3$.

(c) $a_n = 3^n/n^4$
Solution: $limit_{n\to\infty} a_n = limit_{x\to\infty} 3^x/x^4$. This last limit can be computed by L'Hopital's rule (used 4 times) to be ∞. Thus there is no finite limit.

(d) $a_n = n^2/n!$
Solution: The limit exists and is equal to 0.

(5) Use any method to determine whether or not each of the following series $\sum_{n=1}^{\infty} a_n$ converges.

(a) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(1 + n^{-2})/n$
Solution: The series diverges (compare to the harmonic series).

(b) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(n^2 + n + 2)/(n - 8n^2)$ **Solution:** $limit_{n\to\infty} a_n = -1/8 \neq 0$, so the series does not converge.

(c) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n+1}(2 + cos(n))/n^2$
Solution: This series converges absolutely (compare it to the series $\sum_{n=0}^{\infty} 3/n^2$).

(d) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n(2 + e^{-n})/n$
Solution: This series is an alternating series with summands whose absolute values decrease to 0. Thus it converges.

(e) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^3/2^n$
Solution: This series converges by the ratio test.

(6) The differential equation

$$P' = 0.2P(1 - P/1000)$$

describes the change in population of wild Dachshunds over time.

(a) Find the equilibrium solutions for this differential equation.
Solution: The equilibrium solutions are $P = 0$ and $P = 1000$.

(b) Sketch the direction field for this differential equation; be sure to indicate the equilibrium solutions in your sketch.

(7) Consider the function $f(x) = 3x^{-2} + 2x - 1$.

(a) Compute the Taylor series for this function at the number 1.
Solution: The Taylor series at 1 is

$$1-4(x-1)+9(x-1)^2-12(x-1)^3+15(x-1)^4-18(x-1)^5+21(x-1)^6-...$$

(b) Find the radius of convergence for the Taylor series in part (a).
Solution: The radius of convergence is 1 (use the ratio test and the fact that the n'th term of the Taylor series is $(-1)^n 3(n+1)(x-1)^n$, if $n \geq 2$.) .

(c) Find the interval of convergence for the Taylor series in part (a).
Solution: $(0, 2)$.

(8) A bacteria culture grows with constant relative growth rate. At the outset there are 500 bacteria.

(a) If $y(t)$ denotes the number of bacteria present after t-hours, write down an initial value problem which y satisifies.
Solution: $y' = ky$ and $y(0) = 500$.

(b) If after 3 hours there are 2400 bacteria, then how many bacteria are there after 2 hours? **Solution:** We have that

$$2400 = y(3) = 500e^{3k}$$

from which we deduce that

$$k = (ln(24/5))/3 \quad .$$

Thus

$$y(2) = 500e^{ln(24/5)2/3} = 500((24/5)^{2/3}) \quad .$$