MAE 501 IN CLASS EXPONENTIAL FUNCTIONS: IN CLASS THURSDAY/TUESDAY, OCTOBER 20/25

Let f denote a function with the following defining properties:

(a) For all x, $y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$.

(b) f is defined and continuous for all $x \in \mathbb{R}$.

(c) f is not constant. That is, there exist two real numbers a_1 and a_2 with $f(a_1) = f(a_2)$

(d) f is differentiable at zero.

(1) Prove that $f(0) = 1$.

(2) Prove that, for all x, $f(x)f(-x) = 1$.

(3) Prove that, for all x, $f(x) \neq 0$.

(4) Prove that, for all x, $f(x) > 0$.

(5) Prove that, for al x, $f(-x) = \frac{1}{f(x)}$.

(6) Prove that, for all x, $(f(x))^2 = f(2x)$.

(7) Use mathematical induction to prove that, for all x, for every natural number n, $(f(x))^n = f(nx)$.

(8) Prove that, for all x and for all natural numbers n, $(f(x))^{1/n} = f(x/n)$.

(9) Prove that, for all x, for every rational number r, $(f(x))^r = f(rx)$.

This project was extracted/adapted from course notes for MAE 301/501 by Bernie Maskit.

(10) Use the definition of the derivative to prove that, since f is differentiable at 0, it must be differentiable for every $x \in \mathbb{R}$. Further, show that $f'(x) = f'(0)f(x)$.

(11) Prove that, since $f(x)$ is not constant, $f'(0) \neq 0$.

(12) Prove that f has derivatives of every order at every point.

(13) Prove that, if $f'(0) > 0$, then f is increasing at every x; if $f'(0) < 0$, then f is decreasing at every x.

Set $a := f(1)$. We know that $f(0) = 1$, and we know that $f(x)$ is either increasing or decreasing. It follows that either:

$$0 < f(1) < 1,$$

or

$$f(1) > 1.$$

(14) Show that, if $f'(0) > 0$, then $a > 1$, and if $f'(0) < 0$, then $a < 1$.

(15) Show that, for every rational number r, $f(r) = a^r$.

So far you proved that, for every rational input r, our function may be defined by $f(r) = a^r$, where $a = f(1)$.

We now specify that, for our function f, $f'(0) = 1$, and we set $e := f(1)$.

Then for every $x \in \mathbb{R}$, we define the function e^x to be the solution (why can we do this?) to the initial value problem:

$$f'(x) = f(x)$$

and

$$f(0) = 1.$$

Can you show that this function, defined as a solution to the differential equation, indeed satisfies the property:

$$e^{(x+y)} = e^x e^y,$$

for all real numbers x and y?

Why does the function e^x satisfy the other three properties?

(16) Since e^x has derivatives of every order, we can write down its Taylor series at the origin:

$$\sum_{n=0}^{\infty} a_n x^n.$$

We have $f(0) = 1$, so $e^0 = a_0 = 1$.

- Differentiate term by term to show that
$$a_n = \frac{a_{n-1}}{n}.$$
- Show that
$$a_n = \frac{1}{n!}$$
for all n.

(17) Show that the Taylor series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

converges for all real numbers x, so that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$