Lectures on Tensor Categories and Modular Functor

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Introduction

The goal of this book is to give a comprehensive exposition of the relations among the following three topics: tensor categories (such as a category of representations of a quantum group), 3-dimensional topological quantum field theory (which, as will be explained below, includes invariants of links), and 2-dimensional modular functor (which arises in 2-dimensional conformal field theory).

The idea that these subjects are somehow related first appeared in physics literature, in the study of quantum field theory. The pioneering works of Witten [**W1, W2**] and Moore and Seiberg [**MS1**] triggered a real avalanche of papers, both physical and mathematical, exploring various aspects of these relations. Among the more important milestones we should name the papers of Segal [**S**], Reshetikhin–Turaev [**RT1, RT2**], Tsuchiya–Ueno–Yamada [**TUY**], Drinfeld [**Dr3, Dr4**], Beilinson–Feigin–Mazur [**BFM**], and many others.

By late 1990s it had become a commonplace that these topics are closely related. However, when the second author decided to teach (and the first author to take) a class on this topic at MIT in the Spring of 1997, they realized that finding precise statements in the existing literature was not easy, and there were some gaps to be filled. Moreover, the only work giving a good exposition of all these notions was Turaev's book [**T**], which unfortunately didn't cover some important (from our point of view) topics, such as complex-analytic approach to modular functor, based on connections on the moduli spaces. Another excellent reference was the manuscript [**BFM**], which unfortunately is still unfinished, and it is not known when (and if) it will be published. Thus, it was natural that after the course was completed, we decided to turn it into a book which would provide a comprehensive exposition. Needless to say, this book is almost completely expository, and contains no new results — our only contribution was putting known results together, filling the gaps, and sometimes simplifying the proofs.

To give the reader an idea of what kind of relations we are talking about, we give a quick introduction. Let us first introduce the main objects of our study:

Tensor categories: These are abelian categories with associative tensor product, unit object, and some additional properties, such as rigidity (existence of duals). We will be interested only in semisimple categories, such as a category of complex representations of a compact group. However, we weaken the commutativity condition: namely, we require existence of functorial isomorphism $\sigma_{VW}\colon V\otimes W \to W\otimes V$, but — unlike the classical representation theory — we do not require $\sigma^2 = \mathrm{id}$. The best known example of such a category is a category of representations of a quantum group; however, there are also other examples.

We will also need special class of tensor categories which are called *modular tensor categories* (MTC); these are semisimple tensor categories with finite number of simple objects and certain non-degeneracy properties. The main example of such categories is provided by a suitable semisimple quotient of the category of representation of a quantum group at a root of unity.

3-dimensional topological quantum field theory (3D TQFT): Despite its physical name, this is a completely mathematical object (to such an extent that some physicists question whether it has any physical meaning at all). A simple definition is that a TQFT is a rule which assigns to every 2-dimensional manifold N a finite-dimensional vector space $\tau(N)$, and to every cobordism — i.e. a 3-manifold M such that its boundary ∂M is written as $\partial M = \overline{N}_1 \sqcup N_2$ — a linear operator $\tau(N_1) \to \tau(N_2)$ (here $\overline{N}$ is N with reversed orientation). In particular, this should give a linear operator $\tau(M)\colon \mathbb{C} \to \mathbb{C}$, i.e., a complex number, for every closed 3-manifold M.

We will, however, need a somewhat more general definition. Namely, we will allow 2-manifolds to have marked points with some additional data assigned to them, and 3-manifolds to have framed tangles inside, which should end at the marked points. In particular, taking a 3-sphere with a link in it, we see that every such extended 3D TQFT defines invariants of links.

2-dimensional modular functor (2D MF): topological definition: By definition, a topological 2D modular functor is the assignment of a finite-dimensional vector space to every 2-manifold with boundary and some additional data assigned to the boundary components, and assignment of an isomorphism between the corresponding vector spaces to every homotopy class of homeomorphisms between such manifolds. In addition, it is also required that these vector spaces behave nicely under *gluing*, i.e., the operation of identifying two boundary circles of a surface to produce a new surface.

