Lectures on Tensor Categories and Modular Functor

(Work in progress; may contain errors. Use at your own risk)

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Contents

Introduction

The goal of this book is to give a comprehensive exposition of the relations among the following three topics: tensor categories (such as a category of representations of a quantum group), 3-dimensional topological quantum field theory (which, as will be explained below, includes invariants of links), and 2-dimensional modular functor (which arises in 2-dimensional conformal field theory).

The idea that these subjects are somehow related first appeared in physics literature, in the study of quantum field theory. The pioneering works of Witten [**W1, W2**] and Moore and Seiberg [**MS1**] triggered a real avalanche of papers, both physical and mathematical, exploring various aspects of these relations. Among the more important milestones we should name the papers of Segal [**S**], Reshetikhin–Turaev [**RT1, RT2**], Tsuchiya–Ueno–Yamada [**TUY**], Drinfeld [**Dr3, Dr4**], Beilinson–Feigin–Mazur [**BFM**], and many others.

By late 1990s it had become a commonplace that these topics are closely related. However, when the second author decided to teach (and the first author to take) a class on this topic at MIT in the Spring of 1997, they realized that finding precise statements in the existing literature was not easy, and there were some gaps to be filled. Moreover, the only work giving a good exposition of all these notions was Turaev's book [**T**], which unfortunately didn't cover some important (from our point of view) topics, such as complex-analytic approach to modular functor, based on connections on the moduli spaces. Another excellent reference was the manuscript [**BFM**], which unfortunately is still unfinished, and it is not known when (and if) it will be published. Thus, it was natural that after the course was completed, we decided to turn it into a book which would provide a comprehensive exposition. Needless to say, this book is almost completely expository, and contains no new results — our only contribution was putting known results together, filling the gaps, and sometimes simplifying the proofs.

To give the reader an idea of what kind of relations we are talking about, we give a quick introduction. Let us first introduce the main objects of our study:

Tensor categories: These are abelian categories with associative tensor product, unit object, and some additional properties, such as rigidity (existence of duals). We will be interested only in semisimple categories, such as a category of complex representations of a compact group. However, we weaken the commutativity condition: namely, we require existence of functorial isomorphism $\sigma_{VW}\colon V\otimes W\to W\otimes V$, but — unlike the classical representation theory — we do not require $\sigma^2=\mathrm{id}$. The best known example of such a category is a category of representations of a quantum group; however, there are also other examples.

We will also need special class of tensor categories which are called *modular tensor categories* (MTC); these are semisimple tensor categories with finite number of simple objects and certain non-degeneracy properties. The main example of such categories is provided by a suitable semisimple quotient of the category of representation of a quantum group at a root of unity.

3-dimensional topological quantum field theory (3D TQFT): Despite its physical name, this is a completely mathematical object (to such an extent that some physicists question whether it has any physical meaning at all). A simple definition is that a TQFT is a rule which assigns to every 2-dimensional manifold N a finite-dimensional vector space $\tau(N)$, and to every cobordism — i.e. a 3-manifold M such that its boundary ∂M is written as $\partial M = \overline{N}_1 \sqcup N_2$ — a linear operator $\tau(N_1) \to \tau(N_2)$ (here $\overline{N}$ is N with reversed orientation). In particular, this should give a linear operator $\tau(M)\colon \mathbb{C} \to \mathbb{C}$, i.e., a complex number, for every closed 3-manifold M.

We will, however, need a somewhat more general definition. Namely, we will allow 2-manifolds to have marked points with some additional data assigned to them, and 3-manifolds to have framed tangles inside, which should end at the marked points. In particular, taking a 3-sphere with a link in it, we see that every such extended 3D TQFT defines invariants of links.

2-dimensional modular functor (2D MF): topological definition: By definition, a topological 2D modular functor is the assignment of a finite-dimensional vector space to every 2-manifold with boundary and some additional data assigned to the boundary components, and assignment of an isomorphism between the corresponding vector spaces to every homotopy class of homeomorphisms between such manifolds. In addition, it is also required that these vector spaces behave nicely under *gluing*, i.e., the operation of identifying two boundary circles of a surface to produce a new surface.

2-dimensional modular functor (2D MF): complex-analytic definition: A complex-analytic modular functor is a collection of vector bundles with flat connection on the moduli spaces of complex curves with marked points, plus the gluing axiom which describes the behavior of these flat connections near the boundary of the moduli space (in Deligne–Mumford compactification). Such structures naturally appear in conformal field theory: every rational conformal field theory gives rise to a complex-analytic modular functor. The most famous example of a rational conformal field theory — and thus, of a modular functor — is the Wess–Zumino–Witten model, based on representations of an affine Lie algebra.

The main result of this book can be formulated as follows: *the notions of a modular tensor category, 3D TQFT and 2D MF (in both versions) are essentially equivalent.*

Below we will provide a simple example that illustrates how one fact — the quantum Yang–Baxter equation — looks in each of these setups. Let us fix a semisimple abelian category $\mathcal{C}$ and a collection of objects $V_1, \dots, V_n \in \mathcal{C}$.

Tensor category setup. Assume that we have a structure of a tensor category on $\mathcal{C}$. Denote by σ_i the commutativity isomorphisms

$$\sigma_{V_i V_{i+1}}\colon V_1 \otimes \dots \otimes V_i \otimes V_{i+1} \otimes \dots \otimes V_n \to V_1 \otimes \dots \otimes V_{i+1} \otimes V_i \otimes \dots \otimes V_n.$$

Then the axioms of a tensor category imply that

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

where both sides are isomorphisms $\cdots \otimes V_i \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \to \cdots \otimes V_{i+2} \otimes V_{i+1} \otimes V_i \otimes \cdots$. This identity is known as the *quantum Yang–Baxter equation.*

3D TQFT setup. Consider the 2-sphere $S^2 = \mathbb{R}^2 \cup \infty$ with marked points $p_1 = (1,0), p_2 = (2,0), \dots, p_n = (n,0)$ and with the objects $V_1, \dots, V_n$ assigned to these points; this defines a vector space $\tau(S^2; V_1, \dots, V_n)$. Consider the 3-manifold $M = S^2 \times [0,1]$ with a tangle inside as shown in Figure 0.1 (which only shows two planes; to get the sphere, the reader needs to add an infinite point to them).

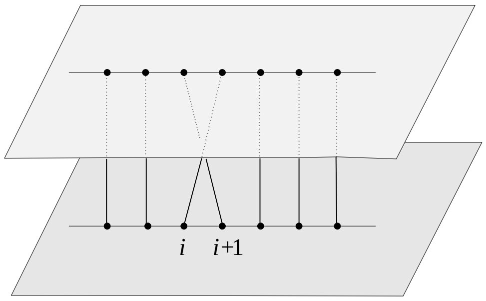
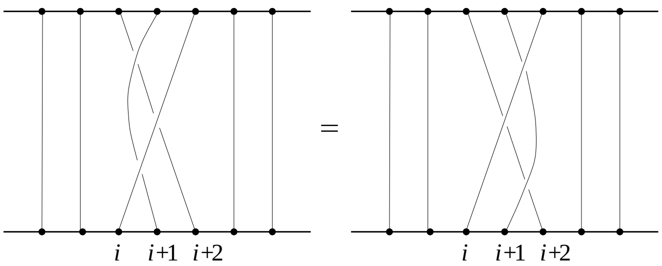


FIGURE 0.1. A 3-manifold with a braid inside.

This gives an operator

$$\sigma^{\mathrm{TQFT}} \colon \tau(S^2; V_1, \dots, V_n) \to \tau(S^2; V_1, \dots, V_{i+1}, V_i, \dots, V_n)$$

which also satisfies the quantum Yang–Baxter equation; this follows from the fact that the following 3-manifolds with tangles inside are homeomorphic:



2D MF (topological) setup. Here, again, we take $N = S^2 = \mathbb{R}^2 \cup \infty$ with small disks around the points $p_1, \dots, p_n$ removed, and with objects $V_1, \dots, V_n$ assigned to the boundary circles. The corresponding vector space is again $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \cdots \otimes V_n)$. Consider the homeomorphism b_i shown in Figure 0.2. This defines an operator

$$(b_i)_* \colon \tau(S^2; V_1, \dots, V_n) \to \tau(S^2; V_1, \dots, V_{i+1}, V_i, \dots, V_n)$$

which also satisfies the quantum Yang–Baxter equation. Now this follows from the fact that the homeomorphisms $b_i b_{i+1} b_i$ and $b_{i+1} b_i b_{i+1}$ are homotopic.

2D MF (complex-analytic) setup. We consider the moduli space of spheres with n marked points. A 2D MF defines a local system on this moduli space; denote

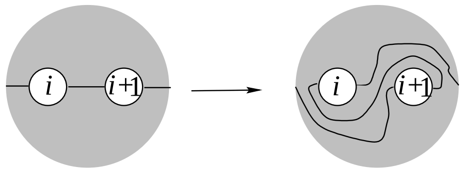


FIGURE 0.2. Braiding for topological modular functor.

the fiber of the corresponding vector bundle over the surface $\Sigma = \mathbb{P}^1$ with marked points $p_1 = 1, \dots, p_n = n$ by $\tau(S^2; V_1, \dots, V_n)$. Then the operator of holonomy along the path b_i, shown in Figure 0.3, gives a map

$$(b_i)_*\colon \tau(\mathbb{P}^1; V_1, \dots, V_n) \to \tau(\mathbb{P}^1; V_1, \dots, V_{i+1}, V_i, \dots, V_n)$$

and the quantum Yang–Baxter equation follows from the identity $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ in the fundamental group of the moduli space of punctured spheres.

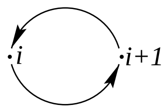


FIGURE 0.3. Braiding for complex-analytic modular functor.

This simple example should convince the reader that indeed there is some common algebraic structure playing pivotal role in all of these subjects. In this particular example, it is not too difficult to show that this underlying algebraic structure is nothing but the braid group. However, when we try to include the notion of dual representation on the tensor category side and higher genus surfaces on the topological side, situation gets more complicated. Still, the main result holds: under some (not too restrictive) assumptions, the notions of modular tensor category (MTC), 3D TQFT and 2D MF (topological and complex-analytic) are essentially equivalent. Schematically, this can be expressed by the following diagram:

$$\begin{tikzcd}
\text{MTC} \arrow[r, leftrightarrow] \arrow[dr] & \text{topological 2D MF} \arrow[r, leftrightarrow] \arrow[d, dotted, shift left] & \text{complex-analytic 2D MF} \\
& \text{3D TQFT} \arrow[u, shift left] &
\end{tikzcd}.$$

Here is a brief description of these equivalences, along with precise references:

Tensor categories→3D TQFT: This equivalence is given by Reshetikhin–Turaev invariants of links and 3-manifold [**RT1, RT2**] and their generalization to surfaces with boundaries [**T**]. In particular, in the example of a sphere with n marked points described above, the correspondence is given by $\tau(S^2; V_1, \dots, V_n) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \dots \otimes V_n), \sigma_i = \sigma_i^{\mathrm{TQFT}}$. Precise statements can be found in Chapter 4, in particular, in Theorem 4.4.3. These invariants have a long history, which we can't describe here; suffices to say that the idea that path integrals in conformal field theory give rise to invariants of links was suggested by Witten [**W1, W2**]. Unfortunately, path integral technique is still far from being rigorous from mathematical point of view, so Reshetikhin and Turaev do not use it; instead, they use presentation of a 3-manifold as a result of surgery along a framed link.

3D TQFT→2D MF (topological): This arrow is almost tautological: all the axioms of 2D MF are contained among the axioms of 3D TQFT, except for the gluing axiom, which is also rather easy to prove. Details are given in Section 5.8.

2D MF(topological)→3D TQFT: A complete construction of such a map is not yet known (at least to the authors); some partial results in this direction, due to Crane[**C**] and Kohno [**Ko**], are given in Section 5.8. They are based on the Heegard splitting.

2D MF (topological)↔tensor categories: This is based on the results of Moore and Seiberg [**MS1**], who showed (with gaps, which were filled in [**BK**], [**FG**]) that the axioms of a modular tensor category, when rewritten in terms of the vector spaces of homomorphisms, almost coincide with the axioms of 2D topological modular functor. (The word "almost" refers to a minor difficulty in dealing with duality, or rigidity, in a tensor category.) This is discussed in detail in Chapter 5; in particular, the main result is given in Theorem 5.5.1, or, in a more abstract language, in Theorem 5.6.19.

2D MF (topological)↔2D MF (complex-analytic): This is based on the Riemann–Hilbert correspondence, which in particular claims that the categories of local systems (=locally constant sheaves) and vector bundles with flat connections with regular singularities are equivalent. Applying this to the moduli space of Riemann surfaces with marked points, and using the fact that the fundamental group of this moduli space is exactly the mapping class group, we get the desired equivalence. We also have to check that this equivalence preserves the gluing isomorphisms. All this is done in Chapter 6; in particular, the main result is contained in Theorem 6.4.2.

The book is organized as follows.

In Chapter 1, we give basic definitions related to braided tensor categories, such as commutativity and associativity isomorphisms, and state various coherence theorems. We also give two basic examples of tensor categories: the category $\mathcal{C}(\mathfrak{g})$ of representations of a quantum group $U_q(\mathfrak{g})$ (for formal q, i.e., over the field of rational functions in q) and Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa), \varkappa \notin \mathbb{Q}$, which as abelian category coincides with the category of finite-dimensional complex representations of a simple Lie algebra $\mathfrak{g}$, but has commutativity and associativity isomorphisms defined in terms of asymptotics of the Knizhnik–Zamolodchikov equations.

In Chapter 2, we continue the study of the theory of tensor categories. We define the notion of ribbon category (in other terminology, rigid balanced braided tensor category) as a category in which every object has a dual satisfying some natural properties and there are functorial isomorphisms $V^{**} \simeq V$ compatible with the tensor product. We develop the "graphic calculus", allowing one to present morphisms in a ribbon category by ribbon (framed) tangles. In particular, this shows that every ribbon category gives rise to invariants of links (Reshetikhin–Turaev invariants). We also show that both examples of Chapter 1 — that is, the categories $\mathcal{C}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g}, \varkappa)$ — are ribbon.

In Chapter 3, we introduce one more refinement of the notion of a tensor category — that of a modular tensor category. By definition, this is a semisimple ribbon category with a finite number of simple objects satisfying a certain non-degeneracy condition. It turns out that these categories have a number of remarkable properties; in particular, we prove that in such a category one can define a projective action of the group $\mathrm{SL}_2(\mathbb{Z})$ on an appropriate object, and that one can express the tensor product multiplicities (fusion coefficients) via the entries of the S-matrix (this is known as Verlinde formula). We also give two examples of modular tensor categories. The first one, the category $\mathcal{C}(\mathfrak{g},\varkappa)$, $\varkappa \in \mathbb{Z}_+$, is a suitable semisimple subquotient of the category of representation of the quantum group $U_q(\mathfrak{g})$ for q being root of unity: $q = e^{\pi \mathrm{i}/m\varkappa}$. The second one is the category of representations of a quantum double of a finite group G, or equivalently, the category of G-equivariant vector bundles on G. (We do not explain here what is the proper definition of Drinfeld's category $\mathcal{D}(\mathfrak{g},\varkappa)$ for $\varkappa \in \mathbb{Z}_+$, which would be a modular category — this will be done in Chapter 7.)

In Chapter 4, we first move from algebra (tensor categories) to topology, namely, to invariants of 3-manifolds and topological quantum field theory (TQFT). We start by showing how one can use Reshetikhin–Turaev invariants of links to define, for every modular tensor category, invariants of closed 3-manifolds with a link inside. This construction is based on presenting a manifold as a result of surgery of S^3 along a framed link, and then using Kirby's theorem to check that the resulting invariant does not depend on the choice of such presentation. Next, we give a general definition of a topological quantum field theory in any dimension and consider a "baby example" of a 2D topological quantum field theory. After this, we return again to dimension 3 case and define "extended" 3D TQFT, in which 3-manifolds may contain framed tangles, whose ends must be on the boundary; thus, the boundary becomes a surface with marked points and non-zero tangent vectors assigned to them. The main result of this chapter is that every MTC defines an extended 3D TQFT (up to a suitable "central extension"). This, in particular, explains the action of $\mathrm{SL}_2(\mathbb{Z})$ which was introduced in Chapter 3: this action corresponds to the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus with one marked point.

In Chapter 5, we introduce topological 2D modular functor and discuss its relation with the mapping class groups. We also introduce the proper formalism — that of towers of groupoids. The main part of this chapter is devoted to describing the tower of mapping class groups — and thus, the modular functor — by generators and relations, as suggested by Moore and Seiberg. Our exposition follows the results of [**BK**]. Once such a description is obtained, as an easy corollary we get that every modular tensor category defines a 2D topological modular functor (with central charge — see below), and conversely, every 2D MF defines a tensor category which is "weakly rigid". Unfortunately, we were unable to prove — and we do not know of such a proof — that the tensor category defined by a 2D MF is always rigid. However, if it is actually rigid, then we prove that it is an MTC.

In Chapter 5, we also describe accurately the central charge phenomena. As was said before, an MTC defines only a projective representation of $\mathrm{SL}_2(\mathbb{Z})$, or, equivalently, a representation of a central extension of $\mathrm{SL}_2(\mathbb{Z})$, while a 2D MF should give a true representation of $\mathrm{SL}_2(\mathbb{Z})$ and all other mapping class groups. To account for projective representations, we introduce the notion of a modular functor with a central charge, which can be thought of as a "central extension" of the modular functor, and show how the central charge can be calculated for a given MTC.

In Chapter 6, we introduce the complex-analytic version of modular functor. We start by giving all the necessary preliminaries, both about flat connections with regular singularities (mostly due to Deligne) and about the moduli space of punctured curves and its compactification (Deligne–Mumford). Unfortunately, this presents a technical problem: the moduli space is not a manifold but an algebraic stack; we try to avoid actually defining algebraic stacks, thus making our exposition accessible to people with limited algebraic geometry background.

After this, we define the complex-analytic MF as a collection of local systems with regular singularities on the moduli spaces of punctured curves, formulate the gluing axiom, which now becomes the statement that these local systems "factorize" near the boundary of the moduli space (the accurate definition uses the specialization functor), and prove that the notions of topological and complex analytic MF are equivalent. We also return to the Drinfeld category $\mathcal{D}(\mathfrak{g}, \varkappa)$, and show that its definition in terms of Knizhnik–Zamolodchikov equations is nothing but an example of the complex-analytic modular functor in genus zero.

Finally, in Chapter 7 we consider the most famous example of a modular functor, namely the one coming from the Wess–Zumino–Witten model of conformal field theory. This modular functor is based on integrable representations of an affine Lie algebra $\widehat{\mathfrak{g}}$, and the vector bundle with flat connection is defined as coinvariants with respect to the action of the Lie algebra of meromorphic $\mathfrak{g}$-valued functions (in physics literature, this bundle is known as the bundle of conformal blocks). The main result of this chapter is proving that this bundle indeed satisfies the axioms of complex-analytic modular functor. The most difficult part is proving the regularity of the connection at the boundary of the moduli space, which was first done by Tsuchiya, Ueno, and Yamada [**TUY**]. The proof presented in this chapter is based on the results of the unpublished manuscript [**BFM**], with necessary changes.

History. Even though the theory described in this book is relatively new, the number of related publications is now measured in thousands, if not tens of thousands. We tried to list some of the most important references in the beginning of each chapter; however, this selection is highly subjective and does not pretend to be complete in any way. If you find that we missed some important result or gave an incorrect attribution, please let us know and we will gladly correct it in the next edition.

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CHAPTER 1

Braided Tensor Categories

In this chapter, we give basic definitions related to braided tensor categories, such as commutativity and associativity isomorphisms, and state various coherence theorems. We also give two basic examples of tensor categories: the category $\mathcal{C}(\mathfrak{g})$ of representations of a quantum group $U_q(\mathfrak{g})$ (for formal q, i.e., over the field of rational functions in q) and Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa), \varkappa \notin \mathbb{Q}$, which as abelian category coincides with the category of finite-dimensional complex representations of a simple Lie algebra $\mathfrak{g}$, but has commutativity and associativity isomorphisms defined in terms of asymptotics of the Knizhnik–Zamolodchikov equations.

1.1. Monoidal tensor categories

We will work over a field k of characteristic 0. Recall the following definition (for details, see, e.g., [**Mac**]).

