PRACTICE MIDTERM FOR MAT 341

(1) Consider the function $f(x) = 2x - 1, \quad 0 < x < 2$.

(a) Compute the Fourier sine series for $f(x)$. At each $x \in [-4, 6]$ compute the value of this series.
Hint: The Fourier sine series looks like $\sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{2}x)$, where formulae for the coefficients b_n are given on page 70. Since the Fourier sine series for $f(x)$ is equal the usual Fourier series for the odd extension $f_o(x)$ of $f(x)$, the Theorem on page 76 may be applied to answer the convergence question.

(b) Compute the Fourier cosine series for $f(x)$. At each $x \in [-4, 6]$ compute the value of this series.
Hint: The Fourier cosine series looks like $\sum_{n=0}^{\infty} a_n cos(\frac{n\pi}{2}x)$, where formulae for the coefficients a_n are given on page 70. Since the Fourier cosine series for $f(x)$ is equal to the usual Fourier series for the even extension $f_e(x)$ of $f(x)$, the Theorem on page 76 may be applied to answer the convergence question.

(2) Set $f(x) = 2 + \sum_{n=1}^{\infty} \frac{cos(nx)}{n^4} + \sum_{n=1}^{\infty} \frac{sin(nx)}{n^2}$.

(a) Explain why $f(x)$ converges for each value of x.
Hint: Use Theorem 1 on page 82, and the fact that $\sum_{n=1}^{\infty} \frac{1}{n^q}$ converges if $q > 1$.

(b) Explain why $f(x)$ is a periodic function of period 2π.

(c) Explain why $f(x)$ is a continuous function.
Hint: Use Theorem 1 on page 82 and the remarks about uniform convergence given at the bottom of page 79; in class we proved that if a series of continuous functions converges uniformly on an interval (a, b) to a function $f(x)$ then $f(x)$ must be continuous.

(d) For each positive integer n find the value of the integral $\int_{-\pi}^{\pi} f(x) cos(nx) dx$.
Hint: You could use the series definition of for $f(x)$, and then integrate the corresponding series representation for $f(x)cos(nx)$ term by term (using Table 1 on page 61). Or you could note that this integral is equal to πa_n (when $n \geq 1$), where a_n is the coefficient of $cos(nx)$ in the Fourier series for $f(x)$. What is the value of a_n?

(3) Set $w(x,t) = sin(rx)e^{st}$.

(a) Show that $w(x,t)$ statisfies

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}$$

iff $-r^2 = \frac{s}{k}$.

(b) Show that $w(x,t)$ satisfies all of the following equalities

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}$$

$$w(0,t) = 0, w(a,t) = 0$$

iff $r = \frac{n\pi}{a}$ and $s = -\frac{kn^2\pi^2}{a^2}$.

(4) Find $u(x,t)$ which satisfies all the following equalities:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$

$$u(0,t) = 0, \quad u(\pi,t) = 1$$

$$u(x,0) = \frac{x}{\pi} + 3sin(2x) - sin(7x).$$

What is a physical situation which these equations describe?
Hint: $u(x,t) = w(x,t) + v(x)$ where $v(x)$ is the steady state solution. Note that $w(x,t) = \sum_{n=1}^{\infty} b_n sin(nx)e^{-2n^2t}$, where the coefficients b_n must be choosen so that the initial condition $w(x,0) = u(x,0) - v(x)$ holds. Physical conditions represented by these equations are discussed on pages 135-139 (see in particular 138-139).

(5) Consider the equations

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(0,t) = T_0, \quad -\kappa\frac{\partial u}{\partial x}(a,t) = (u(a,t) - T_1)h$$

where κ, h are positive constants and T_0, T_1 are constants.

(a) What is a physical situation that these equations describe?
Hint: See pages 135-139.

(b) Find the "steady state" solution to these equations.
Hint: $v(x) = \alpha x + \beta$. The first boundary condition becomes $v(0) = T_0$; thus $\beta = T_0$. The second boundary condition becomes $-\kappa v'(a) = (v(a) - T_1)h$; thus $-\kappa\alpha = (\alpha a + T_0 - T_1)h$, from which is deduced $\alpha = \frac{(T_0 - T_1)h}{-\kappa - ah}$.

(c) Why is the "steady state" solution important – both physically and mathematically?
Hint: If the problem has a steady state solution $v(x)$ then its physical importance is $limit_{t\to\infty} u(x,t) = v(x)$. Its mathematical importance is that $w(x,t) = u(x,t) - v(x)$ satisfies homogeneous boundary conditions.