2-dimensional modular functor (2D MF): complex-analytic definition: A complex-analytic modular functor is a collection of vector bundles with flat connection on the moduli spaces of complex curves with marked points, plus the gluing axiom which describes the behavior of these flat connections near the boundary of the moduli space (in Deligne–Mumford compactification). Such structures naturally appear in conformal field theory: every rational conformal field theory gives rise to a complex-analytic modular functor. The most famous example of a rational conformal field theory — and thus, of a modular functor — is the Wess–Zumino–Witten model, based on representations of an affine Lie algebra.

The main result of this book can be formulated as follows: *the notions of a modular tensor category, 3D TQFT and 2D MF (in both versions) are essentially equivalent.*

Below we will provide a simple example that illustrates how one fact — the quantum Yang–Baxter equation — looks in each of these setups. Let us fix a semisimple abelian category $\mathcal{C}$ and a collection of objects $V_1, \dots, V_n \in \mathcal{C}$.

Tensor category setup. Assume that we have a structure of a tensor category on $\mathcal{C}$. Denote by σ_i the commutativity isomorphisms

$$\sigma_{V_i V_{i+1}}\colon V_1 \otimes \dots \otimes V_i \otimes V_{i+1} \otimes \dots \otimes V_n \to V_1 \otimes \dots \otimes V_{i+1} \otimes V_i \otimes \dots \otimes V_n.$$

Then the axioms of a tensor category imply that

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

where both sides are isomorphisms $\cdots\otimes V_i\otimes V_{i+1}\otimes V_{i+2}\otimes\cdots \to \cdots\otimes V_{i+2}\otimes V_{i+1}\otimes V_i\otimes\cdots$. This identity is known as the *quantum Yang–Baxter equation.*

3D TQFT setup. Consider the 2-sphere $S^2 = \mathbb{R}^2\cup\infty$ with marked points $p_1 = (1,0), p_2 = (2,0), \dots, p_n = (n,0)$ and with the objects $V_1,\dots,V_n$ assigned to these points; this defines a vector space $\tau(S^2;V_1,\dots,V_n)$. Consider the 3-manifold $M = S^2\times[0,1]$ with a tangle inside as shown in Figure 0.1 (which only shows two planes; to get the sphere, the reader needs to add an infinite point to them).

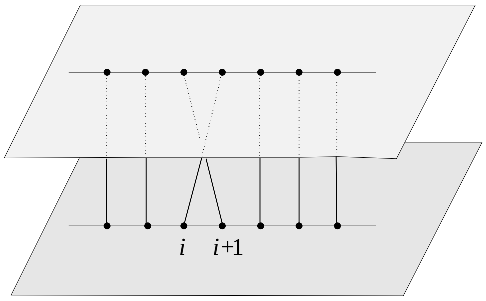
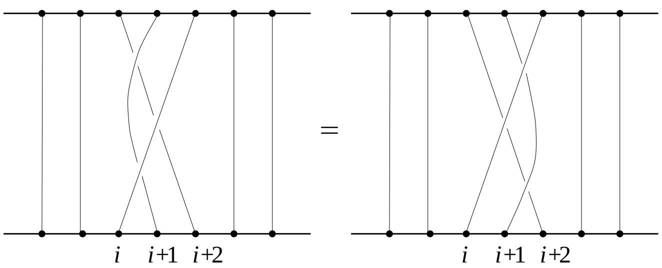


FIGURE 0.1. A 3-manifold with a braid inside.