DEFINITION 1.1.1. A category $\mathcal{C}$ is an *additive category* over k if the following conditions are satisfied:

(i) All $\mathrm{Hom}_{\mathcal{C}}(U, V) \equiv \mathrm{Mor}_{\mathcal{C}}(U, V)$ are k-vector spaces and the compositions

$$\mathrm{Hom}_{\mathcal{C}}(V, W) \times \mathrm{Hom}_{\mathcal{C}}(U, V) \to \mathrm{Hom}_{\mathcal{C}}(U, W), \qquad (\varphi, \psi) \mapsto \varphi \circ \psi$$

are k-bilinear ($U, V, W \in \mathrm{Ob}\, \mathcal{C}$).

(ii) There exists a zero object $0 \in \mathrm{Ob}\, \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(0, V) = \mathrm{Hom}_{\mathcal{C}}(V, 0) = 0$ for all $V \in \mathrm{Ob}\, \mathcal{C}$.

(iii) Finite direct sums exist in $\mathcal{C}$.

An additive category $\mathcal{C}$ is called *abelian* if it satisfies the following condition:

(iv) Every morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(U, V)$ has a kernel $\ker \varphi \in \mathrm{Mor}\, \mathcal{C}$ and a cokernel $\mathrm{coker}\, \varphi \in \mathrm{Mor}\, \mathcal{C}$. Every morphism is a composition of an epimorphism followed by a monomorphism. If $\ker \varphi = 0$, then $\varphi = \ker(\mathrm{coker}\, \varphi)$; if $\mathrm{coker}\, \varphi = 0$, then $\varphi = \mathrm{coker}(\ker \varphi)$.

Informally speaking, an abelian category is an additive category in which we can use the notions of a kernel and a cokernel of a morphism in the same way as, say, in the category of vector spaces over k.

Functors between additive categories will always be assumed to be k-linear on the spaces of morphisms.

EXAMPLE 1.1.2. The following categories are abelian:

(i) The category of k-vector spaces $\mathcal{V}ec(k)$ and the category of finite dimensional k-vector spaces $\mathcal{V}ec_f(k)$.

(ii) The category $\mathcal{R}ep(A)$ of representations of a k-algebra A.

(iii) The category $\mathcal{R}ep(G)$ of representations of a group G over k.

DEFINITION 1.1.3. An object U in an abelian category $\mathcal{C}$ is called *simple* if any injection $V \hookrightarrow U$ is either 0 or isomorphism.

An abelian category $\mathcal{C}$ is called *semisimple* if any object V is isomorphic to a direct sum of simple ones:

$$V \simeq \bigoplus_{i \in I} N_i V_i$$

where V_i are simple objects, I is the set of isomorphism classes of non-zero simple objects in $\mathcal{C}$, $N_i \in \mathbb{Z}_+$ and only a finite number of N_i are non-zero.

Throughout the book we will consider only semisimple categories satisfying the following additional property:

$$\operatorname{End} V_i = k \quad \text{for all } i \in I. \tag{1.1.1}$$

This automatically holds if the base field k is algebraically closed (Schur's Lemma). It easily follows from the definition that

$$\operatorname{Hom}(V_i, V_j) = 0, \quad i \neq j. \tag{1.1.2}$$

Let $\mathcal{C}$ be an abelian category. We want to define a notion of tensor product $\otimes$ on $\mathcal{C}$ with natural associativity and commutativity properties. Let us first digress to a well-known example. Recall that for a k-vector space A, a bilinear map $\cdot\colon A \times A \to A$ is called *associative* if any two expressions

$$(a_1 \cdot a_2) \cdot \big((a_3 \cdot a_4) \cdots a_n\big), \tag{1.1.3}$$

obtained by placing brackets in the product $a_1 \cdot a_2 \cdots a_n$, are equal. This definition of associativity is equivalent to the usual one.

THEOREM 1.1.4. *A bilinear map* $\cdot\colon A \times A \to A$ *is associative if and only if*

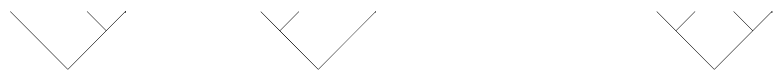
$$(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3) \tag{1.1.4}$$

for all $a_1, a_2, a_3 \in A$.

Of course, this fact is well-known. Nevertheless, we will give a proof because its method will be useful later.

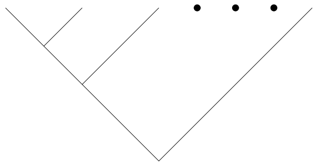
PROOF. Let A_n be the set of all ways of placing brackets (grammatically correctly) in the product $a_1 \cdot a_2 \cdots a_n$, i.e., the set of all expressions of the form (1.1.3). Let us connect two points in A_n if they can be obtained from one another by applying (once) the relation (1.1.4). Our goal is to show that this makes A_n a *connected* graph. It is easy to see that the elements of A_n are in 1-1 correspondence with binary trees with n leaves, e.g.

$a_1 \cdot (a_2 \cdot a_3)$ $\qquad$ $(a_1 \cdot a_2) \cdot a_3$ $\qquad$ $(a_1 \cdot a_2) \cdot (a_3 \cdot a_4)$

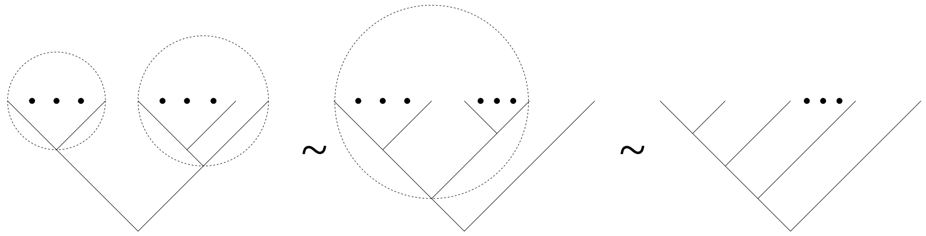


We will prove by induction that every tree can be reduced by applying (1.1.4) to the tree

$$(\cdots((a_1 \cdot a_2) \cdot a_3) \cdots) \cdot a_n$$



This is clear by the figure



The first equivalence $\sim$ is obtained by applying the inductive assumption to the two smaller subtrees and then the elementary relation (1.1.4); the second one—by the inductive assumption. $\square$

EXERCISE 1.1.5. Prove that $|A_n| = \frac{1}{n}\binom{2n-2}{n-1}$. This is known as the $(n-1)$st *Catalan number* c_{n-1}.
Hint: $|A_n| = \sum_{k=1}^{n-1} |A_k||A_{n-k}|$.

Now, what kind of associativity to require for a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for an abelian category $\mathcal{C}$? We cannot assert that

$$(U \otimes V) \otimes W = U \otimes (V \otimes W)$$

for all $U, V, W \in \mathcal{C}$ because this is not true even in the category of vector spaces $\mathcal{V}ec(k)$. We may ask that instead

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$$

but now this condition is too weak. For example, in $\mathcal{V}ec_f(k)$ every two vector spaces of equal dimension are isomorphic. What we need is the existence of a canonical isomorphism.

Recall that for two functors $F, G : \mathcal{C} \to \mathcal{C}'$ a *functorial morphism* $\varphi : F \to G$ is a collection of morphisms

$$\varphi_U : F(U) \to G(U), \quad U \in \operatorname{Ob} \mathcal{C},$$

such that for every $f \in \operatorname{Hom}_{\mathcal{C}}(U, V)$ the following diagram is commutative:

(1.1.5)
$$\begin{tikzcd} F(U) \arrow[r, "F(f)"] \arrow[d, "\varphi_U"'] & F(V) \arrow[d, "\varphi_V"] \\ G(U) \arrow[r, "G(f)"'] & G(V) \end{tikzcd} .$$

Functorial morphisms are also called "natural transformations". Sometimes they are also referred to as "canonical morphisms"; however, we will use the latter term in a slightly different situation (see below).

EXAMPLE 1.1.6. (i) In $\mathcal{V}ec_f(k)$ there exists a functorial isomorphism between a vector space and its double dual, $V \simeq V^{**}$, but there is no functorial isomorphism between V and V^*.

(ii) Let G be a group, $g \in Z(G)$ (center of G), V be a G-module. Then the action of g is a functorial isomorphism $V \simeq V$ (of the identity functor in $\mathcal{R}ep(G)$).

DEFINITION 1.1.7. A *monoidal category* is a category $\mathcal{C}$ with the following additional data:

(i) a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

(ii) a functorial isomorphism

$$\alpha_{UVW} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W) \tag{1.1.6}$$

(*associativity isomorphism*) of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

(iii) a *unit object* $\mathbf{1} \in \operatorname{Ob} \mathcal{C}$ and functorial isomorphisms

$$\lambda_V : \mathbf{1} \otimes V \xrightarrow{\sim} V, \tag{1.1.7}$$

$$\rho_V : V \otimes \mathbf{1} \xrightarrow{\sim} V \tag{1.1.8}$$

for $V \in \operatorname{Ob} \mathcal{C}$.

They have to satisfy the following

(iv) *Associativity axiom.* Suppose X_1 and X_2 are two expressions obtained from $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by inserting $\mathbf{1}$'s and brackets, e.g.,

$$(V_1 \otimes \mathbf{1}) \otimes \big((V_2 \otimes V_3) \otimes \cdots \otimes V_n\big). \tag{1.1.9}$$

Then all isomorphisms $\varphi\colon X_1 \xrightarrow{\sim} X_2$, composed of α's, λ's, ρ's and their inverses, have to be equal.

Usually, we will consider additive monoidal categories. In this case, we additionally require that $\otimes$ is bilinear on the spaces of morphisms and that

(v) $\mathbf{1}$ is a simple object in $\mathcal{C}$ and $\operatorname{End}_{\mathcal{C}}(\mathbf{1}) = k$.

The associativity axiom 1.1.7(iv) implies that we have a uniquely defined functorial isomorphism $X_1 \xrightarrow{\sim} X_2$. We will call such an isomorphism "canonical". Note that a composition of two such canonical isomorphisms is again a canonical isomorphism.

EXAMPLE 1.1.8. The following categories are monoidal:

(i) $\mathcal{V}ec(k)$ and $\mathcal{V}ec_f(k)$;

(ii) The category $\mathcal{R}ep(\mathfrak{g})$ (respectively $\mathcal{R}ep_f(\mathfrak{g})$) of representations (respectively finite dimensional representations) of a Lie algebra $\mathfrak{g}$ over k.

(iii) Let A be a *bialgebra* over k, i.e., a k-algebra provided with algebra homomorphisms $\Delta\colon A \to A \otimes A$ (comultiplication) and $\varepsilon\colon A \to k$ (counit) satisfying

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \tag{1.1.10}$$

$$(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}\,. \tag{1.1.11}$$

Then the category $\mathcal{R}ep(A)$ of representations of A (as a k-algebra) is a monoidal category. ($\otimes$ is the tensor product of vector spaces and $\mathbf{1} = k$ with the following action of A: $x(v \otimes w) := \Delta(x)(v \otimes w)$, $xc := \varepsilon(x)c$ for $x \in A$, $v \in V$, $w \in W$, $c \in k$, $V, W \in \operatorname{Ob} \mathcal{R}ep(A)$.)

In fact, the axioms (1.1.10, 1.1.11) can be deduced from the requirement that the category $\mathcal{R}ep(A)$ be monoidal.

THEOREM 1.1.9 (MacLane Coherence Theorem). *Suppose we are given the data $(\mathcal{C}, \otimes, \alpha, \lambda, \rho)$ as above. Then $\mathcal{C}$ is a monoidal category if and only if the following properties are satisfied.*

(i) Pentagon axiom. *For any $V_i \in \operatorname{Ob}\mathcal{C}$ $(i = 1, 2, 3, 4)$ the following diagram is commutative*

$$\begin{tikzcd}[column sep=tiny]
& ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl, "\alpha_{1,2,3}\otimes \mathrm{id}_4"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "\alpha_{1,23,4}"] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d, "\alpha_{1,2,34}"'] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr, "\mathrm{id}_1 \otimes \alpha_{2,3,4}"'] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd}$$

(ii) Triangle axiom. *For any $V_1, V_2 \in \operatorname{Ob}\mathcal{C}$ the diagram*

$$\begin{tikzcd}
(V_1 \otimes \mathbf{1}) \otimes V_2 \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes \mathrm{id}"'] & & V_1 \otimes (\mathbf{1} \otimes V_2) \arrow[dl, "\mathrm{id}\otimes\lambda"] \\
& V_1 \otimes V_2 &
\end{tikzcd}$$

is commutative.

We do not want to give the proof of this theorem here as it is rather technical and does not help in any way to understand the structure of a monoidal category; instead, we refer the reader to [**Mac**, Sect. VII.2]. Also, we want to stress that this theorem is in a sense just a technical tool, similar to describing a group by generators and relations. Of course, such a description may be useful, but a "global" description—such as a group of automorphisms of some object—is much more important, and in many cases, more useful. For monoidal categories, such a global description is given by the associativity axiom 1.1.7(iv).

REMARK 1.1.10. In analogy with Theorem 1.1.4, we can reformulate the statement of the above theorem as follows (forgetting about $\mathbf{1}$). Suppose that we are given $(\mathcal{C}, \otimes, \alpha)$ as above, satisfying the pentagon axiom 1.1.9(i). Recall the definition of the set A_n from the proof of Theorem 1.1.4; then Theorem 1.1.4 claims that if we add to A_n an edge for every elementary associativity equality (1.1.4) then A_n becomes a connected graph: $\pi_0(A_n) = 1$.

Let us construct a 2-dimensional complex by gluing to A_n pentagons corresponding to each commutative diagram of the form 1.1.9(i) and squares corresponding to the functoriality condition (1.1.5) for α. Then the Coherence Theorem 1.1.9 states that the resulting 2-complex is simply connected: $\pi_1(A_n) = 0$. It is easy to see that this implies that the category $\mathcal{C}$ is monoidal. In fact, Stasheff has shown that this 2-complex is a 2-skeleton of the sphere S^{n-3}.

Similarly, the Coherence Theorem in the form stated—with the unit object and isomorphisms λ, ρ—is equivalent to the statement that the 2-complex M_n whose vertices are labeled by the expressions of the form (1.1.9), is simply connected. Note that M_n has infinite number of vertices; however, this does not cause any problems.

In a monoidal category we can use the canonical isomorphisms to "identify" all expressions of the form (1.1.9) and write the tensor product without brackets, in the same way we do with tensor product of vector spaces. Here is an appropriate formalism (warning: this terminology is not standard).

DEFINITION 1.1.11. For a category $\mathcal{C}$, define its *universal cover* $\hat{\mathcal{C}}$ to be the category with:

Objects: An object of $\hat{\mathcal{C}}$ is a collection $(V_\alpha, \varphi_{\alpha\alpha'})_{\alpha,\alpha'\in A}$, where A is any set, V_α are objects of $\mathcal{C}$, and $\varphi_{\alpha\alpha'} \in \operatorname{Hom}_{\mathcal{C}}(V_{\alpha'}, V_\alpha)$ are isomorphisms satisfying $\varphi_{\alpha_1\alpha_2}\varphi_{\alpha_2\alpha_3} = \varphi_{\alpha_1\alpha_3}$.

Morphisms: For two objects $X = (V_\alpha, \varphi_{\alpha\alpha'})_{\alpha,\alpha'\in A}$ and $Y = (V_\beta, \varphi_{\beta\beta'})_{\beta,\beta'\in B}$ of $\hat{\mathcal{C}}$, the space of morphisms $\operatorname{Hom}_{\hat{\mathcal{C}}}(X, Y)$ is defined to be the set of all collections $(f_{\alpha\beta} \in \operatorname{Hom}_{\mathcal{C}}(V_\alpha, V_\beta))_{\alpha\in A,\beta\in B}$ satisfying $f_{\alpha\beta}\varphi_{\alpha\alpha'} = f_{\alpha'\beta}$ and $\varphi_{\beta'\beta}f_{\alpha\beta} = f_{\alpha\beta'}$.

In other words, objects of $\hat{\mathcal{C}}$ are collections of objects of $\mathcal{C}$ related by canonical isomorphisms, and morphisms are collections of morphisms compatible with these isomorphisms. In particular, taking collections containing only one object, we see that $\mathcal{C}$ itself is a full subcategory of $\hat{\mathcal{C}}$.

LEMMA 1.1.12. *The category $\hat{\mathcal{C}}$ is equivalent to $\mathcal{C}$.*

The proof of this lemma is left to the reader as an exercise.

Now we can say that in a monoidal category, for given $V_1, \dots, V_n$, the collection of all expressions of the form (1.1.9) with the canonical isomorphisms between them forms an object of the category $\hat{\mathcal{C}}$, which we will denote $V_1 \otimes \dots \otimes V_n$. Since the categories $\hat{\mathcal{C}}$ and $\mathcal{C}$ are equivalent, we can as well think of $V_1 \otimes \dots \otimes V_n$ as an object of $\mathcal{C}$ without bothering about brackets. In other words, in a monoidal category iterations of $\otimes$ give a functor $\mathcal{C}^{\times n} \to \mathcal{C}$, which we also denote by $\otimes$. From now on, we will frequently use this remark and omit parentheses and the associativity and unit isomorphisms in our formulas. This is no less rigorous than omitting the parentheses in tensor products of vector spaces, which usually is considered as too trivial to mention. Readers who feel uneasy about this can spell out all the formulas, writing all the associativity and unit isomorphisms explicitly.

More standard (but in our opinion, more artificial) way to deal with the same problem is to use the notion of a strict category.

DEFINITION 1.1.13. A monoidal category $\mathcal{C}$ is called *strict* if

$$V \otimes \mathbf{1} = V, \quad \mathbf{1} \otimes V = V, \quad (V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3) \tag{1.1.12}$$

for all objects $V, V_i \in \operatorname{Ob}\mathcal{C}$ and all α's, λ's, ρ's are the identity isomorphisms.

In a strict monoidal category one can write multiple tensor products $V_1 \otimes V_2 \otimes \dots \otimes V_n$ without bothering about brackets.

Note that the category of vector spaces $\mathcal{V}ec(k)$ is not strict. However, we have the following result.

THEOREM 1.1.14 (MacLane). *Every monoidal category is equivalent to a strict one.*

PROOF. See, e.g., [**Ka**, Sect. XI.5] or [**Mac**]. □

Finally, let us note that if $\mathcal{C}$ is an additive monoidal category, then the functor $\otimes\colon \mathcal{C}^{\times n} \to \mathcal{C}$ is polylinear on the spaces of morphisms. For many applications, it is convenient to define the notion of "tensor product of additive categories" as below.

DEFINITION 1.1.15. Let $\mathcal{C}_1, \mathcal{C}_2$ be additive categories over k. Their tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is the category with the following objects and morphisms:

$$\mathrm{Ob}(\mathcal{C}_1 \boxtimes \mathcal{C}_2) = \text{finite sums of the form } \bigoplus X_i \boxtimes Y_i, \quad X_i \in \mathrm{Ob}\,\mathcal{C}_1, Y_i \in \mathrm{Ob}\,\mathcal{C}_2,$$

$$\mathrm{Hom}_{\mathcal{C}_1 \boxtimes \mathcal{C}_2}(\bigoplus X_i \boxtimes Y_i, \bigoplus X'_j \boxtimes Y'_j) = \bigoplus_{i,j} \mathrm{Hom}(X_i, X'_j) \otimes \mathrm{Hom}(Y_i, Y'_j).$$

One easily sees that $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is again an additive category, and that for any additive category $\mathcal{C}$ there is a natural bijection between additive functors $\mathcal{C}_1 \boxtimes \mathcal{C}_2 \to \mathcal{C}$ and *bilinear* functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$. In a similar way, one can define $\mathcal{C}_1 \boxtimes \dots \boxtimes \mathcal{C}_n$. More generally, for every finite set A (not necessarily ordered) and a collection of additive categories $\mathcal{C}_a, a \in A$, we can define $\boxtimes_{a \in A} \mathcal{C}_a$. As usual, we will use the notation $\mathcal{C}^{\boxtimes n}$, $\mathcal{C}^{\boxtimes A}$ if all the $\mathcal{C}_i$ are equal to $\mathcal{C}$ (respectively, all the $\mathcal{C}_a$ are equal to $\mathcal{C}$). It is also convenient to define $\mathcal{C}^0 = \mathcal{C}^{\emptyset} = \mathcal{V}ec_f(k)$.

Using this definition, we see that a structure of a monoidal category on an additive category $\mathcal{C}$ gives rise to a collection of functors $\otimes\colon \mathcal{C}^{\boxtimes n} \to \mathcal{C}$, satisfying some natural compatibility conditions (see [**De2**]). We will discuss this in detail later.

