SOLUTIONS TO MIDTERM II; MAT 127

(1) (20 points total) After 3 days a sample of radon-222 decayed to 58% of its original amount.

(a) (10 points) What is the half-life of radon-222?
Solution: Let $y(t)$ denote the amount of radon-222 at time t. Then $y(t) = ce^{kt}$ where c denotes the amount at time $t = 0$.
The "half-life" is the the time t it takes to cut in half the original amount of radon-222; so the half-life t is the solution to the equation

$$(i) \qquad ce^{kt} = c/2 \quad .$$

To solve this equation divide both sides by c, then apply the natural log to both sides, and finally divide each side by k to get that

$$(ii) \qquad t = ln(1/2)/k \quad .$$

This gives the half life "in terms of k"; so to find the half life it remains to compute k. To solve for k we use the information tht the amount of radon-222 left after 3 days ($y(3) = ce^{3k}$) is 58% of of the initial amount (.58c). We thus have the equation

$$(iii) \qquad ce^{3k} = .58c \quad .$$

Dividing this equation by c, applying the natural log to each side, and then dividing by 3 gives

$$(iv) \qquad k = ln(.58)/3 \quad .$$

Now combining forulae (ii),(iv) gives the following forula for the half-life:

$$(v) \qquad t = 3ln(1/2)/ln(.58) \quad .$$

(b) (10 points) How long would it take the sample to decay to 10% of its original amount?
Solution: We are looking for the time t which satisfies

$$(vi) \qquad ce^{kt} = .1c \quad .$$

Divide each side of (vi) by c, then apply the natural log to each side and divide each side of the resulting equation by k to get

$$(vii) \qquad t = ln(.1)/k \quad .$$

Combining forulae (iv) and (vii) yields

$$(vi) \qquad t = 3ln(.1)/ln(5.8) \quad .$$

(2) (20 points total) The differential equation

$$P' = .08P(1 - P/500)$$

describes the growth of a population of rabbits in a fenced in area of the South Dakota prairie.

(a) (10 points) Find the equilibrium solutions for this differential equation.
Solution: $P = 0$ and $P = 500$ are the equilibrium solutions.

(b) (5 points) Explain (in common sense terms) what the equilibrium solutions tell us about the growth of this rabbit population.
Solution: If there are fewer than 500 rabbits (but more than 0 rabbits) then the number of rabbits will increase towards 500 but never actually reach 500; the rate of increase is becoming less and less as the number approaches 500. If there are more than 500 rabbits then the number will decrease towards 500, but never actully reach 500; the rate of decrease is becoming less and less as this number approaches 500.

(c) (5 points) What changes would need to be made to the preceeding differential equation if the habitat of our rabbit population suddenly experienced some abnormally dry weather? (Give a common sense explanation for each change you make to the the differential equation.)
Solution: The coefficient .08 would have to be made smaller; the capacity 500 would also have to be made smaller.

(3) (20 points total) Determine which of the following sequences $\{a_n\}$ converge and which diverge (in either case give reasons or show work). If a sequence converges then find its limit.

(a) (7 points) $a_n = (3 + 5n^2)/n + n^2$
Solution: By dividing both the top and bottom of this fraction by n^2 we get that

$$a_n = (5 + 3/n^2)/(1 + 1/n) \quad .$$

Thus we have that

$$lim_{n\to\infty} a_n = (lim_{n\to\infty}(5 + 3/n^2))/(lim_{n\to\infty}(1 + 1/n)) = 5/1 \quad .$$

(b) (7 points) $a_n = n^2/2^n$
Solution: We have that

$$lim_{n\to\infty} n^2/2^n = lim_{x\to\infty} x^2/2^x \quad .$$

This right hand limit is computed by using L'Hopital's rule twice:

$$lim_{x\to\infty} x^2/2^x = lim_{x\to\infty} = 2/(ln(2)^2 2^x) = 0 \quad .$$

(c) (6 points) $a_1 = 2$ and $a_n = 1 + a_{n-1}/2$ for all $n = 2, 3, 4, ...$
Solution: Note that $a_n = 2$ for all n; so this sequence converges to 2.

(4) (20 points total) Determine which of the following series converge and which diverge (in either case give reasons or show work).

(a) (7 points) $\sum_{n=1}^{\infty} -3(.01)^{n/2}$
Solution: We have that

$$\sum_{n=1}^{\infty} -3(.01)^{n/2} = -3(\sum_{n=1}^{\infty} r^n) \quad ,$$

where $r = .1$. Note that $\sum_{n=1}^{\infty} r^n$ is a geometric series which converges to $(r/(1-r)) = .1/.9 = 1/9$. Thus the given series converges to $-3(1/9) = -1/3$.

(b) (6 points) $\sum_{n=1}^{\infty} n^2/(8n^2+2)$
Solution: Note that

$$limit_{n\to\infty} n^2/(8n^2+2) = 1/8 \neq 0 \quad .$$

Thus this series must diverge.

(c) (7 points) $\sum_{n=1}^{\infty} 1/(2^n+8)$
Solution: Set $a_n = 1/(2^n+8)$ and set $b_n = 1/2^n$. Note that $0 < a_n < b_n$ holds for all $n = 1, 2, 3...$; thus by the comparison test we see that the series $\sum_{n=1}^{\infty} a_n$ converges if the series $\sum_{n=1}^{\infty} b_n$ converges. The seires $\sum_{n=1}^{\infty} 1/2^n$ is a geometric series with $r = 1/2$, and thus converges to $r/(1-r) = 1/2/(1-1/2) = 1$.

(5) (20 points total)

(a) (15 points) (Use the integral test to show the convergence of the series

$$\sum_{n=1}^{\infty} (2n-3n^2)/n^5 \quad .$$

Solution: We have that

$$\sum_{n=1}^{\infty} (2n-3n^2)/n^5 = 2(\sum_{n=1}^{\infty} a_n) - 3(\sum_{n=1}^{\infty} b_n$$

where $a_n = 1/n^4$ and $b_n = 1/n^3$. The a_n series converges iff the improper integral $\int_1^{\infty} 1/x^4 dx$ converges; and the b_n series converges iff the improper integral $\int_1^{\infty} 1/x^3 dx$ converges. Note that

$$\int_1^{\infty} 1/x^4 dx = lim_{t\to\infty}(-1/3x^3) \mid_1^t = 1/3$$

and

$$\int_1^\infty 1/x^3 ds = lim_{t\to\infty}(-/2x^2) \mid_1^t = 1/2 \quad .$$

(b) (5 points) How large must the positive integer m be in order for the m'th partial sum of the series in part (a) approximates the actual value of that series to within .01?

Solution: Using the notation of part (a), let S^a and S^a_m denote the actual sum and the m'th partial sum of the series $\sum_{n=1}^\infty a_n$; and let S^b and S^b_m denote the acutal sum and the m'th partial sum of the series $\sum_{n=1}^\infty b_n$. Then we have that

$$S^a - S^a_m < \int_m^\infty 1/x^4 dx = 1/3m^3$$

and

$$S^b - S^b_m < \int_m^\infty 1/x^3 dx = 1/2m^2 \quad .$$

Thus, if S and S_m denote the actual sum and the m'th partial sum of the series given in part (a) then we have that

$$\mid S - S_m \mid < 2/3m^3 + 3/2m^2 \quad .$$

So $\mid S - S_m \mid$ will be less than .01 if m is choosen so that

$$2/3m^3 + 3/2m^2 < .01 \quad .$$