MIDTERM: MAT 310,FALL 2010

Instructions: First fill in your name, ID number and and circle your recitation section directly below. Then complete each of the following six problems in the spaces provided. Be sure to support your answers (to problems #1,2,3,5,6) by showing your work and/or by giving a reasoned explanation.

Print Name:

ID Number:

Recitation section: R01-Tues 3:50-4:45pm; R02-Mond 9:35-10:30am

(1) (15 points) Let $T : V \longrightarrow W$ denote a linear transformation between the two vector spaces V, W over the field F. A theorem in the text book states that T *is one to one iff the null space of* T *is equal to the zero vector of* V. Prove this theorem from basics.

Solution: Assume that T is one to one: this means that if $T(\mathbf{v}) = T(\mathbf{w})$ then $\mathbf{v} = \mathbf{w}$. Suppose that $\mathbf{v} \in N(T)$, then $T(\mathbf{v}) = \mathbf{0}$. Note that $\mathbf{0} \in N(T)$, so $T(\mathbf{0}) = \mathbf{0}$. Since $T(\mathbf{v}) = \mathbf{v} = T(\mathbf{0})$, it follows (from T being one to one) that $\mathbf{v} = \mathbf{0}$; thus $N(T) = \{\mathbf{0}\}$.

Next assume that $N(T) = \mathbf{0}$. If $T(\mathbf{v}) = T(\mathbf{w})$ then we have the following implications:

$$\begin{aligned} T(\mathbf{v}) = T(\mathbf{w}) \quad &\Rightarrow T(\mathbf{v}) - T(\mathbf{w}) = \mathbf{0} \quad \Rightarrow \\ T(\mathbf{v} - \mathbf{w}) = \mathbf{0} \quad &\Rightarrow \mathbf{v} - \mathbf{w} \in N(T) \quad \Rightarrow \\ \mathbf{v} - \mathbf{w} = \mathbf{0} \quad &\Rightarrow \mathbf{v} = \mathbf{w} \end{aligned}$$

This shows that T is one to one.

(2) Let V denote a 4-dimensional subspace of the vector subspace of $P_6(\mathbb{Q})$, where $\mathbb{Q}$ denotes the rational numbers.

(a) (10 points) Explain why there is another subspace $W \subset P_6(\mathbb{Q})$ such that $V \oplus W = P_6(\mathbb{Q})$; i.e. given V, explain how you would obtain W.
Solution: Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ denote a basis for V. Since $dim(P_6(\mathbb{Q}) = 7$, a theorem in the text states that the independent set α extends to a basis $\beta = \alpha \cup \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for all of $P_6(\mathbb{R})$. Set $W = span(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(b) (7 points) Compute $dim(W)$.
Solution: W has the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$; so $dim(W) = 3$.

(3) (15 points) Let C^∞ denote the vector space (over the complex numbers) of infinitely differentiable complex valued functions $x(t)$ defined on the real line (here t denotes the real variable). Find a basis for the null space for the linear transformation $T : C^\infty \longrightarrow C^\infty$ which is defined by

$$T(x(t)) = \frac{d^5}{dt^5}x(t) + 2\frac{d^4}{dt^4}x(t) + 2\frac{d^3}{dt^3}x(t).$$

Solution: Note that $T(x(t)) = p(D)(x(t))$, where the polynomial $p \in P(\mathbb{C})$ is defined by $p(u) = u^5 + 2u^4 + 2u^3$. p factors as

$$p(u) = (u + 0)^3(u + 1 + i)(u + 1 - i).$$

By a theorem in the text the solution space for the homogeneous linear differential equation $P(D)(x(t)) = \mathbf{0}$ has the following functions for basis: $\{1, t, t^2, e^{(-1-i)t}, e^{(-1+i)t}\}$.

(4) Determine whether each of the following statements is true or false.

(a) (3 points) Let $A, B \in M_{3\times 4}(F)$ denote two matrices in reduced row echelon form (F=field). If $rank(A) = rank(B)$ then $A = B$.
Solution: False. (Everyone gets 3 points for this one, since — as a student pointed out to me in the test — it covers material covered in sections beyond 3.2 in the text.)

(b) (3 points) Let V denote a vector space (corrected during test to read: "Let V denote a non-zero vector space") over the field F and suppose that V has a finite generating (spanning) set. Then V is isomorphic to F^n for some positive integer n.
Solution: True

(c) (3 points) For any $B \in M_{2\times 2}(F)$ and any diagonal matrix $A \in M_{2\times 2}(F)$, we must have AB=BA. (F=field.)
Solution: False.

(d) (3 points) For any $A, B \in M_{n\times n}(F)$ if AB is invertible then A is also invertible. (F=field.)
Solution: True.

(e) (3 points) Let $T : V \longrightarrow V$ denote any linear operator on the 3-dimensional vector space V. Then $R(T) \subset R(T^2)$.
Solution: False

(f) (3 points) If $A, B \in M_{n\times n}(\mathbb{R})$ have the same rank then A is similar to B.
Solution: False

(g) (3 points) If the matrix $A \in M_{n\times n}(F)$ has rank n then the linear transformation $L_A : F^n \longrightarrow F^n$ is an isomorphism.
True: True

(5) (15 points) Let A denote the following 3×3 matrix over the real numbers:

$$\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix}$$

Write A^{-1} as a finite product of elementary matrices.
Solution: The identity can be obtained from A by applying the following sequence of elementary operations e_1, e_2, e_3: e_1 adds $-2r_2$ to r_1; e_2 adds r_3 to r_1; e_3 adds $-2r_3$ to r_2. It follows that $A^{-1} = E_3E_2E_1$ where E_1, E_2, E_3 are the elemetary matrices

$$\begin{matrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{matrix}$$

(6) Consider the linear transformation $T : P_3(\mathbb{R}) \longrightarrow M_{1\times 4}(\mathbb{R})$ defined by $T(p(x)) = (\int_0^1 p(x)dx, p(0), p(1), \frac{d^3}{dx^3}p(x))$.

(a) (9 points) Compute the matrix $[T]_\alpha^\beta$ for T, where α denotes the ordered basis $\{x^2, x-1, x^3+x+1, x^2+x^3\}$ for $P_3(\mathbb{R})$ and where β denotes the standard basis for $M_{1\times 4}(\mathbb{R})$.
Solution: We have the following computations:

$$T(x^2) = (\frac{1}{3}, 0, 1, 0)$$
$$T(x-1) = (-\frac{1}{2}, -1, 0, 0)$$
$$T(x^3+x+1) = (\frac{7}{4}, 1, 3, 6)$$
$$T(x^2+x^3) = (\frac{7}{12}, 0, 2, 6)$$

Thus the matix $[T]_\alpha^\beta$ is equal to the 4×4 matrix

$$\begin{matrix} \frac{1}{3} & -\frac{1}{2} & \frac{7}{4} & \frac{7}{12} \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 6 & 6 \end{matrix}$$

(b) (8 points) Is T an isomorphism?
Solution: Yes T is an isomorphism. To see this recall that T being an isomorphism simply means that its inverse T^{-1} exits. Moreover T^{-1} exists iff $[T]_\alpha^\beta$ is invertible; and $[T]_\alpha^\beta$ is invertible iff its rank is equal to 4. To verify that rank$([T]_\alpha^\beta) = 4$ perform some row operations on it until the resulting matrix B obviously has 4 independent rows (or verify that its reduced row echelon form is the 4×4 identity matrix.)