1.2. Braided tensor categories

In the category of vector spaces $\mathcal{V}ec(k)$ we also have a commutativity isomorphism $\sigma_{VW}\colon V \otimes W \xrightarrow{\sim} W \otimes V$. It is naturally compatible with the isomorphisms α, λ, ρ, and $\sigma^2 = \mathrm{id}$. We would like to axiomatize this kind of structure; however, we want to allow that $\sigma^2 \neq \mathrm{id}$, since this is what happens in most interesting examples. It turns out that the simplest way to formulate the compatibility axioms is based on the so-called braid group.

DEFINITION 1.2.1. A *braid* in n strands is an isotopy class of a union of n nonintersecting segments of smooth curves (strands) in $\mathbb{R}^3$ with end points $\{1, \dots, n\} \times \{0\} \times \{0, 1\}$, such that for each of these strands the third coordinate z is strictly increasing from 0 to 1 (so strands are considered as "going up").

An example of a braid is depicted in Figure 1.1 below. All braids form a group called the *braid group* in n strands B_n.

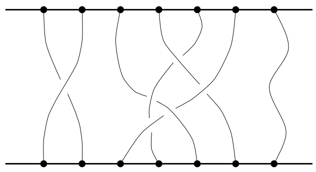


FIGURE 1.1. A braid in 7 strands.

We multiply two braids by putting one of them on the top of the other:

$$b'b'' = \begin{array}{|c|}\hline b' \\ \hline b'' \\ \hline\end{array} \quad . \tag{1.2.1}$$

The unit element is the braid shown in Figure 1.2.

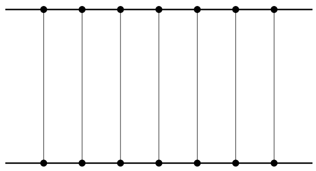


FIGURE 1.2. The unit of B_7.

THEOREM 1.2.2 (E. Artin). *The braid group B_n has generators b_i, $i = 1, \dots, n-1$ (see Figure 1.3) and relations* (braid relations):

$$b_i b_j = b_j b_i, \quad |i-j| > 1, \tag{1.2.2}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{1.2.3}$$

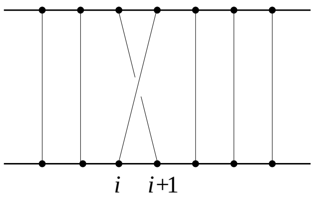
$b_i =$ 

FIGURE 1.3. Generators of the braid group.

PROOF. See, e.g., [**B1**, Theorem 1.8]. □

Let $\mathcal{C}$ be a monoidal category with functorial isomorphisms

$$\sigma_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V \tag{1.2.4}$$

for all objects $V, W \in \operatorname{Ob} \mathcal{C}$. The functoriality of σ means that

$$\sigma_{V'W'}(f \otimes g) = (g \otimes f)\sigma_{VW} \tag{1.2.5}$$

for any two morphisms $f : V \to V'$ and $g : W \to W'$.

For given objects $V_1, \dots, V_n$ in $\mathcal{C}$, we consider all expressions of the form

$$\big((V_{i_1} \otimes V_{i_2}) \otimes (\mathbf{1} \otimes V_{i_3})\big) \otimes \cdots \otimes V_{i_n}, \tag{1.2.6}$$

obtained from $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$ by inserting some $\mathbf{1}$'s and brackets, where $(i_1, \dots, i_n)$ is a permutation of $\{1, \dots, n\}$. To any composition of α's, λ's, ρ's, σ's and their inverses, acting on the element (1.2.6), we assign an element of the braid group B_n as follows. To α, λ and ρ we assign 1, to $\sigma_{V_{i_k} V_{i_{k+1}}}$ the generator b_k.

For example, both isomorphisms

$$(V_1 \otimes V_2) \otimes V_3 \xrightarrow{\sim} (V_2 \otimes V_1) \otimes V_3$$

and

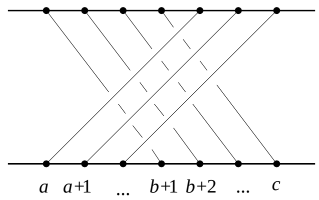
$$(V_3 \otimes V_2) \otimes V_1 \xrightarrow{\sim} (V_2 \otimes V_3) \otimes V_1$$

correspond to the element b_1.

More generally, to the isomorphism

$$\begin{aligned}(1.2.7)\quad \sigma_{AB}\colon \cdots\otimes(V_{i_a}\otimes\cdots\otimes V_{i_b})\otimes(V_{i_{b+1}}\otimes\cdots\otimes V_{i_c})\otimes\cdots\\ \xrightarrow{\sim}\cdots\otimes(V_{i_{b+1}}\otimes\cdots\otimes V_{i_c})\otimes(V_{i_a}\otimes\cdots\otimes V_{i_b})\otimes\cdots\end{aligned}$$

flipping two blocks A and B we assign the braid

$b_{AB} =$ 

.

To a composition of morphisms we associate the product of the corresponding braids. In view of (1.2.1), one can say that "operators act from bottom to top".

Definition 1.2.3. A *braided tensor category* (BTC) $\mathcal{C}$ is a category with $\otimes$, $\mathbf{1}$, α, λ, ρ, σ as above, such that for any two expressions X_1, X_2 of the form (1.2.6) and $\varphi\colon X_1 \xrightarrow{\sim} X_2$ obtained by composing α's, λ's, ρ's, σ's and their inverses, φ depends only on its image in the braid group B_n.

The functorial isomorphism σ is called the *commutativity isomorphism.*

Unless stated otherwise, we will always assume that $\mathcal{C}$ is abelian, in which case we also require that $\otimes$ is bilinear and that $\mathbf{1}$ is simple, with $\mathrm{End}(\mathbf{1}) = k$.

Remark 1.2.4. Any braided tensor category is a monoidal category.

Theorem 1.2.5 (Coherence Theorem). *The data* $(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda,\rho,\sigma)$ *constitute a braided tensor category iff they satisfy the following axioms*:

(i) Pentagon axiom 1.1.9(i).

(ii) Triangle axiom 1.1.9(ii).

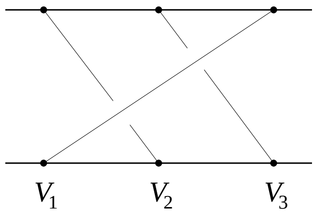
(iii) Hexagon axioms:

(a) *For any* $V_i \in \mathrm{Ob}\,\mathcal{C}$ $(i = 1, 2, 3)$ *the following diagram is commutative*

$$\begin{tikzcd}
& V_1\otimes(V_2\otimes V_3) \arrow[r, "\sigma_{1,23}"] & (V_2\otimes V_3)\otimes V_1 \arrow[dr, "\alpha_{2,3,1}"] & \\
(V_1\otimes V_2)\otimes V_3 \arrow[ur, "\alpha_{1,2,3}"] \arrow[dr, "\sigma_{1,2}\otimes\mathrm{id}_3"'] & & & V_2\otimes(V_3\otimes V_1) \\
& (V_2\otimes V_1)\otimes V_3 \arrow[r, "\alpha_{2,1,3}"'] & V_2\otimes(V_1\otimes V_3) \arrow[ur, "\mathrm{id}_2\otimes\sigma_{1,3}"'] &
\end{tikzcd}$$

(b) *The same as* (a) *but with* σ^{-1} *instead of* σ*.*

Proof. It is easy to see that these axioms are necessary; for example, the hexagon axiom claims equality of two isomorphisms, both corresponding to the following element of the braid group:



We refer the reader to [**Mac**] for the proof that these axioms are sufficient. □

EXERCISE 1.2.6. Show that in a BTC, the braid relation (1.2.3) gives rise to identity of two morphisms $V_1 \otimes V_2 \otimes V_3 \xrightarrow{\sim} V_3 \otimes V_2 \otimes V_1$ and deduce this identity from the pentagon and hexagon axioms. This identity is called the *Yang–Baxter equation*.

DEFINITION 1.2.7. A braided tensor category $\mathcal{C}$ is called a *symmetric tensor category* (STC) if all isomorphisms σ satisfy $\sigma_{WV}\sigma_{VW} = \mathrm{id}_{V\otimes W}$.

EXAMPLE 1.2.8. (i) The categories of vector spaces $\mathcal{V}ec(k)$ and $\mathcal{V}ec_f(k)$ are symmetric tensor categories.

(ii) Let A be a *cocommutative* bialgebra (i.e., a bialgebra with $\Delta = \Delta^{\mathrm{op}} := P\Delta$, where $P(a \otimes b) = b \otimes a$). Then the category $\mathcal{R}ep(A)$ of A-modules is a symmetric tensor category, with the same σ as in $\mathcal{V}ec(k)$ (that is, $\sigma = P$).

(iii) Let A be a *quasi-triangular* bialgebra, i.e., a bialgebra possessing a *universal R-matrix* R: an invertible element $R \in A \otimes A$ satisfying

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1}, \qquad x \in A, \tag{1.2.8}$$

$$(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}, \tag{1.2.9}$$

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}. \tag{1.2.10}$$

(Here we use the standard notation $R_{13} := \sum a_i \otimes 1 \otimes b_i \in A \otimes A \otimes A$, etc., for $R = \sum a_i \otimes b_i \in A \otimes A$.)

Then, taking $\sigma_{VW} = PR\colon V \otimes W \xrightarrow{\sim} W \otimes V$ (where $P(v \otimes w) = w \otimes v$), we see that $\mathcal{R}ep(A)$ is a braided tensor category. As in Example 1.1.8(iii), the axioms for R are in fact equivalent to the requirement that $\mathcal{R}ep(A)$ be a BTC (for details, see, e.g., [**Ka**]).

EXERCISE 1.2.9. Show that the universal R-matrix satisfies

$$(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1, \tag{1.2.11}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.2.12}$$

and interpret these identities in terms of the BTC structure of $\mathcal{R}ep(A)$. Eq. (1.2.12) is called the *quantum Yang–Baxter equation* (QYBE), cf. Exercise 1.2.6.

Finally, we will need a notion of functors between braided tensor categories which agree in a certain sense with the tensor product.

DEFINITION 1.2.10. Let $\mathcal{C}_1, \mathcal{C}_2$ be braided tensor categories. A *tensor functor* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a pair (F, J), where F is a functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ and J is a functorial isomorphism

$$J_{U,V}\colon F(U \otimes V) \xrightarrow{\sim} F(U) \otimes F(V)$$

such that:

(i) $F(\alpha_1) = \alpha_2$, where α_1, α_2 are the associativity isomorphisms in $\mathcal{C}_1, \mathcal{C}_2$, respectively, and

$$F(\alpha_1)\colon F((V_1 \otimes V_2) \otimes V_3) \to F(V_1 \otimes (V_2 \otimes V_3))$$

is considered as an operator

$$(F(V_1) \otimes F(V_2)) \otimes F(V_3) \to F(V_1) \otimes (F(V_2) \otimes F(V_3))$$

using J.

(ii) $F(\sigma_1) = \sigma_2$, where σ_1, σ_2 are the commutativity isomorphisms in $\mathcal{C}_1, \mathcal{C}_2$.

The notions of *functorial isomorphisms* between two tensor functors and of *equivalence* of braided tensor categories are defined in an obvious way.

In the next two sections we will give two major examples of braided tensor categories: the category of representations of a quantum group and the Drinfeld category arising from the Knizhnik–Zamolodchikov equations.

1.3. Quantum groups

We assume that the reader is familiar with the basics of representation theory of simple Lie algebras and quantum groups, so our exposition is very brief. More detailed information can be found in [**Hum**] (classical theory) and in [**L2**], [**CP**], [**Jan**] (quantum groups). Let us first fix the notation.

$\mathfrak{g}$ will be a finite dimensional simple Lie algebra over $\mathbb{C}$,

$\mathfrak{h}$—its Cartan subalgebra,

$\Delta \subset \mathfrak{h}^*$—the root system,

$\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$—the set of simple roots,

$h_i = \alpha_i^\vee \in \mathfrak{h}$—the dual roots (coroots),

$A = (a_{ij})_{1 \le i,j \le r}$—the Cartan matrix, $a_{ij} = (\alpha_i^\vee, \alpha_j)$,

$P \subset \mathfrak{h}^*$—the weight lattice,

$P_+ \subset P$—the cone of dominant integer weights,

$Q \subset \mathfrak{h}^*$—the root lattice,

$Q^\vee \subset \mathfrak{h}$—the dual root lattice (coroot lattice).

Let $\langle\langle \cdot, \cdot \rangle\rangle$ be an invariant bilinear form on $\mathfrak{g}$ normalized by $\langle\langle \alpha, \alpha \rangle\rangle = 2$ for *short* roots α. Then $d_i := \langle\langle \alpha_i, \alpha_i \rangle\rangle / 2 \in \mathbb{Z}_+$ for all $i = 1, \dots, r$.

Finally, $\mathbb{C}_q$ will be the field $\mathbb{C}(q^{1/|P/Q|})$ where q is a formal variable.

DEFINITION 1.3.1. The *quantum group* $U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{C}_q$ with generators e_i, f_i $(i = 1, \dots, r)$, q^h $(h \in Q^\vee)$ and relations

$$q^{h'} q^{h''} = q^{h'+h''}, \quad q^0 = 1, \quad h', h'' \in Q^\vee, \tag{1.3.1}$$

$$q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i, \tag{1.3.2}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}, \tag{1.3.3}$$

and the Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0, \qquad i \neq j, \tag{1.3.4}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0, \qquad i \neq j, \tag{1.3.5}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_i := \frac{[n]_i!}{[k]_i!\,[n-k]_i!}, \quad [n]_i! := [1]_i[2]_i \cdots [n]_i, \quad [n]_i := \frac{q^{d_i n} - q^{-d_i n}}{q^{d_i} - q^{-d_i}}. \tag{1.3.6}$$

$U_q(\mathfrak{g})$ has the following *Hopf algebra* structure:

comultiplication

$$\Delta(q^h) = q^h \otimes q^h, \tag{1.3.7}$$

$$\Delta(e_i) = e_i \otimes q^{d_i h_i} + 1 \otimes e_i, \tag{1.3.8}$$

$$\Delta(f_i) = f_i \otimes 1 + q^{-d_i h_i} \otimes f_i, \tag{1.3.9}$$

counit

$$\varepsilon(q^h) = 1, \;\; \varepsilon(e_i) = \varepsilon(f_i) = 0, \tag{1.3.10}$$

and antipode

$$\gamma(q^h) = q^{-h}, \;\; \gamma(e_i) = -e_i q^{-d_i h_i}, \;\; \gamma(f_i) = -q^{d_i h_i} f_i. \tag{1.3.11}$$

It also has a *quasi-triangular* structure: an invertible element R in a certain completed tensor product $U_q(\mathfrak{g}) \widehat{\otimes} U_q(\mathfrak{g})$ satisfying (1.2.8–1.2.10). R has the form

$$R = q^{\sum x_i \otimes x^i} (1 + \cdots), \tag{1.3.12}$$

where $\{x_i\}$ and $\{x^i\}$ are dual bases in $\mathfrak{h}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and the terms in the brackets belong to $U_q(\mathfrak{g})^+ \otimes U_q(\mathfrak{g})^-$. Here $U_q(\mathfrak{g})^+$ (respectively $U_q(\mathfrak{g})^-$) is the subalgebra of $U_q(\mathfrak{g})$ generated by the elements q^h, e_i (respectively q^h, f_i).

Let $\mathcal{C}(\mathfrak{g})$ be the category of finite dimensional representations of $U_q(\mathfrak{g})$ over $\mathbb{C}_q$ which have a weight decomposition:

$$V = \bigoplus_{\lambda \in P} V^\lambda, \qquad q^h|_{V^\lambda} = q^{(h,\lambda)} \operatorname{id}_{V^\lambda}. \tag{1.3.13}$$

Then $\mathcal{C}(\mathfrak{g})$ is a braided tensor category with the usual $\otimes$, $\mathbf{1} = \mathbb{C}_q$, α, λ, ρ — the same as in the category of vector spaces, and

$$\sigma_{VW} = PR_{VW} \colon V \otimes W \xrightarrow{\sim} W \otimes V. \tag{1.3.14}$$

This is a well-defined operator even though R lies in a completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, i.e., the sum in (1.3.12) becomes finite when applied to any vector from $V \otimes W$.

The category $\mathcal{C}(\mathfrak{g})$ is semi'simple with the same simple objects as $\mathcal{R}ep_f(\mathfrak{g})$, i.e., V_λ ($\lambda \in P_+$), and moreover,

$$V_\lambda \otimes V_\mu = \sum_{\nu \in P_+} N_{\lambda\mu}^\nu V_\nu \tag{1.3.15}$$

with the same multiplicities $N_{\lambda\mu}^{\nu}$ as in $\mathcal{R}ep_f(\mathfrak{g})$ [**L1**]. However, $\mathcal{C}(\mathfrak{g})$ is not equivalent to $\mathcal{R}ep_f(\mathfrak{g})$ as a BTC (or even as a monoidal category—see Remark 1.3.3 below).

There is a version of $\mathcal{C}(\mathfrak{g})$ (due to Lusztig [**L2**]) in which we allow q to be a complex number instead of a formal variable. Let $\mathcal{A} = \mathbb{Z}[q^{\pm 1/|P/Q|}]$ and $U_q(\mathfrak{g})_{\mathbb{Z}}$ be the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$e_i^{(n)} = \frac{e_i^n}{[n]_i!},\ f_i^{(n)} = \frac{f_i^n}{[n]_i!},\ q^h, \qquad n = 1, 2, \dots,\ i = 1, \dots, r,\ h \in Q^\vee. \tag{1.3.16}$$

Fix $\varkappa \in \mathbb{C}^\times$ and consider $\mathbb{C}$ as an $\mathcal{A}$-module via the homomorphism

$$\mathcal{A} \to \mathbb{C}, \quad q^a \mapsto e^{a\pi \mathrm{i}/m\varkappa}$$

where

$$m := \max d_i = \frac{\langle\!\langle \alpha, \alpha \rangle\!\rangle}{2} \quad \text{for a long root } \alpha. \tag{1.3.17}$$

Note that $m = 1$ when the Lie algebra $\mathfrak{g}$ is simply-laced.

Now we define

$$U_q(\mathfrak{g})|_{q=e^{\pi \mathrm{i}/m\varkappa}} := U_q(\mathfrak{g})_{\mathbb{Z}} \otimes_{\mathcal{A}} \mathbb{C}. \tag{1.3.18}$$

The quasi-triangular Hopf algebra structure of $U_q(\mathfrak{g})$ can be defined for $U_q(\mathfrak{g})_{\mathbb{Z}}$ and hence for all algebras $U_q(\mathfrak{g})|_{q=e^{\pi \mathrm{i}/m\varkappa}}$ [**L2**]. Thus, we can define a braided tensor category $\mathcal{C}(\mathfrak{g}, \varkappa)$—the category of finite dimensional representations of $U_q(\mathfrak{g})|_{q=e^{\pi \mathrm{i}/m\varkappa}}$ over $\mathbb{C}$ possessing a weight decomposition. Note that there is a subtlety in defining the notion of weight decomposition for $\varkappa \in \mathbb{Q}$; we will discuss it in detail in Section 3.3.

THEOREM 1.3.2 (Lusztig [**L2**]). *If $\varkappa \notin \mathbb{Q}$ then the category $\mathcal{C}(\mathfrak{g}, \varkappa)$ is semisimple with the same simple objects and multiplicities $N_{\lambda\mu}^{\nu}$ (1.3.15) as $\mathcal{R}ep_f(\mathfrak{g})$.*

REMARK 1.3.3. $\mathcal{C}(\mathfrak{g}, \varkappa)$ is not equivalent to $\mathcal{R}ep_f(\mathfrak{g})$ as a monoidal category. In fact, for $\mathfrak{g} = \mathfrak{sl}_n$, $\varkappa \notin \mathbb{Q}$, the categories $\mathcal{C}(\mathfrak{g}, \varkappa)$ and $\mathcal{C}(\mathfrak{g}, \varkappa')$ are equivalent as monoidal categories iff $\varkappa = \pm\varkappa'$. (See [**KW**].)

1.4. Drinfeld category

In this section we use the same notation as in the previous one. Let $\langle \cdot, \cdot \rangle$ be an invariant bilinear form on $\mathfrak{g}$ normalized by $\langle \alpha, \alpha \rangle = 2$ for *long* roots α. Define the element

$$\Omega := \sum a_i \otimes a^i \in U(\mathfrak{g}) \otimes U(\mathfrak{g}), \tag{1.4.1}$$

where $\{a_i\}$ and $\{a^i\}$ are dual bases in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. Recall that the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has a comultiplication determined by

$$\Delta(x) := x \otimes 1 + 1 \otimes x \in U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad x \in \mathfrak{g}. \tag{1.4.2}$$

LEMMA 1.4.1. (i) $[\Delta(x), \Omega] = 0$ *for all* $x \in \mathfrak{g}$.

