PRACTICE FINAL EXAM FOR MAT 312 AND AMS 351

(1)
- (a) Use the Euclidean algorithm to compute $gcd(a, b)$ for the integers $a = 124$ and $b = 38$.
- (b) Find integers c, d such that $gcd(a, b) = ca + db$.
- (b) Is 38 invertible in Z_{124}?
- (c) Is the ring Z_{124} a field?

(2)
- (a) Use the Euclidean algorithm to compute $gcd(a(x), b(x))$ for the polynomials $a(x) = x^5 + 4x^2 - x + 4$ and $b(x) = x^6 - x^2$ in the polynomial ring $\mathbb{R}[x]$ — where $\mathbb{R}$=real numbers.
- (b) Is $a(x)$ invertible in the quotient ring $\mathbb{R}[x]/b(x)$?
- (c) Is the quotient ring $\mathbb{R}[x]/b(x)$ a field?

(3) Solve the equation $[20]_n[x]_n = [1]_n$, for $n = 263$. (This is a congruence problem mod n.)

(4) Let G denote a group satisfying $\mid G \mid= 23$, i.e. G contains 23 members. Explain why G must be an abelian group.

(5) Let G denote a group satisfying $\mid G \mid= 24$, and let $g \in G$. Explain why $g^{73} = g$.
(6) State Lagrange's Theorem. State Burnside's Theorem.

(7) Consider the permutations $\sigma = (1, 3, 5)$ and $\tau = (4, 2)$ in the symmetric group on 5 letters S_5; and let G denote the smallest subgroup of S_5 which contains both σ and τ.
- (a) Verify that G is isomorphic to the cyclic group Z_6.
- (b) How many left cosets of G are there in S_5.
- (c) List any 2 of the left cosets of G in S_5.

(8)

(a) Find a third degree polynomial $p(x)$ in $Z_5[x]$ such that the quotient ring $Z_5[x]/p(x)$ is a field. (In what follows we denote this quotient ring by K.)
(b) Explain how Z_5 may regarded as a subset of K; thus $p(x) \in K[x]$.
(c) Verify that $p(x)$ is not a prime polynomial in $K[x]$.

(9) Let $G \subset S_8$ be any subgroup of the symmetric group on 8 letters S_8. Define a relation $\sim$ on the set $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ by setting $x \sim y$ iff there is $g \in G$ such that $g(x) = y$.

(a) Verify that $\sim$ is an equivalence relation on X (called *G-equivalence* on X).
(b) For each $x \in X$ and $g \in G$ define G_x (the *stabilizer of* x) and X_g (the *fixed point set* of g). Show that G_x is a subgroup of G.
(c) Show that $x \in X_g$ iff $g \in G_x$.
(d) Let g denote the permutation represented by the 2×8 matrix

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 5 & 2 & 1 & 7 & 6 & 8 \end{matrix}$$

and let $G =< g >$. Then list all the distinct G-equivalences classes $[x]$. Also for each $x \in X$ compute G_x and for each $g \in G$ compute X_g.
(e) Let F denote the set of all functions $X \longrightarrow Z_2$ from X to the binary numbers Z_2. Explain how any subgroup $G \subset S_8$ is also a subgroup of the permutation group of the set F. For the specific G given in part (d) compute how many G-equivalences classes there are in F.

(10) Let $Q[x]$ denote the ring of polynomials over the rational numbers Q, and consider the polynomials $p(x) = x^3 - x^2 + 3x - 1$, $a(x) = x^5 - x^3 + 2x$, $b(x) = -x^7 + x^6 - 2$ in $Q[x]$.

(a) Do $a(x)$ and $b(x)$ represent the same element in the quotient ring $Q[x]/p(x)$?
(b) Find polynomials $\alpha(x)$ and $\beta(x)$ in $Q[x]$, both having degree ≤ 2, such that $\alpha(x)$ and $a(x)$ represent the same element in the quotient ring $Q[x]/p(x)$; and such that $\beta(x)$ and $b(x)$ also represent the same element in $Q[x]/p(x)$.