This gives an operator

$$\sigma^{\mathrm{TQFT}}\colon \tau(S^2;V_1,\dots,V_n)\to\tau(S^2;V_1,\dots,V_{i+1},V_i,\dots,V_n)$$

which also satisfies the quantum Yang–Baxter equation; this follows from the fact that the following 3-manifolds with tangles inside are homeomorphic:



2D MF (topological) setup. Here, again, we take $N = S^2 = \mathbb{R}^2\cup\infty$ with small disks around the points $p_1,\dots,p_n$ removed, and with objects $V_1,\dots,V_n$ assigned to the boundary circles. The corresponding vector space is again $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1\otimes\cdots\otimes V_n)$. Consider the homeomorphism b_i shown in Figure 0.2. This defines an operator

$$(b_i)_*\colon \tau(S^2;V_1,\dots,V_n)\to\tau(S^2;V_1,\dots,V_{i+1},V_i,\dots,V_n)$$

which also satisfies the quantum Yang–Baxter equation. Now this follows from the fact that the homeomorphisms $b_ib_{i+1}b_i$ and $b_{i+1}b_ib_{i+1}$ are homotopic.

2D MF (complex-analytic) setup. We consider the moduli space of spheres with n marked points. A 2D MF defines a local system on this moduli space; denote

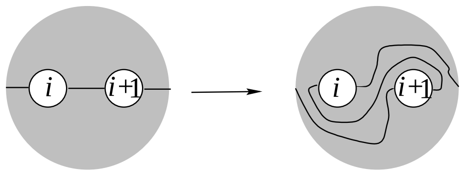


FIGURE 0.2. Braiding for topological modular functor.

the fiber of the corresponding vector bundle over the surface $\Sigma = \mathbb{P}^1$ with marked points $p_1 = 1, \dots, p_n = n$ by $\tau(S^2; V_1, \dots, V_n)$. Then the operator of holonomy along the path b_i, shown in Figure 0.3, gives a map

$$(b_i)_* \colon \tau(\mathbb{P}^1; V_1, \dots, V_n) \to \tau(\mathbb{P}^1; V_1, \dots, V_{i+1}, V_i, \dots, V_n)$$

and the quantum Yang–Baxter equation follows from the identity $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ in the fundamental group of the moduli space of punctured spheres.

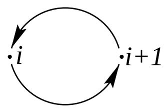


FIGURE 0.3. Braiding for complex-analytic modular functor.

This simple example should convince the reader that indeed there is some common algebraic structure playing pivotal role in all of these subjects. In this particular example, it is not too difficult to show that this underlying algebraic structure is nothing but the braid group. However, when we try to include the notion of dual representation on the tensor category side and higher genus surfaces on the topological side, situation gets more complicated. Still, the main result holds: under some (not too restrictive) assumptions, the notions of modular tensor category (MTC), 3D TQFT and 2D MF (topological and complex-analytic) are essentially equivalent. Schematically, this can be expressed by the following diagram:

$$\begin{tikzcd}
\text{MTC} \arrow[r, leftrightarrow] \arrow[dr] & \text{topological 2D MF} \arrow[r, leftrightarrow] \arrow[d, dotted, shift left] & \text{complex-analytic 2D MF} \\
& \text{3D TQFT} \arrow[u, shift left] &
\end{tikzcd} .$$

Here is a brief description of these equivalences, along with precise references:

Tensor categories→3D TQFT: This equivalence is given by Reshetikhin–Turaev invariants of links and 3-manifold [**RT1, RT2**] and their generalization to surfaces with boundaries [**T**]. In particular, in the example of a sphere with n marked points described above, the correspondence is given by $\tau(S^2; V_1, \dots, V_n) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \cdots \otimes V_n), \sigma_i = \sigma_i^{\mathrm{TQFT}}$. Precise statements can be found in Chapter 4, in particular, in Theorem 4.4.3. These invariants have a long history, which we can't describe here; suffices to say that the idea that path integrals in conformal field theory give rise to invariants of links was suggested by Witten [**W1, W2**]. Unfortunately, path integral technique is still far from being rigorous from mathematical point of view, so Reshetikhin and Turaev do not use it; instead, they use presentation of a 3-manifold as a result of surgery along a framed link.

3D TQFT→2D MF (topological): This arrow is almost tautological: all the axioms of 2D MF are contained among the axioms of 3D TQFT, except for the gluing axiom, which is also rather easy to prove. Details are given in Section 5.8.

2D MF(topological)→3D TQFT: A complete construction of such a map is not yet known (at least to the authors); some partial results in this direction, due to Crane[**C**] and Kohno [**Ko**], are given in Section 5.8. They are based on the Heegard splitting.