(ii) $(1 \otimes \Delta)\Omega = \Omega_{12} + \Omega_{13}$ *in* $U(\mathfrak{g})^{\otimes 3}$, *where* $\Omega_{12} = \sum a_i \otimes a^i \otimes 1$, *etc., as before.*

(iii) *For any* $V, W \in \mathcal{R}ep_f(\mathfrak{g})$, Ω *is diagonalizable on* $V \otimes W$. *If* $V_\nu \subset V_\lambda \otimes V_\mu$ $(\lambda, \mu, \nu \in P_+)$, *then* Ω *acts on* V_ν *as the constant*

$$\frac{1}{2}\big(\langle \nu, \nu + 2\rho \rangle - \langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle\big). \tag{1.4.3}$$

PROOF. Part (i) is a standard exercise; (ii) is obvious; (iii) follows from the identity $\Omega = \frac{1}{2}(\Delta(D) - D \otimes 1 - 1 \otimes D)$, where

$$D = \sum a_i a^i \tag{1.4.4}$$

is the Casimir element. Details can be found, for example, in [**Hum**]. $\square$

Consider the *Knizhnik–Zamolodchikov equation*

$$\varkappa \frac{d}{dt} f(t) = \left(\frac{\Omega_{12}}{t} + \frac{\Omega_{23}}{t-1} \right) f(t) \tag{KZ$_3'$}$$

for a function $f\colon (0,1) \to V_1 \otimes V_2 \otimes V_3$, $V_i \in \mathcal{R}ep_f(\mathfrak{g})$. The equation (KZ_3') has two singular points: $t = 0$ and $t = 1$, both of them regular.

DEFINITION 1.4.2. Let $v \in V_1 \otimes V_2 \otimes V_3$ be an eigenvector of Ω_{12}, i.e., $\Omega_{12} v = \lambda v$, $\lambda \in \mathbb{C}$. We say that a solution $f(t)$ of (KZ_3') is an *asymptotic solution* around 0 corresponding to v, and write $f(t) \sim t^{\lambda/\varkappa} v$, if

$$f(t) = t^{\lambda/\varkappa}(v + r(t)) \tag{1.4.5}$$

for some vector valued function $r(t)$ analytic in a neighborhood of 0 and vanishing at $t = 0$.

LEMMA 1.4.3. *If $\varkappa \notin \mathbb{Q}$ then for every eigenvector $v \in V_1 \otimes V_2 \otimes V_3$ of Ω_{12}, there exists a unique asymptotic solution of (KZ_3') around 0 corresponding to v. Extended by linearity, this correspondence gives an isomorphism*

$$\phi_0 \colon \Gamma(V_1, V_2, V_3) \to V_1 \otimes V_2 \otimes V_3, \tag{1.4.6}$$

where $\Gamma(V_1, V_2, V_3)$ is the space of solutions of (KZ_3') on the interval $(0,1)$.

PROOF. Follows from the standard results on asymptotics of solutions of ordinary differential equations (see, e.g., [**CL**, Chapter 4]) and the fact that Ω_{12} is diagonalizable and eigenvalues of $\Omega_{12}/\varkappa$ can not differ by a non-zero integer. The latter statement follows from Lemma 1.4.1(iii) and the irrationality of $\varkappa$. $\square$

REMARK 1.4.4. The assumption $\varkappa \notin \mathbb{Q}$ is essential, the map ϕ_0 may have poles for rational values of $\varkappa$.

Similarly, we can define the notion of an asymptotic solution of (KZ_3') around 1 and get an isomorphism

$$\phi_1 \colon \Gamma(V_1, V_2, V_3) \to V_1 \otimes V_2 \otimes V_3. \tag{1.4.7}$$

Note that it easily follows from Lemma 1.4.1(i) that $\Gamma(V_1, V_2, V_3)$ is a $\mathfrak{g}$-module, and that ϕ_0, ϕ_1 are morphisms of $\mathfrak{g}$-modules.

THEOREM 1.4.5 (Drinfeld [**Dr1**]). *Let $\mathcal{D}(\mathfrak{g}, \varkappa)$ for $\varkappa \notin \mathbb{Q}$ be the category of all finite dimensional representations of $\mathfrak{g}$ over $\mathbb{C}$ with the usual $\otimes$, $\mathbf{1} = \mathbb{C}$, λ and ρ, but with*

$$\alpha_{V_1 V_2 V_3} = \phi_1 \phi_0^{-1} \colon (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3) \tag{1.4.8}$$

and

$$\sigma_{V_1 V_2} = P e^{\pi \mathrm{i} \Omega / \varkappa} \colon V_1 \otimes V_2 \to V_2 \otimes V_1. \tag{1.4.9}$$

Then $\mathcal{D}(\mathfrak{g}, \varkappa)$ is a braided tensor category.

The proof of this theorem will be sketched below. The associativity morphism α (1.4.8) is called the *Drinfeld associator*. The following theorem relates the Drinfeld category $\mathcal{D}(\mathfrak{g}, \varkappa)$ with the category of representations of quantum group, considered in the previous section.

THEOREM 1.4.6 (Drinfeld, Kazhdan–Lusztig). *For $\varkappa \notin \mathbb{Q}$, the categories $\mathcal{D}(\mathfrak{g}, \varkappa)$ and $\mathcal{C}(\mathfrak{g}, \varkappa)$ are equivalnet as braided tensor categories.*

The proof of this theorem is extremely difficult. It was proved by Drinfeld ([**Dr1**], [**Dr4**]) over the ring of formal power series in $1/\varkappa$ (which corresponds to the infinitesimal neighborhood of the point $q = 1$). Kazhdan and Lusztig in their series of papers [**KL**] proved that for simply-laced Lie algebras this theorem holds for numeric values of $\varkappa$ provided that $\varkappa \notin \mathbb{Q}_+$. Note that for $\varkappa \in \mathbb{Q}_-$, our definition of the category $\mathcal{D}(\mathfrak{g}, \varkappa)$ does not work; Kazhdan and Lusztig use a different definition, based on representation theory of affine Lie algebras, which works for all $\varkappa \neq 0$, and which coincides with our definition for $\varkappa \notin \mathbb{Q}$. Finally, the case of non-simply laced Lie algebras was treated by Lusztig in [**L3**].

Let us explain why Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa)$ is indeed a BTC, and why the equation (KZ_3') is so special. To do so, we will need to introduce more general *Knizhnik–Zamolodchikov equations*

$$\varkappa \frac{\partial}{\partial z_i} f = \Big(\sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Big) f, \qquad 1 \leq i \leq n, \tag{KZ_n}$$

for a function $f(z_1, \ldots, z_n)$ with values in $V_1 \otimes \cdots \otimes V_n$, $V_i \in \mathcal{R}ep_f(\mathfrak{g})$.

LEMMA 1.4.7. *The KZ equations are compatible. In other words, this system of equations defines a flat connection: locally, every $v \in V_1 \otimes \cdots \otimes V_n$ can be extended uniquely to a solution.*

PROOF. We have to check that:

$$\Big[\varkappa \frac{\partial}{\partial z_i} - \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \, , \, \varkappa \frac{\partial}{\partial z_k} - \sum_{j \neq k} \frac{\Omega_{kj}}{z_k - z_j} \Big] = 0.$$

This is verified by an explicit calculation, based on the identity $[\Omega_{12} + \Omega_{13}, \Omega_{23}] = 0$, which easily follows from Lemma 1.4.1 (i), (ii). $\square$

Knizhnik–Zamolodchikov equations were introduced in [**KZ**] in the study of Wess–Zumino–Witten model of Conformal Field Theory. These equations also play important role in representation theory of affine Lie algebras and have been studied in many papers (see [**EFK**] and references therein). We will return to these equations later when we discuss the relations between tensor categories and modular functors.

Here is another argument which explains why these equations are so important. Suppose we want to write a system of differential equations of the form

$$\frac{\partial}{\partial z_i} f = \Big(\sum_{j \neq i} r_{ij}(z_i, z_j) \Big) f, \tag{1.4.10}$$

where f takes values in the tensor product $V_1 \otimes \cdots \otimes V_n$ and $r_{ij}(z_i, z_j)$ acts on $V_i \otimes V_j$. Moreover, let us assume that V_i are representations of $\mathfrak{g}$ and that we have some function $r(z, w) \in \mathfrak{g} \otimes \mathfrak{g}$ such that $r_{ij}(z_i, z_j)$ is just the action of $r(z_i, z_j)$ in $V_i \otimes V_j$. All such $r(z, w)$ giving a flat connection were determined by Belavin and

Drinfeld [**BD**]. In particular, they showed that up to certain equivalences, all non-degenerate $r(z,w)$ which give a compatible system of differential equations (1.4.10) are of the form

$$r(z,w) = \frac{\Omega}{z-w} + \text{regular}.$$

Therefore, in a sense, (KZ_n) is the simplest compatible system of the form (1.4.10).

REMARK 1.4.8. Let f be a solution of (KZ_n). Then it is easy to see that:

(i) $(\sum_{i=1}^n \partial/\partial z_i)f = 0$, hence f is translation invariant: $f(z_1+c,\dots,z_n+c) = f(z_1,\dots,z_n)$ for all $c\in\mathbb{C}$.

(ii) $\varkappa(\sum_{i=1}^n z_i(\partial/\partial z_i))f = (\sum_{i<j}\Omega_{ij})f$ and the operator $\sum_{i<j}\Omega_{ij}$ commutes with all Ω_{ij}. Therefore, $f(cz_1,\dots,cz_n) = c^{\varkappa^{-1}\sum\Omega_{ij}}f(z_1,\dots,z_n)$.

This remark shows that (KZ_n) can be reduced to an equation in $n-2$ variables. For example, when $n=3$, setting $z_1=0$, $z_2=t$, $z_3=1$, we see that (KZ_3) reduces to (KZ'_3).

Now, let us return to explaining why the category $\mathcal{D}(\mathfrak{g},\varkappa)$ is a BTC. We will try to convey the main idea, referring the reader to the original papers of Drinfeld for the formal proof (which is not difficult).

Let $V_1,\dots,V_n$ be finite-dimensional representations of $\mathfrak{g}$. Define

$$\begin{aligned}\Gamma(V_1,V_2,\dots,V_n) &= \{\text{space of solutions of the equations } (\mathrm{KZ}_n) \text{ on } \Delta^n\},\\ \Delta^n &= \{(z_1,\cdots,z_n)\in\mathbb{R}^n \mid z_1<\cdots<z_n\}.\end{aligned} \tag{1.4.11}$$

As before, we note that $\Gamma(V_1,V_2,\dots,V_n)$ is a $\mathfrak{g}$-module, and that for every point $z^0=(z_1^0,\dots,z_n^0)\in\Delta^n$, the map $f\mapsto f(z^0)$ gives an isomorphism $\Gamma(V_1,V_2,\dots,V_n)\to V_1\otimes\cdots\otimes V_n$.

Our next goal is to define an analogue of the morphisms ϕ_0,ϕ_1, corresponding to taking asymptotics of a solution. Unlike the case $n=3$, when our system can be reduced to an ODE with two regular singularities, for general n we have more choices.

Let X be any bracket arrangement in the tensor product $V_1\otimes\cdots\otimes V_n$, e.g., $X=(V_1\otimes V_2)\otimes(V_3\otimes V_4)$. As was discussed in the proof of Theorem 1.1.4, such expressions are in bijection with binary trees. For each such X, we define an isomorphism

$$\phi_X\colon \Gamma(V_1,V_2,\dots,V_n)\to V_1\otimes\cdots\otimes V_n \tag{1.4.12}$$

as follows. First, we choose a curve

$$\gamma\colon (0,1)\to\{z=(z_1,\dots,z_n)\in\mathbb{R}^n | z_1<\cdots<z_n\}$$

such that as $t\to 0$, we have $z_i(t)-z_j(t)\sim t^{d-d_{ij}}$, where

$$\begin{aligned} d_{ij} &= \text{depth of the minimal subtree in } X \text{ containing both } V_i \text{ and } V_j,\\ d &= \max d_{ij} = \text{depth of } X.\end{aligned}$$

In other words, the closer are V_i and V_j in the tree, the faster z_i-z_j approaches zero. For example, for $X=(V_1\otimes V_2)\otimes(V_3\otimes V_4)$, we can take the curve $\gamma(t)=(0,t,1,1+t)$. For $X=((V_1\otimes V_2)\otimes V_3)\otimes V_4$, we can take $\gamma(t)=(0,t^2,t,1)$.

Now, let us restrict the system of equations (KZ_n) to this curve, i.e., rewrite them in terms of the variable t. We claim that this gives an ODE with a regular singularity at $t=0$; thus, we can define an isomorphism ϕ_X as the operation of taking asymptotics as $t\to 0$, similarly to Lemma 1.4.3. Moreover, this isomorphism does not depend on the choice of the curve as long as it satisfies the conditions formulated above. Details can be found in [**EFK**, Lecture 8].

EXERCISE 1.4.9. Prove that for $n = 3, X = (V_1 \otimes V_2) \otimes V_3$, the morphism ϕ_X coincides with the morphism ϕ_0 defined in Lemma 1.4.3, and for $X = V_1 \otimes (V_2 \otimes V_3)$, ϕ_X coincides with ϕ_1.

Now we are ready to prove the pentagon axiom 1.1.9(i). Let us consider the following diagram:

(1.4.13)

$$\begin{tikzcd}
 & ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl] \arrow[dr] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d] & \Gamma(V_1, V_2, V_3, V_4) \arrow[l] \arrow[r] \arrow[u] \arrow[dl] \arrow[dr] & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd}$$

where the arrows forming the outer sides of the pentagon are the same as in the pentagon axiom, with the associativity given by the Drinfeld associator (1.4.8), and the arrows originating at $\Gamma(V_1, V_2, V_3, V_4)$ are the morphisms ϕ_X defined above.

LEMMA 1.4.10. *Each of the five triangles forming the diagram above is commutative.*

Obviously, this lemma immediately implies that the Drinfeld associator satisfies the pentagon axiom. Let us prove, for example, that the following triangle is commutative:

(1.4.14)

$$\begin{tikzcd}
 & \Gamma(V_1, V_2, V_3, V_4) \arrow[dl] \arrow[dr] & \\
((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[rr] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4)
\end{tikzcd} .$$

Commutativity of this triangle is equivalent to saying that the map

$$\phi_Y \phi_X^{-1} : X = ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \to Y = (V_1 \otimes V_2) \otimes (V_3 \otimes V_4),$$

obtained from comparing asymptotics of the KZ equations in 4 variables, coincides with the map

$$\phi_{Y'} \phi_{X'}^{-1} : X' = (V_1' \otimes V_3) \otimes V_4 \to Y' = V_1' \otimes (V_3 \otimes V_4),$$

obtained from the KZ equations in 3 variables, with $V_1' = V_1 \otimes V_2$.

Let us make two observations. First, the maps

$$\begin{aligned}
\phi_X &: \Gamma(V_1, V_2, V_3, V_4) \to ((V_1 \otimes V_2) \otimes V_3) \otimes V_4, \\
\phi_Y &: \Gamma(V_1, V_2, V_3, V_4) \to (V_1 \otimes V_2) \otimes (V_3 \otimes V_4)
\end{aligned}$$

are determined only by the behavior of the solutions for $|z_1 - z_2| \ll |z_1 - z_3|$, $|z_1 - z_2| \ll |z_1 - z_4|$. More formally, let us define

$$D_\varepsilon = \{(z_1, \ldots, z_4) \in \mathbb{R}^4 \,|\, z_1 < \cdots < z_4, |z_1 - z_2| < \varepsilon |z_1 - z_3|, \ |z_1 - z_2| < \varepsilon |z_1 - z_4|\}.$$

Then for every $\varepsilon > 0$, the curves $\gamma(t)$ used in the definition of the operators ϕ_X, ϕ_Y satisfy $\gamma(t) \in D_\varepsilon$ for t small enough.

Second, let us consider the following system of equations:

$$
(\mathrm{KZ}_4') \qquad
\begin{aligned}
\varkappa\frac{\partial}{\partial z_1} f &= \Big(\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{13}}{w_2 - z_3} + \frac{\Omega_{14}}{w_2 - z_4}\Big) f,\\
\varkappa\frac{\partial}{\partial z_2} f &= \Big(\frac{\Omega_{12}}{z_2 - z_1} + \frac{\Omega_{23}}{w_2 - z_3} + \frac{\Omega_{24}}{w_2 - z_4}\Big) f,\\
\varkappa\frac{\partial}{\partial z_3} f &= \Big(\frac{\Omega_{13}}{z_3 - w_2} + \frac{\Omega_{23}}{z_3 - w_2} + \frac{\Omega_{34}}{z_3 - z_4}\Big) f,\\
\varkappa\frac{\partial}{\partial z_4} f &= \Big(\frac{\Omega_{14}}{z_4 - w_2} + \frac{\Omega_{24}}{z_4 - w_2} + \frac{\Omega_{34}}{z_4 - z_3}\Big) f,
\end{aligned}
$$

where $w_2 := (z_1 + z_2)/2$. The only difference between these equations and the equations (KZ_4) is that in the right hand side we have replaced everywhere z_1 and z_2 with w_2, except in the terms $\Omega_{12}/(z_1 - z_2)$ and $\Omega_{12}/(z_2 - z_1)$. One can show that this is still a compatible system of equations. Moreover, since—as was mentioned before—asymptotics only depend on the behavior of solutions for $|z_1 - z_2| \ll |z_1 - z_3|$, $|z_1 - z_2| \ll |z_1 - z_4|$, this replacement does not change the asymptotics: the operator $\phi_Y\phi_X^{-1}$ defined from the equations (KZ_4') coincides with that obtained from (KZ_4). (Of course, this requires a rigorous proof which is not too difficult.)

Let $w_1 = (z_1 - z_2)/2$, $w_2 = (z_1 + z_2)/2$, then $\frac{\partial}{\partial w_1} = \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}$ and $\frac{\partial}{\partial w_2} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$. The system (KZ_4') is equivalent to the following system of equations:

$$
\varkappa\frac{\partial}{\partial w_1} f = 2\frac{\Omega_{12}}{w_1} f + \Big(\frac{\Omega_{13} - \Omega_{23}}{w_2 - z_3} + \frac{\Omega_{14} - \Omega_{24}}{w_2 - z_4}\Big) f,
$$

and

$$
\begin{aligned}
\varkappa\frac{\partial}{\partial w_2} f &= \Big(\frac{\Omega_{13} + \Omega_{23}}{w_2 - z_3} + \frac{\Omega_{14} + \Omega_{24}}{w_2 - z_4}\Big) f,\\
\varkappa\frac{\partial}{\partial z_3} f &= \Big(\frac{\Omega_{13} + \Omega_{23}}{z_3 - w_2} + \frac{\Omega_{34}}{z_3 - z_4}\Big) f,\\
\varkappa\frac{\partial}{\partial z_4} f &= \Big(\frac{\Omega_{14} + \Omega_{24}}{z_4 - w_2} + \frac{\Omega_{34}}{z_4 - z_3}\Big) f.
\end{aligned}
$$

Let us call (KZ_4'') the system obtained from this one by discarding the last term in the first equation. Again, this does not change the asymptotics for $|z_1 - z_2| \ll |z_1 - z_3|$, $|z_1 - z_2| \ll |z_1 - z_4|$. Then the first equation becomes

$$
\varkappa\frac{\partial}{\partial w_1} f = 2\frac{\Omega_{12}}{w_1} f
$$

which is equivalent to the KZ system in 2 variables. Noting that $\Omega_{13} + \Omega_{23} = \Omega_{V_1\otimes V_2, V_3}$, we see that this system can be written as a sum of two systems of equations:

$$
\mathrm{KZ}_4''(V_1, V_2, V_3, V_4) = \mathrm{KZ}_2(V_1, V_2) \otimes \mathrm{id} + \mathrm{KZ}_3(V_1 \otimes V_2, V_3, V_4). \tag{1.4.15}
$$

Moreover, by virtue of Lemma 1.4.1(i), the right-hand sides of these two systems of equations commute. It is not difficult to show from this that the operators $\phi_Y\phi_X^{-1}$ defined from the equations $\mathrm{KZ}_4''(V_1, V_2, V_3, V_4)$ coincide with those defined from the equations $\mathrm{KZ}_3(V_1 \otimes V_2, V_3, V_4)$. This proves that the triangle (1.4.14) is commutative. In a similar way, one proves commutativity of all the other triangles in (1.4.13) and thus, proves the lemma.

