REVIEW FOR FINAL EXAM; MAT 312 (SPRING, 08)

(1) Let G denote a finite group. Prove that each element $g \in G$ appears exacty once in every row and in every column of the multiplication table for G.

(2)

(a) Show that $S(3)$ is not a cyclic group.
(b) Show that $A(3)$ is a cyclic group.

(3) Let $\sigma \in S(7)$ denote the permutation $\sigma = (1, 2, 3)(7, 5, 3)(2, 5, 4)$ and let H denote the cyclic subgroup of $S(7)$ generated by σ. Compute $\mid H \mid$.

(4) Let G, H denote two groups.

(a) Show that the direct product groups $G \times H$ and $H \times G$ are isomorphic.
(b) Show that the direct product group $G \times H$ is abelian iff both G and H are abelian.

(5) Show that the direct product group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the symmetry group of a (non-square) rectangle.

(6) Show that the direct product group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ iff $(m, n) = 1$. (Hint: if $(m, n) = 1$ then show that $([1]_m, [1]_n)$ generates $\mathbb{Z}_m \times \mathbb{Z}_n$.)

(7) Show that no two of the following groups are isomorphic: $(\mathbb{Z}_2)^3$, $\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_8$.

(8) Let G denote a finite group with order equal n. Show that G is isomorphic to a subgroup of $S(n)$. (Hint: Note that the permutations of the set G may be identitfied with $S(n)$. Now, for each $g \in G$, consider the map $L_g : G \longrightarrow G$ which sends each $x \in G$ to gx — i.e. left multiplication by g.)

(9) Let $H \subset G$ denote a subgroup of the group G. Show that, for any $g \in G$, left multiplication by g — denoted by $L_g : G \longrightarrow G$ — permutes the left cosets of H.

(10)) Let G denote a group and let $a, b \in G$.

(a) Suppose that a, b "commute", i.e. $ab = ba$; and also suppose that the $order(a), order(b)$ are relatively prime. Then show that $order(ab) = order(a)order(b)$.
(b) Give an example of G and $a, b \in G$ (with order(a) and order(b) relatively prime)such that $order(ab) \neq order(a)order(b)$. (Note: in your example a, b will note commute).

(11) Let G denote a finite group such that $g^2 = e$ holds for all $g \in G$. Show that G is isomorphic to $(\mathbb{Z}_2)^n$ for some positive integer n. (Hint: By theorem 5.3.4 G is abelian. So we may us "+" to denote the group operation and let "0" denote the group identity; then $g^2 = e$ becomes $2g = 0$. Show that G is a finite dimensional vector space over the field $\mathbb{Z}_2$.

(12) #2,3,4 on page 252-253.