2D MF (topological)↔tensor categories: This is based on the results of Moore and Seiberg [**MS1**], who showed (with gaps, which were filled in [**BK**], [**FG**]) that the axioms of a modular tensor category, when rewritten in terms of the vector spaces of homomorphisms, almost coincide with the axioms of 2D topological modular functor. (The word "almost" refers to a minor difficulty in dealing with duality, or rigidity, in a tensor category.) This is discussed in detail in Chapter 5; in particular, the main result is given in Theorem 5.5.1, or, in a more abstract language, in Theorem 5.6.19.

2D MF (topological)↔2D MF (complex-analytic): This is based on the Riemann–Hilbert correspondence, which in particular claims that the categories of local systems (=locally constant sheaves) and vector bundles with flat connections with regular singularities are equivalent. Applying this to the moduli space of Riemann surfaces with marked points, and using the fact that the fundamental group of this moduli space is exactly the mapping class group, we get the desired equivalence. We also have to check that this equivalence preserves the gluing isomorphisms. All this is done in Chapter 6; in particular, the main result is contained in Theorem 6.4.2.

The book is organized as follows.

In Chapter 1, we give basic definitions related to braided tensor categories, such as commutativity and associativity isomorphisms, and state various coherence theorems. We also give two basic examples of tensor categories: the category $\mathcal{C}(\mathfrak{g})$ of representations of a quantum group $U_q(\mathfrak{g})$ (for formal q, i.e., over the field of rational functions in q) and Drinfeld's category $\mathcal{D}(\mathfrak{g},\varkappa), \varkappa \notin \mathbb{Q}$, which as abelian category coincides with the category of finite-dimensional complex representations of a simple Lie algebra $\mathfrak{g}$, but has commutativity and associativity isomorphisms defined in terms of asymptotics of the Knizhnik–Zamolodchikov equations.

In Chapter 2, we continue the study of the theory of tensor categories. We define the notion of ribbon category (in other terminology, rigid balanced braided tensor category) as a category in which every object has a dual satisfying some natural properties and there are functorial isomorphisms $V^{**} \simeq V$ compatible with the tensor product. We develop the "graphic calculus", allowing one to present morphisms in a ribbon category by ribbon (framed) tangles. In particular, this shows that every ribbon category gives rise to invariants of links (Reshetikhin–Turaev invariants). We also show that both examples of Chapter 1 — that is, the categories $\mathcal{C}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g},\varkappa)$ — are ribbon.

In Chapter 3, we introduce one more refinement of the notion of a tensor category — that of a modular tensor category. By definition, this is a semisimple ribbon category with a finite number of simple objects satisfying a certain non-degeneracy condition. It turns out that these categories have a number of remarkable properties; in particular, we prove that in such a category one can define a projective action of the group $\mathrm{SL}_2(\mathbb{Z})$ on an appropriate object, and that one can express the tensor product multiplicities (fusion coefficients) via the entries of the S-matrix (this is known as Verlinde formula). We also give two examples of modular tensor categories. The first one, the category $\mathcal{C}(\mathfrak{g}, \varkappa)$, $\varkappa \in \mathbb{Z}_+$, is a suitable semisimple subquotient of the category of representation of the quantum group $U_q(\mathfrak{g})$ for q being root of unity: $q = e^{\pi \mathrm{i}/m\varkappa}$. The second one is the category of representations of a quantum double of a finite group G, or equivalently, the category of G-equivariant vector bundles on G. (We do not explain here what is the proper definition of Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa)$ for $\varkappa \in \mathbb{Z}_+$, which would be a modular category — this will be done in Chapter 7.)