It is clear that the crucial step in the above arguments was that the KZ equations have certain nice "factorization" properties, like (1.4.15). The seeming informality and awkwardness of these arguments can be avoided if we use the appropriate language—namely, of local systems with regular singularities on the appropriate moduli spaces—which we will do later (see Section 6.5).

Finally, we leave it to the reader to prove the unit and hexagon axioms. The proofs are easy once you notice that the Drinfeld's associator is trivial if one of the representations V_1, V_2, V_3 is equal to $\mathbb{C}$, and that the commutativity constraint (1.4.9) relates the asymptotics of a solution of the KZ equations in 2 variables in the zone $z_1 < z_2$ and its analytic continuation to the zone $z_2 < z_1$ via the upper half-plane.

CHAPTER 2

Ribbon Categories

In this chapter, we continue the study of the theory of tensor categories. We define the notion of ribbon category (in other terminology, rigid balanced braided tensor category) as a category in which every object has a dual satisfying some natural properties and there are functorial isomorphisms $V^{**} \simeq V$ compatible with the tensor product. We develop the "graphic calculus", allowing one to present morphisms in a ribbon category by ribbon (framed) tangles. In particular, this shows that every ribbon category gives rise to invariants of links (Reshetikhin–Turaev invariants). We also show that both examples of Chapter 1 — that is, the categories $\mathcal{C}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g}, \varkappa)$ — are ribbon.

2.1. Rigid monoidal categories

Now we will discuss the notion of duality in a tensor category. To motivate the definitions, let us consider again the category $\mathcal{V}ec_f(k)$ of finite dimensional vector spaces over a field k. For each $V \in \operatorname{Ob} \mathcal{V}ec_f(k)$, there is a dual vector space $V^* \in \operatorname{Ob} \mathcal{V}ec_f(k)$ and natural morphisms

$$e_V : V^* \otimes V \to k, \tag{2.1.1}$$

$$i_V : k \to V \otimes V^*. \tag{2.1.2}$$

Here e_V is the evaluation map and $i_V(1) := \sum v_i \otimes v^i$ where $\{v_i\}$ and $\{v^i\}$ are dual bases in V and V^*, i.e., $i_V(1)$ corresponds to id_V via the isomorphism $V \otimes V^* \simeq \operatorname{End}_k(V)$.

DEFINITION 2.1.1. Let $\mathcal{C}$ be a monoidal category and V be an object in $\mathcal{C}$. A *right dual* to V is an object V^* with two morphisms

$$e_V : V^* \otimes V \to \mathbf{1}, \tag{2.1.3}$$

$$i_V : \mathbf{1} \to V \otimes V^*, \tag{2.1.4}$$

such that the composition

$$V \xrightarrow{i_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes e_V} V \tag{2.1.5}$$

is equal to id_V, and the composition

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes i_V} V^* \otimes V \otimes V^* \xrightarrow{e_V \otimes \mathrm{id}_{V^*}} V^* \tag{2.1.6}$$

is equal to id_{V^*}.

The properties (2.1.5) and (2.1.6) are called the *rigidity axioms*.

As mentioned before, we skip the canonical associativity and unit isomorphisms in our formulas. Otherwise, we would have to write the first map in (2.1.5) as follows:

$$V \xrightarrow{\lambda_V^{-1}} \mathbf{1} \otimes V \xrightarrow{i_V \otimes \mathrm{id}_V} (V \otimes V^*) \otimes V \xrightarrow{\alpha} V \otimes (V^* \otimes V).$$

Similarly to 2.1.1, we can define a *left dual* of an object V to be an object ${}^*V$ with morphisms

$$e'_V : V \otimes {}^*V \to \mathbf{1}, \tag{2.1.7}$$

$$i'_V : \mathbf{1} \to {}^*V \otimes V \tag{2.1.8}$$

and similar axioms.

Definition 2.1.2. A monoidal category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ has right and left duals.

Example 2.1.3. As we already discussed, the category of finite dimensional vector spaces $\mathcal{V}ec_f(k)$ has duals satisfying the rigidity axioms. For example, (2.1.5) is equivalent to the well-known identity

$$\sum_i v_i(v^i, v) = v, \qquad v \in V. \tag{2.1.9}$$

Example 2.1.4. Let A be a *Hopf algebra* over a field k, i.e., a bialgebra A with an algebra anti-isomorphism $\gamma\colon A \to A$, called the antipode, satisfying

$$\mu(\mathrm{id} \otimes \gamma)\Delta = \varepsilon = \mu(\gamma \otimes \mathrm{id})\Delta, \tag{2.1.10}$$

where $\mu\colon A \otimes A \to A$ is the multiplication.

Let $\mathcal{C} = \mathcal{R}ep_f(A)$ be the category of finite dimensional representations of A. It is a monoidal category (see Exercise 1.1.8(iii)). For an object V we define its dual V^* to be the dual vector space with the following action of A:

$$(av^*, v) := (v^*, \gamma(a)v) \tag{2.1.11}$$

for $a \in A$, $v^* \in V^*$, $v \in V$. Then the canonical maps of vector spaces $k \to V \otimes V^*$ and $V^* \otimes V \to k$ are morphisms of A-modules, and thus, $\mathcal{C}$ is a rigid monoidal category.

Lemma 2.1.5. *If it exists, the right dual is unique up to a unique isomorphism compatible with e and i, i.e., for any two duals $(V^*_{(1)}, e_{(1)}, i_{(1)})$ and $(V^*_{(2)}, e_{(2)}, i_{(2)})$ of an object V, there is a unique isomorphism $\varphi\colon V^*_{(1)} \xrightarrow{\sim} V^*_{(2)}$ such that the diagrams*

$$\begin{tikzcd} V^*_{(1)} \otimes V \arrow[rr, "\varphi \otimes \mathrm{id}_V"] \arrow[dr, "e_{(1)}"'] & & V^*_{(2)} \otimes V \arrow[dl, "e_{(2)}"] \\ & \mathbf{1} & \end{tikzcd} \qquad \begin{tikzcd} V \otimes V^*_{(1)} \arrow[rr, "\mathrm{id}_V \otimes \varphi"] & & V \otimes V^*_{(2)} \\ & \mathbf{1} \arrow[ul, "i_{(1)}"] \arrow[ur, "i_{(2)}"'] & \end{tikzcd}$$

are commutative.

Proof. Take φ to be the composition

$$V^*_{(1)} \xrightarrow{\mathrm{id} \otimes i_{(2)}} V^*_{(1)} \otimes V \otimes V^*_{(2)} \xrightarrow{e_{(1)} \otimes \mathrm{id}} V^*_{(2)}.$$

The rest of the proof is left as an exercise. ☐

Note that if V^* and ${}^*V$ exist, then there are canonical isomorphisms

$$V = {}^*(V^*) = ({}^*V)^*. \tag{2.1.12}$$

LEMMA 2.1.6. *Suppose that V has a dual V^*. Then there exist canonical isomorphisms*

$$\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, W \otimes V^*), \tag{2.1.13}$$

$$\mathrm{Hom}(U, V \otimes W) = \mathrm{Hom}(V^* \otimes U, W). \tag{2.1.14}$$

PROOF. To $\psi \in \mathrm{Hom}(U \otimes V, W)$ we associate the composition

$$U \xrightarrow{\mathrm{id} \otimes i_V} U \otimes V \otimes V^* \xrightarrow{\psi \otimes \mathrm{id}} W \otimes V^*$$

which is an element of $\mathrm{Hom}(U, W \otimes V^*)$.

Similarly, to $\varphi \in \mathrm{Hom}(U, W \otimes V^*)$ we assign

$$U \otimes V \xrightarrow{\varphi \otimes \mathrm{id}} W \otimes V^* \otimes V \xrightarrow{\mathrm{id} \otimes e_V} W.$$

One can easily check that these two maps are inverse to each other, establishing (2.1.13). The proof of (2.1.14) is similar. □

In particular, if both U and V have duals, then by Lemma 2.1.6

$$\mathrm{Hom}(U, V) = \mathrm{Hom}(V^*, U^*) = \mathrm{Hom}(\mathbf{1}, V \otimes U^*). \tag{2.1.15}$$

(In the language of abstract nonsense, this means that the category $\mathcal{C}$ has internal Hom's when it has duals.) For $f \in \mathrm{Hom}(U, V)$ its image in $\mathrm{Hom}(V^*, U^*)$ via the isomorphism (2.1.15) will be denoted by f^*.

If the right dual $*$ exists for all objects in $\mathcal{C}$, then by (2.1.15) it is a contravariant functor, i.e., a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ where $\mathcal{C}^{\mathrm{op}}$ is the opposite (or dual) category to $\mathcal{C}$. (Recall that $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ but with all arrows reversed.)

EXERCISE 2.1.7. Show that, in a rigid category, $*$ is an equivalence of categories $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.

Rigidity is a very restrictive requirement. As an illustration, let us prove the following proposition.

PROPOSITION 2.1.8. *In an abelian rigid monoidal category the tensor product functor $\otimes$ is exact, i.e., for any short exact sequence $0 \to U \to V \to W \to 0$ and an object X, the sequences*

$$0 \to U \otimes X \to V \otimes X \to W \otimes X \to 0$$

and

$$0 \to X \otimes U \to X \otimes V \to X \otimes W \to 0$$

are exact.

PROOF. The sequence

$$0 \to U \otimes X \to V \otimes X \to W \otimes X$$

is exact iff

$$0 \to \mathrm{Hom}(Y, U \otimes X) \to \mathrm{Hom}(Y, V \otimes X) \to \mathrm{Hom}(Y, W \otimes X)$$

is exact for any object Y. But by (2.1.12, 2.1.13), $\mathrm{Hom}(Y, U \otimes X) = \mathrm{Hom}(Y \otimes {}^*X, U)$. Since the functor $\mathrm{Hom}(Y, -)$ is left exact, it follows that $- \otimes X$ is left exact. Using Exercise 2.1.7 (or repeating the same argument with duals), we see that it is also right exact. □

DEFINITION 2.1.9. For an abelian category $\mathcal{C}$, its *Grothendieck group* $K(\mathcal{C})$ is the quotient of the free abelian group on the set of all isomorphism classes of objects in $\mathcal{C}$ modulo the relations $\langle V\rangle = \langle U\rangle + \langle W\rangle$ for any short exact sequence $0 \to U \to V \to W \to 0$. Here $\langle U\rangle$ denotes the isomorphism class of U. When the category $\mathcal{C}$ is rigid monoidal we can make $K(\mathcal{C})$ a ring—the *Grothendieck ring* of $\mathcal{C}$—by defining $\langle U\rangle\langle V\rangle = \langle U \otimes V\rangle$. Note that we need the exactness of $\otimes$ in order that this be well-defined.

Obviously, $K(\mathcal{C})$ is an associative ring with unit; if in addition $\mathcal{C}$ is braided, then $K(\mathcal{C})$ is commutative.

EXAMPLE 2.1.10. Let $\mathcal{C} = \mathcal{R}ep_f(\mathfrak{sl}_2)$ be the category of finite dimensional representations of $\mathfrak{sl}_2$ over $\mathbb{C}$. It is well-known that every object in $\mathcal{C}$ is a direct sum of simple ones and the simple objects are classified by their dimension: for any $n \in \mathbb{Z}_+$ there is a unique up to isomorphism irreducible $\mathfrak{sl}_2$-module V_n of dimension $n+1$. Therefore the Grothendieck group of $\mathcal{C}$ is

$$K(\mathcal{C}) = \bigoplus_{n=0}^{\infty} \mathbb{Z}\,\langle V_n\rangle. \tag{2.1.16}$$

The ring structure is also well-known:

$$\langle V_m\rangle\langle V_n\rangle = \sum_k N_{mn}^k \langle V_k\rangle, \tag{2.1.17}$$

where

$$N_{mn}^k = \begin{cases} 1 & \text{for } |m-n| \le k \le m+n,\ k+m+n \in 2\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.18}$$

Here is a different description of $K(\mathcal{C})$. Define the map

$$K(\mathcal{C}) \to \mathbb{Z}[x, x^{-1}]^{S_2} = \{f(x) \in \mathbb{Z}[x, x^{-1}] \mid f(x) = f(x^{-1})\} \tag{2.1.19}$$

by

$$V \mapsto \operatorname{tr}_V x^h \qquad \text{for } V \in \operatorname{Ob}\mathcal{C}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2. \tag{2.1.20}$$

Then it is easy to see that this map is an isomorphism.

As we discussed before, the category $\mathcal{C}(\mathfrak{sl}_2, \varkappa)$ of representations of the quantum group $U_q(\mathfrak{sl}_2)$ with $q = e^{\pi \mathrm{i}/\varkappa}$, $\varkappa \notin \mathbb{Q}$, has the same Grothendieck ring.

Let again $\mathcal{C}$ be a rigid monoidal category. One may ask whether the duality is compatible with the monoidal structure $(\otimes, \mathbf{1}, \alpha, \lambda, \rho)$. It turns out that this is true without imposing further restrictions.

LEMMA 2.1.11. *Let $\mathcal{C}$ be a rigid monoidal category. Then*:

(i) $\mathbf{1}^* = \mathbf{1} = {}^*\mathbf{1}$.

(ii) $(V \otimes W)^* = W^* \otimes V^*$.

(iii) $(\alpha_{V_1V_2V_3})^* = \alpha_{V_3^*V_2^*V_1^*}$.

If $\mathcal{C}$ is a BTC then, in addition, we have:

(iv) $(\sigma_{VW})^* = \sigma_{V^*W^*}$.

(v) $e_{V\otimes W} = (e_V \otimes e_W)(\sigma_{W^*,V^*\otimes V} \otimes \mathrm{id})$, *and the same with σ^{-1} instead of σ.*

(vi) $i_{V\otimes W} = (\mathrm{id} \otimes \sigma_{V^*,W\otimes W^*})(i_V \otimes i_W)$, *and the same with σ^{-1} instead of σ.*

(In the statements (i), (ii) equality means existence of a unique isomorphism; these isomorphisms are used in (iii)–(vi).)

PROOF. (i) and (ii) are obvious since the dual is unique up to a unique isomorphism. The statements (iii)–(vi) are also easy, but we find it more instructive to give their proof in Section 2.3 using the pictorial technique developed there. □

2.2. Ribbon categories

In our basic example—the category of finite dimensional k-vector spaces—we also have functorial isomorphisms

$$\delta_V : V \xrightarrow{\sim} V^{**} \tag{2.2.1}$$

with the properties:

$$\delta_{V\otimes W} = \delta_V \otimes \delta_W, \tag{2.2.2}$$

$$\delta_{\mathbf{1}} = \mathrm{id}, \tag{2.2.3}$$

$$\delta_{V^*} = (\delta_V^*)^{-1}, \tag{2.2.4}$$

where for $f \in \mathrm{Hom}(U,V)$, $f^* \in \mathrm{Hom}(V^*,U^*)$ is defined by (2.1.15).

The existence of such isomorphisms δ does not follow from the other axioms of a rigid BTC. Let us introduce the last formal definition of this chapter.

DEFINITION 2.2.1. A *ribbon category* is a rigid braided tensor category with functorial isomorphisms (2.2.1) satisfying (2.2.2—2.2.4) for all objects V and W.

As before, unless otherwise specified we assume that $\mathcal{C}$ is abelian, $\otimes$ is bilinear, $\mathbf{1}$ is simple and $\mathrm{End}(\mathbf{1}) = k$.

The term "ribbon", introduced by Reshetikhin and Turaev [**RT1**], will be explained later. Ribbon BTCs are also called "tortile categories" [**JS**] or "balanced rigid braided tensor categories". The word "balancing" will be explained below.

Note that in any rigid BTC $\mathcal{C}$ one can construct functorial isomorphisms

$$\psi_V : V^{**} \xrightarrow{\sim} V \tag{2.2.5}$$

as the composition

$$V^{**} \xrightarrow{i\otimes\mathrm{id}} V \otimes V^* \otimes V^{**} \xrightarrow{\mathrm{id}\otimes\sigma^{-1}} V \otimes V^{**} \otimes V^* \xrightarrow{\mathrm{id}\otimes e} V. \tag{2.2.6}$$

However, ψ does not satisfy the property (2.2.2) unless $\mathcal{C}$ is symmetric.

LEMMA 2.2.2. *In any rigid braided tensor category $\mathcal{C}$, we have*:

$$\psi_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\psi_V \otimes \psi_W),$$
$$\psi_{\mathbf{1}} = \mathrm{id}\,.$$

If, in addition, $\mathcal{C}$ is ribbon, then

$$\psi_{V^*} = \delta_V^* \psi_V^* \delta_V^*.$$

PROOF. Left as an exercise, which is best done using the pictorial technique of Section 2.3. □

Let $\mathcal{C}$ be a ribbon category. We define functorial isomorphisms

$$\theta_V = \psi_V \delta_V : V \xrightarrow{\sim} V, \qquad V \in \mathrm{Ob}\,\mathcal{C}. \tag{2.2.7}$$

The isomorphisms θ are usually called *balancing isomorphisms*, or *twists*. It follows from Lemma 2.2.2 that they satisfy the *balancing axiom*:

$$\theta_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\theta_V \otimes \theta_W), \tag{2.2.8}$$

$$\theta_{\mathbf{1}} = \mathrm{id}, \tag{2.2.9}$$

$$\theta_{V^*} = (\theta_V)^*. \tag{2.2.10}$$

Conversely, a system of twists satisfying the balancing axioms (2.2.8–2.2.10) uniquely defines δ_V satisfying (2.2.2–2.2.4).

COROLLARY 2.2.3. *If $\mathcal{C}$ is a symmetric tensor category, then $\delta_V = \psi_V^{-1}$ defines a structure of a ribbon category on $\mathcal{C}$.*

A non-trivial example of a ribbon category is provided by the category $\mathcal{R}ep_f U_q(\mathfrak{g})$ of finite dimensional representations of the quantum group $U_q(\mathfrak{g})$ (see Section 1.3 and Example 2.1.4). Here q will be a formal variable. Recalling (2.1.11), we see that $V^{**} \simeq V$ as a vector space, but has a different action of $U_q(\mathfrak{g})$, namely

$$\pi_{V^{**}}(a) = \pi_V(\gamma^2(a)), \qquad a \in U_q(\mathfrak{g}). \tag{2.2.11}$$

These two representations are isomorphic:

PROPOSITION 2.2.4. *The category $\mathcal{R}ep_f U_q(\mathfrak{g})$ is a ribbon category with balancing $\delta_V = q^{2\rho}\colon V \xrightarrow{\sim} V^{**}$, where ρ is half of the sum of positive roots in $\mathfrak{g}$. (On the weight subspace $V^\lambda \subset V$, $\lambda \in \mathfrak{h}^*$, $q^{2\rho}$ acts as a multiplication by $q^{\langle\langle 2\rho,\lambda\rangle\rangle}$ and $\langle\langle \rho,\alpha_i\rangle\rangle := \langle\langle \alpha_i,\alpha_i\rangle\rangle/2 = d_i$.)*

PROOF. Quite trivial and based on the identity

$$\gamma^2(a) = q^{2\rho} a q^{-2\rho}, \qquad a \in U_q(\mathfrak{g}),$$

which can be easily checked on the generators of $U_q(\mathfrak{g})$ (see (1.3.11)). $\square$

Let us calculate ψ and θ in the category $\mathcal{C} = \mathcal{R}ep_f U_q(\mathfrak{g})$. Let R be the universal R-matrix of $U_q(\mathfrak{g})$ (see Example 1.2.8(iii) and (1.3.12)). Write

$$R_{21}^{-1} = \sum a_i \otimes b_i, \quad R_{21} = P(R), \tag{2.2.12}$$

where $P(a\otimes b) := b \otimes a$. Recall that $\sigma = PR = R_{21}P$, so $\sigma^{-1} = PR_{21}^{-1}$. Let $\{v_i\}$, $\{v^i\}$ be dual bases in V, V^*, and let $x \in V^{**}$. Then we compute $\psi(x)$ using (2.1.9), (2.2.6):

$$x \mapsto \sum_i v_i \otimes v^i \otimes x \mapsto \sum_{i,j} v_i \otimes b_j(x) \otimes a_j(v^i) \mapsto$$

$$\mapsto \sum_{i,j} v_i\,(b_j(x), a_j(v^i)) = \sum_{i,j} v_i\,(\gamma^{-1}(a_j)b_j(x), v^i) = \sum_j \gamma^{-1}(a_j)b_j(x).$$

Thus we obtained:

LEMMA 2.2.5. *$\theta = u^{-1}q^{2\rho}$, where $u^{-1} := \sum_j \gamma^{-1}(a_j)b_j$ lies in a certain completion of $U_q(\mathfrak{g})$.*

EXERCISE 2.2.6. Deduce from the previous lemma that θ acts as a multiplication by $q^{\langle\langle \lambda,\lambda+2\rho\rangle\rangle}$ on the irreducible $U_q(\mathfrak{g})$-module V_λ with highest weight λ.

The central element $\theta = u^{-1}q^{2\rho}$ is called a (universal) *Casimir operator* for $U_q(\mathfrak{g})$. The element u was introduced and studied by Drinfeld [**Dr2**].

Our next example of a ribbon category will be the Drinfeld category $\mathcal{D}(\mathfrak{g}, \varkappa)$ (see Section 1.4).

THEOREM 2.2.7 (Drinfeld [**Dr2**]). (i) *The category $\mathcal{D}(\mathfrak{g}, \varkappa)$ is a ribbon category, with V^*, e, i, δ the same as in the category of vector spaces.*

(ii) *In this category, the universal twist is given by $\theta = e^{\pi \mathrm{i} D/\varkappa}$ where D is the Casimir operator of $\mathfrak{g}$ defined by* (1.4.4).

Note that this theorem is not immediately obvious since both the rigidity axioms and the definition of θ involve the Drinfeld associator α (1.4.8), i.e., the asymptotics of solutions of KZ equations. However, it is not too difficult.

The formula for the universal twist given by this theorem agrees with $q^{\langle\langle\lambda, \lambda+2\rho\rangle\rangle}$ if $q = e^{\pi \mathrm{i}/m\varkappa}$ (where m is given by (1.3.17)) and confirms the equivalence of $\mathcal{D}(\mathfrak{g}, \varkappa)$ and $\mathcal{C}(\mathfrak{g}, \varkappa)$ as ribbon categories (cf. Theorem 1.4.6).

Finally, we note that a given rigid BTC may be endowed with different structures of a ribbon category, i.e., the maps δ are not uniquely defined by the associativity and commutativity isomorphisms. Here is an example.

EXAMPLE 2.2.8. Let $\mathcal{C} = \mathcal{R}ep_f(\mathfrak{sl}_2)$ be the category of finite dimensional representations of $\mathfrak{sl}_2$ over $\mathbb{C}$ (see Example 2.1.10). This category is symmetric, and thus, has a structure of ribbon category given by Corollary 2.2.3. This definition of δ coincides with the canonical isomorphism of vector spaces $V \xrightarrow{\sim} V^{**}$.

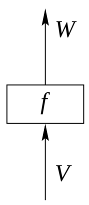
Let $Z = (-1)^h$. It is obvious that for every $\mathfrak{sl}_2$-module V, the map $Z: V \to V$ commutes with the action of $\mathfrak{sl}_2$; thus, Z is a functorial isomorphism (of the identity functor). On the simple module V_n it acts as 1 for even n and as -1 for odd n. Define $\tilde{\delta} = Z\delta$. Then $\tilde{\delta}$ also satisfies axioms (2.2.2–2.2.4), which follows from $\Delta(Z) = Z \otimes Z, Z|_{\mathbb{C}} = 1, \gamma(Z) = Z = Z^{-1}$. Thus, $\tilde{\delta}$ defines a new structure of a ribbon category on $\mathcal{C}$.

EXERCISE 2.2.9. Show that in any ribbon symmetric tensor category $\theta^2 = \mathrm{id}$. This exercise is done much easier using the results of the next section.
Hint: By Lemma 2.2.2, $(\theta_V^2)^* = \psi_{V^*}\psi_V^*$.

2.3. Graphical calculus for morphisms

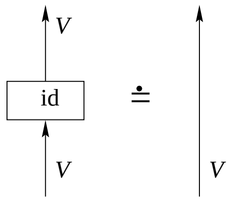
Let $\mathcal{C}$ be a ribbon category. We will introduce a pictorial technique for representing morphisms in $\mathcal{C}$ (cf. [**RT1**], [**T**]).

A morphism $f: V \to W$ in $\mathcal{C}$ will be represented by the figure

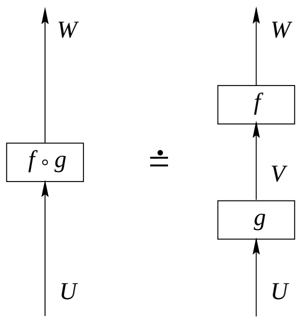


Note that this diagram should be read from bottom to top, as indicated by the arrows. (Some authors use other conventions.)

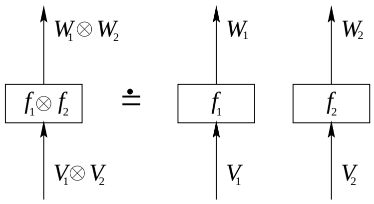
When f is the identity morphism the box will be omitted:



Here and below we relate by $\doteq$ diagrams which give equal morphisms in $\mathcal{C}$. The composition $f \circ g$ of two morphisms is obtained by placing the diagram of f on top of that of g:

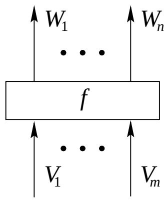


The tensor product of two morphisms f_1 and f_2 will be depicted by placing the diagram of f_1 to the left of the diagram of f_2:



A morphism $f \colon V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ will be depicted as

(2.3.1)

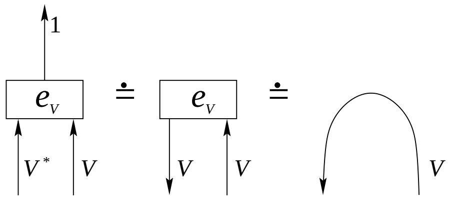


Since $\mathbf{1} \otimes V \simeq V \otimes \mathbf{1} \simeq V$ for any object V, we can add arrows labeled with $\mathbf{1}$'s to any picture without changing the morphism it represents. Hence, the empty picture represents the identity endomorphism of $\mathbf{1}$.

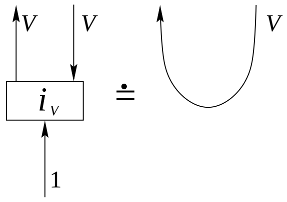
Note that there is ambiguity in defining $V_1 \otimes \cdots \otimes V_m$ for $m/ge3$ when the category $\mathcal{C}$ is not strict. In practice, however, we will use pictures of the form (2.3.1) only for $m, n \leq 2$. An accurate formulation of this formalism, that works for non-strict categories, is provided by Theorem 2.3.9 below.

We will depict duals by simply reversing the arrows and will skip the arrows labeled by $\mathbf{1}$. Also we identify V^{**} with V via δ_V. Then, for example, the morphism

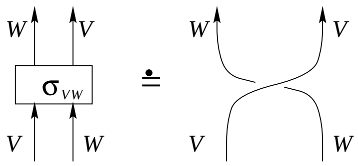
e_V (2.1.3) corresponds to

(2.3.2) 

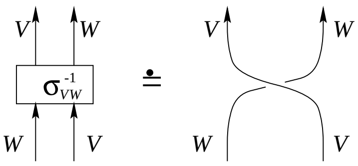
Similarly, i_V (2.1.4) corresponds to

(2.3.3) 

The braiding σ_{VW} (1.2.4) will be depicted as

(2.3.4) 

and its inverse σ_{VW}^{-1} as

(2.3.5) 

Using these pictures as building blocks one can represent arbitrary functorial morphism in $\mathcal{C}$ which is a composition of $\alpha, \lambda, \rho, \sigma, e, i, \delta$ and their inverses.

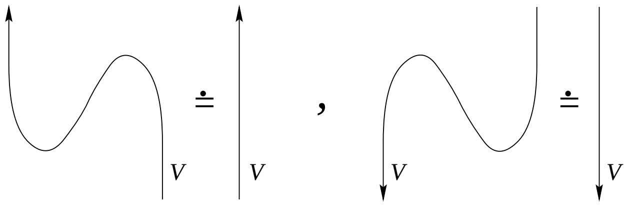
REMARK 2.3.1. The associativity α and the balancing δ are lost in this formalism. A more refined version which keeps track of α was proposed by Bar-Natan [**BN**].

The rigidity axioms (2.1.5, 2.1.6) in the case of a strict category read

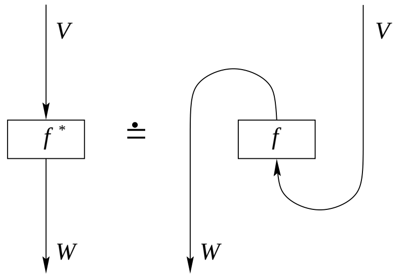
$$(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \tag{2.3.6}$$

$$(e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}, \tag{2.3.7}$$

and can be represented graphically as

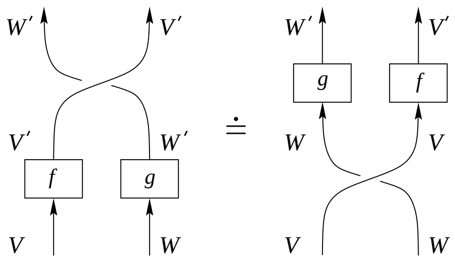


For a morphism $f\colon V \to W$ in $\mathcal{C}$, its dual $f^*\colon W^* \to V^*$ is given by the following picture (see the proof of Lemma 2.1.6):



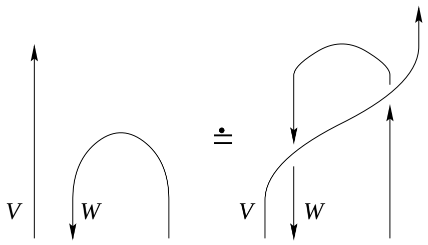
The functoriality of σ (1.2.5) is represented by:

(2.3.8)



Another example is the following lemma.

Lemma 2.3.2.

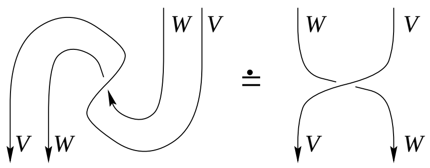


Proof. This is equivalent to the commutative diagram

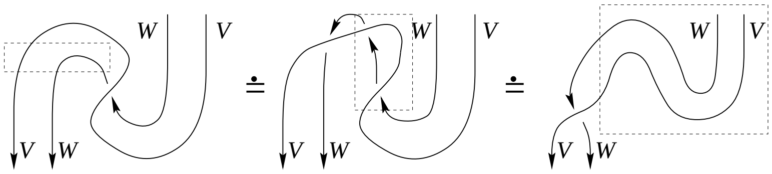
$$\text{(2.3.9)}\qquad \begin{tikzcd} V \otimes W^* \otimes W \arrow[r, "\sigma_{V,W^*\otimes W}"] \arrow[d, "\mathrm{id}_V \otimes e_W"'] & W^* \otimes W \otimes V \arrow[d, "e_W \otimes \mathrm{id}_V"] \\ V \otimes \mathbf{1} \arrow[r, "\sigma_{V,\mathbf{1}}"'] & \mathbf{1} \otimes V \end{tikzcd}$$

which follows from the functoriality of $\sigma_{V,-}$. □

As an immediate corollary we obtain the identity from Lemma 2.1.11(v). Next, let us prove the identity $(\sigma_{VW})^* = \sigma_{V^*W^*}$ from Lemma 2.1.11(iv). Its pictorial representation is



We manipulate the left hand side as follows using Lemma 2.3.2:



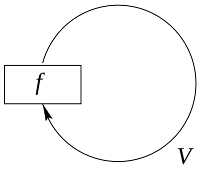
Then applying rigidity, we obtain the right hand side.

It is not difficult to write this proof formally in terms of axioms—it is not long but looks completely mysterious. (We leave it to the suspicious reader.) Representing morphisms with pictures is an intuitive way to visualize the axioms.

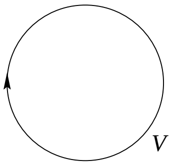
DEFINITION 2.3.3. Let V be an object in a ribbon category $\mathcal{C}$ and f be an endomorphism of V. We define the *trace* of f—$\operatorname{tr} f \in \operatorname{End}_k(\mathbf{1}) \simeq k$—to be the composition

$$\mathbf{1} \xrightarrow{i_V} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{\delta_V \otimes \mathrm{id}} V^{**} \otimes V^* \xrightarrow{e_{V^*}} \mathbf{1}. \tag{2.3.10}$$

Its picture is

(2.3.11) $\operatorname{tr} f =$ 

In particular, for $f = \mathrm{id}_V$, we define the *dimension* of V to be $\dim V := \operatorname{tr} \mathrm{id}_V$

(2.3.12) $\dim V =$ 

EXERCISE 2.3.4. (i) Show that in the category $\mathcal{R}ep_f U_q(\mathfrak{g})$ we have:

$$\operatorname{tr}_q f = \operatorname{tr}_V q^{2\rho} f, \quad \dim_q V = \operatorname{tr}_V q^{2\rho}. \tag{2.3.13}$$

Here we denote by $\dim_q$ and tr_q the above defined dimension and trace to distinguish them from the ordinary dimension and trace of V considered as a vector space.

(ii) Using their pictorial presentation, deduce from the axioms that

$$\operatorname{tr}(f \otimes g) = \operatorname{tr} f \operatorname{tr} g, \quad \operatorname{tr}(f^*) = \operatorname{tr} f, \quad \operatorname{tr}(fg) = \operatorname{tr}(gf). \tag{2.3.14}$$

In particular,

$$\dim(V \otimes W) = \dim V \dim W, \quad \dim V^* = \dim V. \tag{2.3.15}$$

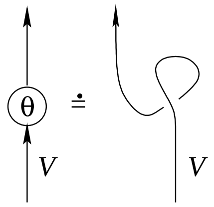
(A solution to this exercise can be found in [**Ka**, Theorem XIV.4.2].)

After the examples given above, it is natural to ask if it is true that any two morphisms giving rise to isotopic pictures are equal. To answer this question, we must finally formalize what we mean by "pictures". As it was with braids, the most natural approach is to consider these pictures as plane projections of certain 3-dimensional objects.

DEFINITION 2.3.5. A *tangle* is an isotopy class of a collection of non-intersecting smooth curves in $\mathbb{R}^2 \times [0,1]$, allowed to have ends only on the lines $\mathbb{R}\times\{0\}\times\{0\}$ and $\mathbb{R}\times\{0\}\times\{1\}$. If all the curves are closed, the tangle is called a *link*; in particular, a link which consists of just one curve is called a *knot*.

In particular, this includes as a special case the notion of a braid (Definition 1.2.1).

Now we can return to the question: is it true that any two morphisms corresponding to isotopic tangles are equal? The answer is "no". For example, the twist
?! $\theta_V\colon V \to V$ defined by (2.2.7) is represented by the picture

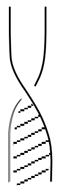


which is obviously isotopic to the picture of id_V, but $\theta \neq \mathrm{id}$ unless the category $\mathcal{C}$ is symmetric. To count the twists, one has to consider so-called framed, or ribbon, tangles.

We will call a *ribbon* a homeomorphic image of a rectangle in $\mathbb{R}^3$. We will always assume that a ribbon has distinguished "bases", i.e., a distinguished pair of opposite edges (in the pictures, they will be usually shown as "short sides"), and a distinguished side—"face side"—which will be shown in white, as opposed to the "back side", which will be shown in gray. We will also allow homeomorphic images of annuli, which, again, must have a distinguished side.

DEFINITION 2.3.6. A *ribbon tangle* (or a *framed tangle*) of n strands is an isotopy class of a union of n non-intersecting ribbons in $\mathbb{R}^2 \times [0,1]$, such that the bases of all ribbons lie on the lines $\mathbb{R}\times\{0\}\times\{0,1\}$, and near these lines, the ribbons are turned with their face side upward. A ribbon tangle composed only of annuli is called a *ribbon* (or *framed*) *link*, and a ribbon link consisting of a single annulus is called a *framed knot*.

Examples of ribbon tangles are shown on the next several pages. Note, however, that by definition, every ribbon must have an even number of twists; in particular, ribbons like this:



are not allowed.

We will call a *coupon* a rectangle in $\mathbb{R}^3$ lying in a plane parallel to $\mathbb{R}\times\{0\}\times\mathbb{R}$ and having edges parallel to $\mathbb{R}\times\{0\}\times\{0\}$ and $\{0\}\times\{0\}\times\mathbb{R}$. Each coupon is provided with a labeling of its sides: "face" and "back", and its edges: "bottom", "left", "top", and "right". The difference between coupons and ribbons is that we do not allow coupons to be twisted.

DEFINITION 2.3.7. A *generalized ribbon tangle* is an isotopy class of a union of several non-intersecting ribbons and several coupons in $\mathbb{R}^2 \times [0,1]$, such that the bases of all ribbons lie on the lines $\mathbb{R} \times \{0\} \times \{0,1\}$ or on edges of coupons parallel to them, and near them ribbons are turned with their face side upward.

So, a generalized ribbon tangle is almost the same as a ribbon tangle, but we allow coupons and allow the ribbons to end on them.

DEFINITION 2.3.8. If $\mathcal{C}$ is a category, a *$\mathcal{C}$-colored ribbon tangle* is a generalized ribbon tangle with the following additional structure:

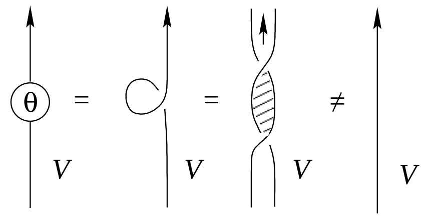
(i) Each ribbon strand is directed.

(ii) Each ribbon strand is labeled (or colored) by an object of $\mathcal{C}$.

(iii) Each coupon is labeled by a morphism of $\mathcal{C}$, so that the following condition is satisfied. For any fixed coupon, let $V_1, \ldots, V_m$ be the labels of the ribbons ending on the bottom edge of the coupon, and let $\varepsilon_i = +$ if the direction of the ith ribbon is "incoming" (i.e., pointing towards the coupon), and $\varepsilon_i = -$ otherwise. To this we associate the object $X = V_1^{\varepsilon_1} \otimes \cdots \otimes V_m^{\varepsilon_m}$, where $V^+ := V$ and $V^- := V^*$ ($X := \mathbf{1}$ if there are no ribbons ending on the bottom edge of the coupon). We do a similar thing for the top edge of the coupon, where now we put $\varepsilon = +$ for the "outgoing" ribbons, and get an object Y. Then we require that the coupon is labeled by a morphism $f \colon X \to Y$.

Note that any plane projection of a usual tangle can be considered as a ribbon tangle which is "lying flat", i.e., always with the face side up, as in Example 2.3.10 (this is called the "blackboard framing"). For technical reasons, we will often draw lines instead of ribbons, always assuming the blackboard framing. Also, we will often omit the arrows pointing up when there is no ambiguity.

Note, however, that different projections of the same (non-framed) tangle now give rise to different framed tangles. For example, the ribbon tangle corresponding to the twist θ_V is no longer isotopic to the trivial tangle. We will insert a circle containing the letter θ (or θ^{-1}) to represent twists:



It turns out that this was the only problem: now it is true that the isomorphisms corresponding to isotopic ribbon tangles are equal. This was proved by Reshetikhin and Turaev in [**RT1**]; see also expositions in [**T, Ka**]. We give here their result in a slightly modified form (cf. [**T**, Theorem 2.5]).

Let $\mathcal{C}$ be a ribbon category. Fix objects $V_1, \ldots, V_n$ in $\mathcal{C}$ and consider all possible expressions of the form

$$X = ((V_1 \otimes V_2) \otimes V_4) \otimes ((V_1^{***} \otimes \mathbf{1}) \otimes {}^*V_2) \otimes \cdots \tag{2.3.16}$$

where we take the tensor product of $V_1, \ldots, V_n$ in arbitrary order and allow repetitions, arbitrary number of left and right stars and $\mathbf{1}$'s.

To each expression X as above we assign a sequence $F(X)$ of arrows and labels by the following rule: to an object ${}^{*\cdots *}V^{*\cdots *}$ we assign $\downarrow_V$ if the total number of stars is odd and $\uparrow_V$ if it is even. All $\mathbf{1}$'s are skipped. For example, to the element (2.3.16) we assign the sequence

$$\uparrow_{V_1} \quad \uparrow_{V_2} \quad \uparrow_{V_4} \quad \downarrow_{V_1} \quad \downarrow_{V_2}$$

For two such expressions X_1 and X_2 consider all morphisms $\varphi\colon X_1 \to X_2$ which can be obtained as a composition of the elementary morphisms $\alpha^{\pm 1}$, $\lambda^{\pm 1}$, $\rho^{\pm 1}$, $\sigma^{\pm 1}$, e, i, $\delta^{\pm 1}$, as well as a number of other morphisms of $\mathcal{C}$. To each such composition of morphisms $\varphi\colon X_1 \to X_2$, we assign a $\mathcal{C}$-colored ribbon tangle $T = F(\varphi)$ with

$$\mathrm{bottom}(T) = F(X_1), \quad \mathrm{top}(T) = F(X_2)$$

by the following rules.