In Chapter 4, we first move from algebra (tensor categories) to topology, namely, to invariants of 3-manifolds and topological quantum field theory (TQFT). We start by showing how one can use Reshetikhin–Turaev invariants of links to define, for every modular tensor category, invariants of closed 3-manifolds with a link inside. This construction is based on presenting a manifold as a result of surgery of S^3 along a framed link, and then using Kirby's theorem to check that the resulting invariant does not depend on the choice of such presentation. Next, we give a general definition of a topological quantum field theory in any dimension and consider a "baby example" of a 2D topological quantum field theory. After this, we return again to dimension 3 case and define "extended" 3D TQFT, in which 3-manifolds may contain framed tangles, whose ends must be on the boundary; thus, the boundary becomes a surface with marked points and non-zero tangent vectors assigned to them. The main result of this chapter is that every MTC defines an extended 3D TQFT (up to a suitable "central extension"). This, in particular, explains the action of $\mathrm{SL}_2(\mathbb{Z})$ which was introduced in Chapter 3: this action corresponds to the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus with one marked point.

In Chapter 5, we introduce topological 2D modular functor and discuss its relation with the mapping class groups. We also introduce the proper formalism — that of towers of groupoids. The main part of this chapter is devoted to describing the tower of mapping class groups — and thus, the modular functor — by generators and relations, as suggested by Moore and Seiberg. Our exposition follows the results of [**BK**]. Once such a description is obtained, as an easy corollary we get that every modular tensor category defines a 2D topological modular functor (with central charge — see below), and conversely, every 2D MF defines a tensor category which is "weakly rigid". Unfortunately, we were unable to prove — and we do not know of such a proof — that the tensor category defined by a 2D MF is always rigid. However, if it is actually rigid, then we prove that it is an MTC.

In Chapter 5, we also describe accurately the central charge phenomena. As was said before, an MTC defines only a projective representation of $\mathrm{SL}_2(\mathbb{Z})$, or, equivalently, a representation of a central extension of $\mathrm{SL}_2(\mathbb{Z})$, while a 2D MF should give a true representation of $\mathrm{SL}_2(\mathbb{Z})$ and all other mapping class groups. To account for projective representations, we introduce the notion of a modular functor with a central charge, which can be thought of as a "central extension" of the modular functor, and show how the central charge can be calculated for a given MTC.

In Chapter 6, we introduce the complex-analytic version of modular functor. We start by giving all the necessary preliminaries, both about flat connections with regular singularities (mostly due to Deligne) and about the moduli space of punctured curves and its compactification (Deligne–Mumford). Unfortunately, this presents a technical problem: the moduli space is not a manifold but an algebraic stack; we try to avoid actually defining algebraic stacks, thus making our exposition accessible to people with limited algebraic geometry background.

After this, we define the complex-analytic MF as a collection of local systems with regular singularities on the moduli spaces of punctured curves, formulate the gluing axiom, which now becomes the statement that these local systems "factorize" near the boundary of the moduli space (the accurate definition uses the specialization functor), and prove that the notions of topological and complex analytic MF are equivalent. We also return to the Drinfeld category $\mathcal{D}(\mathfrak{g}, \varkappa)$, and show that its definition in terms of Knizhnik–Zamolodchikov equations is nothing but an example of the complex-analytic modular functor in genus zero.

Finally, in Chapter 7 we consider the most famous example of a modular functor, namely the one coming from the Wess–Zumino–Witten model of conformal field theory. This modular functor is based on integrable representations of an affine Lie algebra $\widehat{\mathfrak{g}}$, and the vector bundle with flat connection is defined as coinvariants with respect to the action of the Lie algebra of meromorphic $\mathfrak{g}$-valued functions (in physics literature, this bundle is known as the bundle of conformal blocks). The main result of this chapter is proving that this bundle indeed satisfies the axioms of complex-analytic modular functor. The most difficult part is proving the regularity of the connection at the boundary of the moduli space, which was first done by Tsuchiya, Ueno, and Yamada [**TUY**]. The proof presented in this chapter is based on the results of the unpublished manuscript [**BFM**], with necessary changes.

History. Even though the theory described in this book is relatively new, the number of related publications is now measured in thousands, if not tens of thousands. We tried to list some of the most important references in the beginning of each chapter; however, this selection is highly subjective and does not pretend to be complete in any way. If you find that we missed some important result or gave an incorrect attribution, please let us know and we will gladly correct it in the next edition.

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