A morphism $f\colon X_1 \to X_2$ corresponds to a coupon labeled with f, so that the ribbons ending on its bottom edge are labeled by $F(X_1)$, and those ending on its top edge by $F(X_2)$ (cf. Definition 2.3.8(iii)). The morphisms $\alpha, \lambda, \rho, \delta$ and their inverses correspond to the trivial tangles, $F(e)$, $F(i)$, $F(\sigma)$, and $F(\sigma^{-1})$ are given by the right hand sides of (2.3.2), (2.3.3), (2.3.4), and (2.3.5), respectively, and

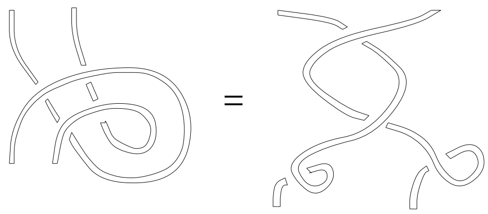
$$F(\varphi_1 \otimes \varphi_2) = \boxed{F(\varphi_1)}\boxed{F(\varphi_2)}, \qquad F(\varphi_1\varphi_2) = \begin{array}{|c|}\hline F(\varphi_1) \\ \hline F(\varphi_2) \\ \hline\end{array}.$$

In other words, we just apply the rules of the graphical calculus introduced earlier, but we use ribbons instead of lines so that we keep track of the twists. Then we have the following crucial result.

Theorem 2.3.9 (Reshetikhin–Turaev [**RT1**]). *The morphism φ depends only on the isotopy class of the tangle $F(\varphi)$, i.e., if $F(\varphi_1)$ and $F(\varphi_2)$ are isotopic as ribbon tangles then $\varphi_1 = \varphi_2$.*

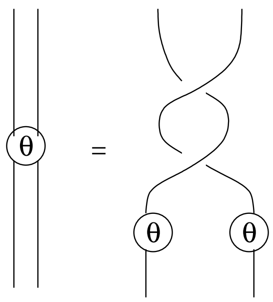
The proof of this theorem will be sketched later, when we give another reformulation.

Example 2.3.10. The identity $\theta_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\theta_V \otimes \theta_W)$ (2.2.8) corresponds to the following isotopic ribbon graphs:



Another way to present this identity, using the conventions formulated before (i.e., drawing lines instead of ribbons and omitting upward arrows), is shown below.

(2.3.17)



An important corollary of Theorem 2.3.9 is that any two isomorphisms $\varphi\colon X_1 \xrightarrow{\sim} X_2$, composed of $\alpha, \lambda, \rho, \delta$ and their inverses, are equal and therefore, if $F(X_1) = F(X_2)$ then there exists a *canonical* isomorphism $X_1 \xrightarrow{\sim} X_2$.

Typically, the Reshetikhin–Turaev Theorem is used in the opposite direction. We will reformulate the theorem in another form which appeared in their original paper [**RT1**].

THEOREM 2.3.11 (Reshetikhin–Turaev [**RT1**]). *Let $\mathcal{C}$ be a ribbon category. Then for every $\mathcal{C}$-colored ribbon tangle T we can define a morphism $F^{-1}(T)\colon X_1 \to X_2$ in $\mathcal{C}$ where*

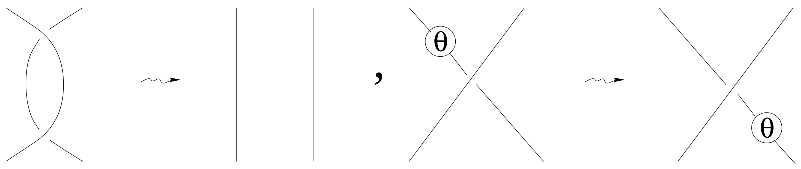
$$X_1 = F^{-1}(\text{bottom}(T)), \quad X_2 = F^{-1}(\text{top}(T)).$$

The objects X_1 and X_2 are defined up to a canonical isomorphism and the morphism $F^{-1}(T)$ depends only on the isotopy type of the tangle T.

OUTLINE OF PROOF. First, one shows that two ribbon tangles are isotopic if and only if the corresponding diagrams in the plane can be obtained one from another by applying a sequence of the following elementary operations:

a) isotopy of $\mathbb{R}^2$ and

b) one of a finite number of "simple moves", such as



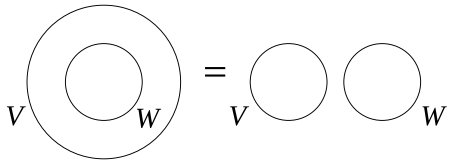
the braid relation and some more—see a complete list in [**RT1**] or [**T, Ka**]. These simple moves generalize the so-called *Reidemeister moves* which play the same role for unframed tangles.

Now, it suffices to check that $F^{-1}(T)$ is unchanged under any of these moves, which is straightforward: for example, the braid relation and the first of these moves follow from the definition of a braided category, and the second relation follows from the functoriality of the commutativity isomorphism. We refer the reader to the original papers for details. □

COROLLARY 2.3.12. *For every $\mathcal{C}$-colored framed link T, $F^{-1}(T) \in k$ is a number which depends only on the isotopy type of T.*

Therefore, every ribbon category gives a number of invariants of links. These invariants are usually called *Reshetikhin–Turaev invariants*. In particular, if we take $\mathcal{C}$ to be the category $\mathcal{C}(\mathfrak{g}, \varkappa)$ of representations of a quantum group then these invariants are rational functions in $q^{1/n}$. If they are rewritten as formal power series in $1/\varkappa$ then every coefficient of such a series is again an invariant; moreover, every such coefficient is an invariant of a very special type— *Vassiliev invariant* (see, for example, [**PS**]). However, it was recently proved that not all Vassiliev invariants can be obtained in this way [**Vo**].

Theorem 2.3.9 is the most effective way of proving identities in ribbon categories— just draw the corresponding pictures and manipulate with them. For example, the identity $\dim(V \otimes W) = \dim V \dim W$ (cf. Exercise 2.3.4) is now obvious from the following isotopy of ribbon tangles:



We also have the following somewhat technical but useful fact.

PROPOSITION 2.3.13. *Let T be a $\mathcal{C}$-colored ribbon tangle. Let T' be obtained from T by reversing the direction of one of the strands and replacing the corresponding to it label V by V^*. Then we get the same morphism $F^{-1}(T) = F^{-1}(T')$.*

The proof is similar to that of Theorem 2.3.9. It is immediate to deduce from Proposition 2.3.13 that $\dim V = \dim V^*$ (cf. Exercise 2.3.4).

2.4. Semisimple categories

In this section we study some properties of semisimple abelian tensor categories and, in particular, semisimple ribbon categories.

Let $\mathcal{C}$ be a semisimple ribbon category. Let I be the set of equivalence classes of non-zero simple objects in $\mathcal{C}$, and let $\{V_i\}_{i\in I}$ be representatives of those classes. Since we assumed that $\mathbf{1}$ is a simple object, it also can be written as V_i for some $i \in I$. Traditionally, this index is denoted by 0: $V_0 = \mathbf{1}$.

Semisimplicity immediately implies a number of properties:

1. For $i \in I$, V_i^* is also simple, hence $V_i^* \simeq V_{i^*}$ for some $i^* \in I$. The map $*: I \to I$ is an involution and $0^* = 0$. (Note, however, that in general there is no canonical way to define the isomorphism $V_i^* \simeq V_{i^*}$, see Remark 2.4.2 below.)

2. We can define the multiplicity coefficients $N_{ij}^k \in \mathbb{Z}_+$ by

$$V_i \otimes V_j \simeq \bigoplus_k N_{ij}^k V_k. \tag{2.4.1}$$

In the physics literature, this formula is called a *fusion rule*, and the coefficients N_{ij}^k are called "fusion coefficients". They satisfy:

$$N_{ij}^k = \dim \operatorname{Hom}(V_k, V_i \otimes V_j) = \dim \operatorname{Hom}(\mathbf{1}, V_i \otimes V_j \otimes V_k^*), \tag{2.4.2}$$

$$N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^*j^*}^{k^*}, \quad N_{ij}^0 = \delta_{ij^*}. \tag{2.4.3}$$

These numbers are nothing but the structure coefficients of the Grothendieck ring $K(\mathcal{C})$, which has the basis $x_i = \langle V_i \rangle$ with the multiplication rule $x_i x_j = \sum_k N_{ij}^k x_k$.

3. Since $\operatorname{End} V_i = k$, we have

$$\theta_{V_i} = \theta_i \operatorname{id}_{V_i}, \qquad \dim V_i = d_i \tag{2.4.4}$$

for some $\theta_i, d_i \in k$. The number d_i is called the *quantum dimension* of V_i, cf. Exercise 2.3.4. We have

$$\theta_0 = 1, \ \ \theta_{i^*} = \theta_i, \tag{2.4.5}$$

$$d_0 = 1, \ \ d_{i^*} = d_i, \ \ d_i d_j = \textstyle\sum_k N_{ij}^k d_k. \tag{2.4.6}$$

LEMMA 2.4.1. *In a semisimple ribbon category, all dimensions $d_i = \dim V_i$ of simple objects are non-zero.*

PROOF. Since $N_{ii^*}^0 = N_{i0}^i = 1$, we can write $V_i \otimes V_i^* = X \oplus X'$, where $X \simeq \mathbf{1}$ (not canonically), and $\mathrm{Hom}(\mathbf{1}, X') = 0$. Thus, the maps $i\colon \mathbf{1} \to V_i \otimes V_i^*$ and $e\colon V_i \otimes V_i^* \to \mathbf{1}$ can be considered as maps $\mathbf{1} \to X$, $X \to \mathbf{1}$. Since both of these maps are non-zero (this follows from rigidity), their composition is also non-zero. □

REMARK 2.4.2. It is tempting (and many authors do so) to choose some identification $V_i^* \xrightarrow{\sim} V_{i^*}$ to simplify the formulas. However, this should be avoided. Not only this cannot be done canonically, but in many cases it cannot be done at all! Indeed, in order for this to be useful we need to choose these isomorphisms $\phi_i\colon V_i^* \to V_{i^*}$ in such a way that the composition

$$V_i \xrightarrow{\phi_{i^*}^{-1}} V_{i^*}^* \xrightarrow{\phi_i^*} V_i^{**}$$

coincides with δ_{V_i}. We leave it as an exercise to the reader that for the category $\mathcal{R}ep_f(\mathfrak{sl}_2)$ (with the standard balancing), considered in Example 2.2.8, this is impossible: regardless of the choice of ϕ_i, the composition above will give $\tilde{\delta}$ rather than δ.

When $\mathcal{C} = \mathcal{R}ep_f G$, G—a finite group, an important role in the representation theory of G is played by the regular representation $R = Fun(G)$. This is a G-bimodule (i.e., an element of $\mathcal{C}^{\boxtimes 2}$); as a bimodule, it is isomorphic to $\bigoplus V_i \boxtimes V_i^*$. More generally, any rigid semisimple abelian category automatically gives rise to an object $R \in \mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$ defined by

$$R = \bigoplus_{i \in I} V_i \boxtimes V_i^*, \tag{2.4.7}$$

where we denote by $\mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$ the category whose objects are infinite sums of the form $\sum A_i \boxtimes B_i$ with $A_i, B_i \in \mathcal{C}$. The object R does not depend on the choice of representatives V_i of the isomorphism classes: if we choose another representative $\tilde{V}_i$, then one has a canonical isomorphism $R \xrightarrow{\sim} \tilde{R}$. In particular, if $\mathcal{C}$ is balanced, then R is symmetric in the following sense: we say that an object $R \in \mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$ is *symmetric* if we are given an isomorphism $s\colon R^{\mathrm{op}} \xrightarrow{\sim} R$ such that $ss^{\mathrm{op}} = \mathrm{id}$. Here the functor op is defined by $(A \boxtimes B)^{\mathrm{op}} = B \boxtimes A$.

For the object R defined by (2.4.7), s can be written explicitly as follows: for every i, choose an isomorphism $\varphi_i\colon V_i^* \xrightarrow{\sim} V_{i^*}$, and define $\psi_i\colon V_i^* \boxtimes V_i \to V_{i^*} \boxtimes V_{i^*}^*$ by $\psi_i = \varphi_i \boxtimes \delta_{V_i}(\varphi_i^*)^{-1}$. This obviously does not depend on the choice of φ_i (since $\mathrm{End}\, V_i = k$). Now, define

$$s = \bigoplus \psi_i\colon R \xrightarrow{\sim} R^{\mathrm{op}}. \tag{2.4.8}$$

We will also frequently use the following object of $\mathrm{ind}{-}\mathcal{C}$ associated with R :

$$H = \otimes(R) = \bigoplus V_i \otimes V_i^*. \tag{2.4.9}$$

The previous arguments show that H is also canonically isomorphic to $\bigoplus V_i^* \otimes V_i$. Also, H is canonically isomorphic to $H^* = \bigoplus V_i^{**} \otimes V_i^*$. Both these properties will be used many times in the sequel.

One of the main goals of the next lectures will be to answer the following question: given a semisimple abelian category $\mathcal{C}$ and a symmetric object $R \in \mathrm{ind}{-}\mathcal{C}^{\boxtimes 2}$, what extra data are needed to reconstruct the structure of a ribbon category on $\mathcal{C}$? We will give an answer to this question in Chapter 5.

Finally, when working with an object $R \in \operatorname{ind-}\mathcal{C}^{\boxtimes 2}$ (not necessarily the one given by (2.4.7)), it is convenient to use the following notation: if $R = \bigoplus A_i \boxtimes B_i$, then we write $\operatorname{Hom}(X, R^{(1)} \otimes Y \otimes R^{(2)})$ instead of $\bigoplus_i \operatorname{Hom}(X, A_i \otimes Y \otimes B_i)$, etc. If R is symmetric, then the superscripts "(1)", "(2)" can be omitted, and we will just write $\operatorname{Hom}(X, R \otimes Y \otimes R)$. Similarly, we can use several copies $R_1, \dots, R_m$ of R and write, for example,

$$\begin{aligned} R_1 \otimes X \otimes R_2 \otimes R_2 \otimes R_1 &= R_1^{(1)} \otimes X \otimes R_2^{(1)} \otimes R_2^{(2)} \otimes R_1^{(2)} \\ &= \bigoplus_{i,j} A_i \otimes X \otimes A_j \otimes B_j \otimes B_i. \end{aligned}$$

CHAPTER 3

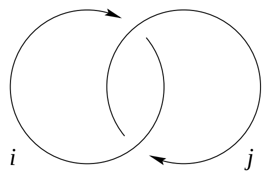
Modular Tensor Categories

In this chapter, we introduce one more refinement of the notion of a tensor category — that of a modular tensor category. By definition, this is a semisimple ribbon category with a finite number of simple objects satisfying a certain non-degeneracy condition. It turns out that these categories have a number of remarkable properties; in particular, we prove that in such a category one can define a projective action of the group $\mathrm{SL}_2(\mathbb{Z})$ on an appropriate object, and that one can express the tensor product multiplicities (fusion coefficients) via the entries of the S-matrix (this is known as Verlinde formula).

We also give two examples of modular tensor categories. The first one, the category $\mathcal{C}(\mathfrak{g}, \varkappa), \varkappa \in \mathbb{Z}_+$, is a suitable semisimple subquotient of the category of representation of the quantum group $U_q(\mathfrak{g})$ for q being root of unity: $q = e^{\pi \mathrm{i}/m\varkappa}$. The second one is the category of representations of a quantum double of a finite group G, or equivalently, the category of G-equivariant vector bundles on G. (We do not explain here what is the proper definition of Drinfeld's category $\mathcal{D}(\mathfrak{g}, \varkappa)$ for $\varkappa \in \mathbb{Z}_+$, which would be a modular category — this will be done in Chapter 7.)

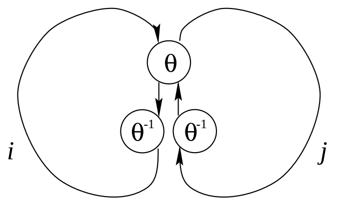
3.1. Modular tensor categories

In this section we will study ribbon categories with some additional properties. Let $\mathcal{C}$ be a semisimple ribbon category. We will use the same notation as in Section 2.4. Define the numbers $\tilde{s}_{ij} \in k = \mathrm{End}\, \mathbf{1}$ $(i, j \in I)$ by the following picture:

(3.1.1) $\tilde{s}_{ij} =$ 

.

Here and below, we will often label strands of tangles by the indices $i \in I$ meaning by this V_i. Note that (2.3.17) implies

(3.1.2) $\tilde{s}_{ij} =$ 

$$= \theta_i^{-1}\theta_j^{-1} \operatorname{tr} \theta_{V_i^* \otimes V_j} = \theta_i^{-1}\theta_j^{-1} \sum_{k \in I} N_{i^* j}^k \theta_k d_k.$$

Also, it is easy to see that

$$\tilde{s}_{ij} = \tilde{s}_{ji} = \tilde{s}_{i^*j^*} = \tilde{s}_{j^*i^*}, \quad \tilde{s}_{i0} = d_i = \dim V_i. \tag{3.1.3}$$

Definition 3.1.1. A *modular* (*tensor*) *category* (MTC for short) is a semisimple ribbon category $\mathcal{C}$ satisfying the following properties:

(i) $\mathcal{C}$ has only a finite number of isomorphism classes of simple objects: $|I| < \infty$.

(ii) The matrix $\tilde{s} = (\tilde{s}_{ij})_{i,j \in I}$, where $\tilde{s}_{ij}$ is defined by (3.1.1), is invertible.

Remark 3.1.2. If $\mathcal{C}$ is symmetric, one can change overcrossing and undercrossing, hence $\tilde{s}_{ij} = d_i d_j$. Unless $|I| = 1$, this matrix $\tilde{s}$ is singular, therefore $\mathcal{C}$ is not modular.

Remarks 3.1.3. (i) Many authors (for example, Turaev [**T**]) impose weaker conditions, not necessarily requiring semisimplicity in our sense. We are only interested in the simplest case; thus the above definition is absolutely sufficient for our purposes. We refer the reader to [**Ke**], [**Lyu2**] for a discussion of the non-semisimple case.

(ii) The name "modular" is justified by the fact that in this case we can define a projective action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on certain objects in our category, as we will show below. To the best of our knowledge, this construction first appeared (in rather vague terms) in a paper of Moore and Seiberg [**MS2**]; later it was formalized by Lyubashenko [**Lyu1**] and others. Our exposition follows the book of Turaev [**T**].

(iii) The appearance of the modular group in tensor categories may seem mysterious; however, there is a simple geometrical explanation, based on the fact that to each modular tensor category one can associate a 2+1-dimensional Topological Quantum Field Theory. This also shows that in fact we have an action of the mapping class group of any closed oriented 2-dimensional surface on the appropriate objects in MTC. This is the key idea of the book [**T**], and will be discussed in detail in Chapter 4.

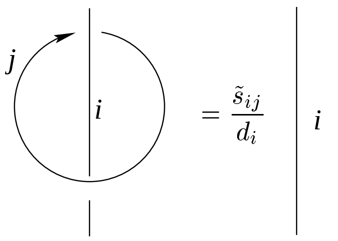
From now on, let us adopt the following convention:

(3.1.4) If some (closed) strand in a picture is left unlabeled then we assume summation over all labels $i \in I$ each taken with the weight $d_i = \dim V_i$.

Since $d_{i^*} = d_i$, we can drop the arrow of such a strand. Recall also that we omit the upward arrow when there is no ambiguity. Then we have the following propositions. (Their statements and proofs can be written explicitly in terms of σ, i, e, δ, etc., but we will prefer to use the pictorial presentation.)

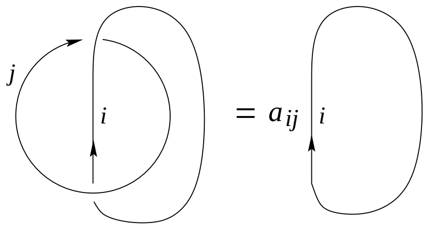
Lemma 3.1.4. *In any semisimple ribbon category we have*

(3.1.5)



Recall that by Lemma 2.4.1, $d_i \neq 0$.

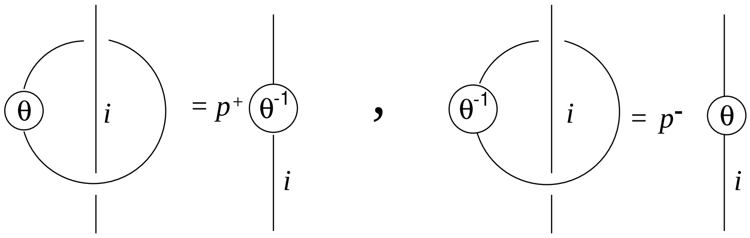
PROOF. The left hand side is an element of $\mathrm{End}(V_i) = k$, i.e., it is equal to $a_{ij}\,\mathrm{id}_{V_i}$ for some $a_{ij} \in k$. Taking a trace (i.e., closing the diagram), we obtain



The left hand side is equal to $\tilde{s}_{ij}$, while the right hand side to $a_{ij} d_i$. □

LEMMA 3.1.5. *We have the following identities*:

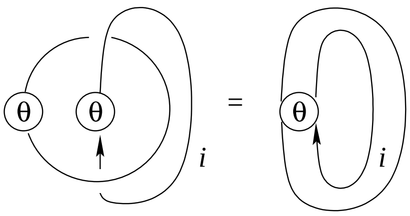
(3.1.6)



where

(3.1.7)
$$p^{\pm} := \sum_{i \in I} \theta_i^{\pm 1} d_i^2.$$

PROOF. We will consider only the case of plus sign, the case of minus sign is similar. Again the left hand side is an element of $\mathrm{End}(V_i) = k$, we take the trace of this element and multiply it with θ_i. Then, using (2.3.17), we get



Now decompose the tensor product $V_j \otimes V_i$ as in (2.4.1) to get

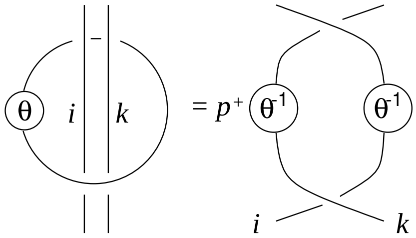
$$\theta_i \,\mathrm{tr}(\mathrm{lhs}) = \sum_j d_j \,\mathrm{tr}_{V_j \otimes V_i} \theta = \sum_{j,k} N_{ji}^k d_j d_k \theta_k.$$

Using (2.4.3) and (2.4.6), we obtain

$$\theta_i \,\mathrm{tr}(\mathrm{lhs}) = \sum_k \Big(\sum_j N_{ik^*}^{j^*} d_j\Big) d_k \theta_k = \sum_k d_i d_{k^*} d_k \theta_k = \Big(\sum_k \theta_k d_k^2\Big) d_i = p^+ d_i,$$

as desired. □

Corollary 3.1.6.



Proof. Since any object is a direct sum of simple ones, (3.1.6) holds if we replace V_i by any object V. Apply this identity for $V = V_i \otimes V_k$ and use (2.3.17). $\square$

Theorem 3.1.7. *Define the matrices* $\tilde{s} = (\tilde{s}_{ij})$, $t = (t_{ij})$ *and* $c = (c_{ij})$ *("charge conjugation matrix") by* (3.1.1) *and*

$$t_{ij} = \delta_{ij}\theta_i, \tag{3.1.8}$$

$$c_{ij} = \delta_{ij^*}. \tag{3.1.9}$$

Then we have:

$$(\tilde{s}t)^3 = p^+\tilde{s}^2, \tag{3.1.10}$$

$$(\tilde{s}t^{-1})^3 = p^-\tilde{s}^2 c, \tag{3.1.11}$$

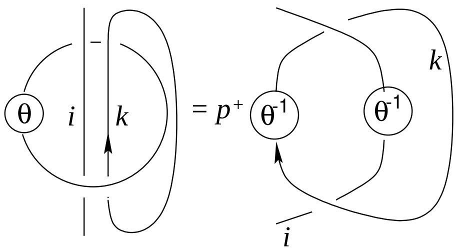
$$ct = tc, \quad c\tilde{s} = \tilde{s}c, \quad c^2 = 1, \tag{3.1.12}$$

where $p^\pm$ *are defined by* (3.1.7). *Moreover, when* $\tilde{s}$ *is invertible, we have*

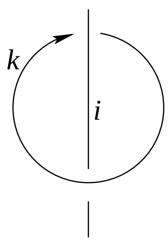
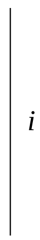
$$\tilde{s}^2 = p^+p^-c. \tag{3.1.13}$$

Proof. The fact that c commutes with $\tilde{s}$ and t follows from (3.1.3) and (2.4.5); and $c^2 = 1$ because $i^{**} = i$. To prove the non-trivial relations (3.1.10, 3.1.11), consider first the identity

(3.1.14)

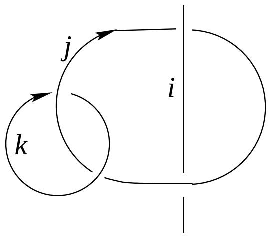


obtained from Corollary 3.1.6. The right hand side is equal to

$p^+\theta_i^{-1}\theta_k^{-1}$  $= p^+\theta_i^{-1}\theta_k^{-1}\dfrac{\tilde{s}_{ik}}{d_i}$ 

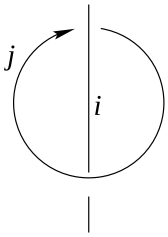
where we used Lemma 3.1.4. We can rewrite the left hand side of (3.1.14) as

$$\sum_j d_j\theta_j$$

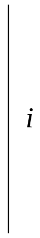


Applying Lemma 3.1.4 twice we obtain

$$\sum_j d_j\theta_j\frac{\tilde{s}_{jk}}{d_j}$$



$$=\sum_j \theta_j\tilde{s}_{jk}\frac{\tilde{s}_{ij}}{d_i}$$



This gives the identity

$$\sum_j \tilde{s}_{ij}\theta_j\tilde{s}_{jk} = p^+\theta_i^{-1}\tilde{s}_{ik}\theta_k^{-1}$$

which is equivalent to

$$\tilde{s}t\tilde{s} = p^+t^{-1}\tilde{s}t^{-1},$$

proving (3.1.10). Similarly, using the analogue of Corollary 3.1.6 with minus sign, one can prove

$$\tilde{s}t^{-1}\tilde{s} = p^-t\tilde{s}tc,$$

which implies (3.1.11).

When the matrix $\tilde{s}$ is non-singular, it is a matter of pure algebra to deduce Eq. (3.1.13) from (3.1.10)–(3.1.12). □

COROLLARY 3.1.8. *In an MTC, p^+ and p^- are non-zero.*

Now assume that the category $\mathcal{C}$ is modular, and introduce the notation

$$D := \sqrt{p^+p^-}, \quad \zeta := (p^+/p^-)^{1/6} \tag{3.1.15}$$

(assuming that they exist in k, otherwise we can always pass to a certain algebraic extension). Define the renormalized matrix

$$s := \tilde{s}/D. \tag{3.1.16}$$

Then we can rewrite the relations from Theorem 3.1.7 as follows:

$$(st)^3 = \sqrt{\frac{p^+}{p^-}}s^2 = \zeta^3 s^2, \quad s^2 = c, \quad ct = tc, \quad c^2 = 1. \tag{3.1.17}$$

Recalling the well-known description of $\mathrm{SL}_2(\mathbb{Z})$ as the group generated by the elements

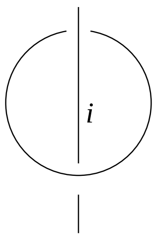
$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.1.18}$$

with relations $(st)^3 = s^2, s^4 = 1$, we see that the matrices s, t give a projective representation of $\mathrm{SL}_2(\mathbb{Z})$. (The fact that $s^2t = ts^2$ follows from $(st)^3 = s^2$.)

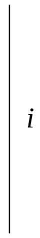
REMARK 3.1.9. Of course, one easily sees that we can replace the matrix t by t/ζ and get a true representation of $\mathrm{SL}_2(\mathbb{Z})$ rather than a projective one. In fact, since $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q}) = 0$, every projective representation of $\mathrm{SL}_2(\mathbb{Z})$ over a field k of characteristic 0 can be trivialized in some algebraic extension of k. However, we prefer not to do it: later we will show that any MTC gives rise to projective representations of more general groups (mapping class groups), of which $\mathrm{SL}_2(\mathbb{Z})$ is the simplest example, and these representations can not be trivialized. Moreover, if we renormalize t now, it will make things only worse later.

COROLLARY 3.1.10. *In an MTC, we have*:

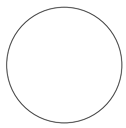
(3.1.19)



$= p^+p^-\delta_{i,0}$

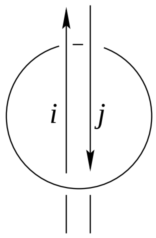


(3.1.20) $p^+p^- = \sum d_i^2 =$

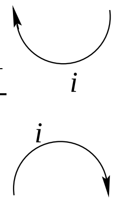


.

(3.1.21)



$= \delta_{ij}\dfrac{p^+p^-}{d_i}$



PROOF. Let us prove the first identity. As before, it suffices to prove that the traces of both sides are equal. By Lemma 3.1.4 the left hand side of (3.1.19) is equal to $\sum_j d_j\tilde{s}_{ij}/d_i \,\mathrm{id}_{V_i}$. Taking a trace, we obtain

$$\sum_j d_j\tilde{s}_{ij} = \sum_j \tilde{s}_{0j}\tilde{s}_{ij} = (\tilde{s})^2_{0i} = p^+p^-c_{0i} = p^+p^-\delta_{i,0}.$$

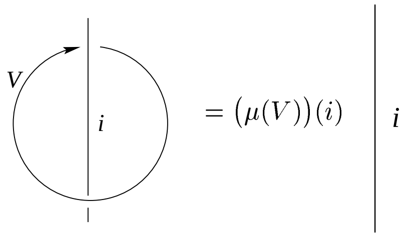
The second identity (3.1.20) easily follows from (3.1.19). The proof of (3.1.21) is similar to the above, using twice Lemma 3.1.4. □

We note that equation (3.1.20), along with the definition of s, give the following formulas for the number $D = \sqrt{p^+p^-}$:

$$D = \sqrt{\sum \dim^2 V_i} = s_{00}^{-1}. \tag{3.1.22}$$

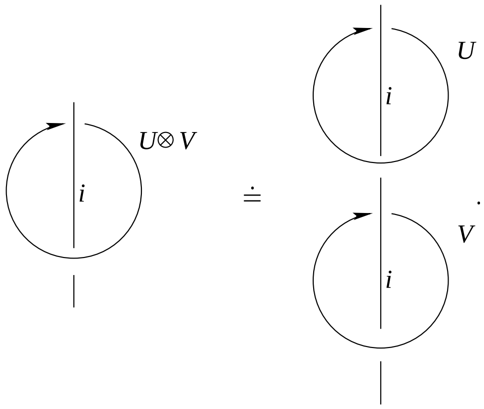
We can easily describe the Grothendieck ring of a modular tensor category. As before, let $\mathcal{C}$ be an MTC and let $K(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$ (see Definition 2.1.9). Then the algebra $K = K(\mathcal{C}) \otimes_{\mathbb{Z}} k$ is a finite dimensional commutative associative algebra with a basis $x_i = \langle V_i \rangle$, $i \in I$, and a unit $1 = x_0$. This algebra is frequently called the *fusion algebra*, or *Verlinde algebra*.

THEOREM 3.1.11. *Let $\mathcal{C}$ be an MTC, $K = K(\mathcal{C}) \otimes_{\mathbb{Z}} k$, and let $F(I)$ be the algebra of k-valued functions on the set I. Define a map $\mu\colon K \to F(I)$ by the picture:*



Then μ is an algebra isomorphism.

PROOF. It is immediate from the results of Section 2.3 that μ is an algebra homomorphism. Indeed,



Choose a basis in $F(I)$ consisting of renormalized delta-functions: $\epsilon_i(j) = \delta_{ij}/s_{0i}$. Then it follows from Lemma 3.1.4 and the obvious identity $\tilde{s}_{ij}/d_i = s_{ij}/s_{0i}$ that the map μ is given by

$$\mu(x_j) = \sum_i s_{ij}\epsilon_i. \tag{3.1.23}$$

Since the matrix s_{ij} is invertible, this completes the proof. □

The importance of this result is that it gives a new basis $\mu^{-1}(\epsilon_i)$ in K in which the multiplication becomes diagonal. For brevity, let us write $\epsilon_i \in K$ instead of $\mu^{-1}(\epsilon_i)$. Then (3.1.23) and $\epsilon_i\epsilon_j = \delta_{ij}\epsilon_i/s_{0i}$ imply that

$$x_i\epsilon_j = \epsilon_j\, s_{ij}/s_{0j}. \tag{3.1.24}$$

Comparing this with the usual formula for the multiplication in the basis x_i:

$$x_i x_j = \sum_k N_{ij}^k x_k, \tag{3.1.25}$$

we get the following proposition.

PROPOSITION 3.1.12. *For a fixed i let N_i be the matrix of multiplication by x_i in the basis $\{x_j\}$, i.e., $(N_i)_{ab} = N_{ib}^a$, and let D_i be the following diagonal matrix: $(D_i)_{ab} := \delta_{ab} s_{ia}/s_{0a}$. Then*

$$sN_i s^{-1} = D_i. \tag{3.1.26}$$

This proposition is usually formulated by saying that "the s-matrix diagonalizes the fusion rules". Another reformulation is the following. Define in K another operation, $*$ (convolution), by the formula

$$x_i * x_j = \delta_{ij} x_i / s_{0i}. \tag{3.1.27}$$

Then:

$$s(xy) = s(x) * s(y), \tag{3.1.28}$$

$$s(x * y) = s(x)s(y). \tag{3.1.29}$$

Therefore, the matrix s can be considered as some kind of a Fourier transform.

Finally, Proposition 3.1.12 immediately implies the following famous formula for the coefficients N_{ij}^k, which was conjectured in [**Ve**] and proved in [**MS1**].

THEOREM 3.1.13 (Verlinde formula).

$$N_{ij}^k = \sum_r \frac{s_{ir} s_{jr} s_{k^* r}}{s_{0r}}. \tag{3.1.30}$$

Before giving the proof, let us note that as a consequence the right hand side of (3.1.30) is a non-negative integer, which is a non-trivial and unexpected fact.

PROOF. Rewrite formula (3.1.26) as $sN_i = D_i s$, or

$$\sum_a N_{ij}^a s_{ar} = \frac{s_{ir} s_{jr}}{s_{0r}}. \tag{3.1.31}$$

Multiplying this identity by s_{rk^*} and summing over r, we get (3.1.30). □

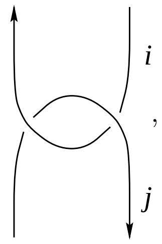
REMARK 3.1.14. If the base field $k = \mathbb{C}$, and the category $\mathcal{C}$ is Hermitian, that is, if it can be endowed with a complex conjugation functor $\overline{}$ satisfying certain compatibility conditions [**T**, Sect. II.5], then it can be shown that the matrices s, t are unitary (see [**Ki**]).

Let $\mathcal{C}$ be a modular tensor category. Recall the object $H = \bigoplus V_i \otimes V_i^* \in \mathcal{C}$ defined in (2.4.9). As was mentioned in Section 2.4, we have canonical isomorphisms $H \simeq H^*$ and $H \simeq \bigoplus V_i^* \otimes V_i$. It also follows from the definition that $\dim H = D^2 = \sum (\dim V_i)^2$.

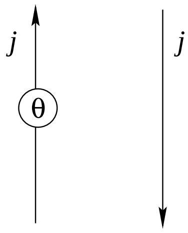
DEFINITION 3.1.15. Define elements $S, T, C \in \operatorname{End} H$ as follows. Write

$$S = \bigoplus_{i,j \in I} S_{ij}, \quad S_{ij} \colon V_j \otimes V_j^* \to V_i \otimes V_i^*$$

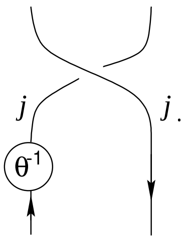
and similarly $T = \bigoplus T_{ij}$, $C = \bigoplus C_{ij}$. Then:

(3.1.32) $$S_{ij} := \frac{d_i}{D}$$ 

,

(3.1.33) $$T_{ij} := \delta_{ij}$$ 

,

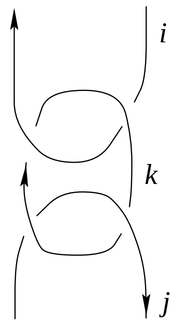
(3.1.34) $$C_{ij} := \delta_{ij^*}$$ 

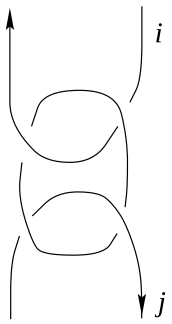
.

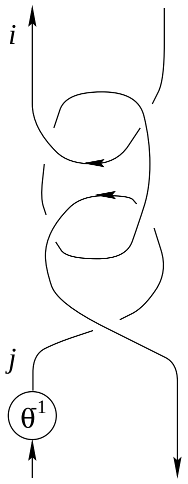
We have the following generalization of Theorem 3.1.7.

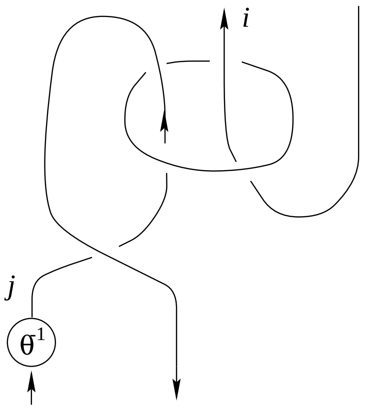
THEOREM 3.1.16. *$S^2 = C$, $C^2 = S^4 = \theta_H^{-1}$, $(ST)^3 = \sqrt{p^+/p^-}S^2$ and the element C is central in* $\operatorname{End} H$.

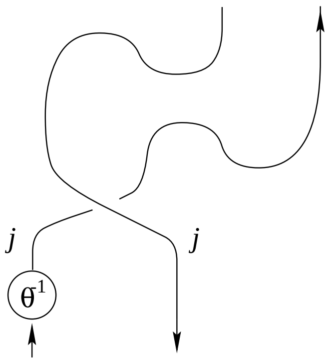
PROOF. Let us first check the identity $S^2 = C$. We have:

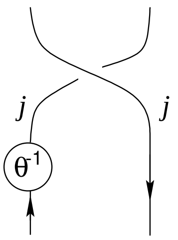
$$(S^2)_{ij} = \sum_k S_{ik}S_{kj} = \sum_k \frac{d_i}{D}\frac{d_k}{D}$$ 

$$= \frac{d_i}{D^2}$$ 

$= \frac{d_i}{D^2}$ 

$= \frac{d_i}{D^2}$ 

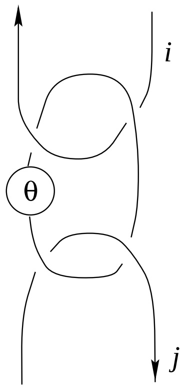
$= \delta_{ij^*}$ 

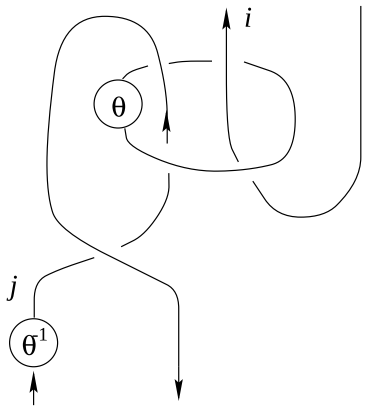
$= \delta_{ij^*}$ 

$= C_{ij}$

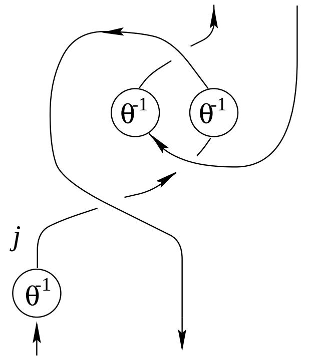
using (3.1.21) and $p^+p^- = D^2$, $d_i = d_{i^*}$.

Similarly, $(STS)_{ij} = \sum_{k,l} S_{ik}T_{kl}S_{lj} = \sum_k S_{ik}(\theta_k \otimes \mathrm{id})S_{kj}$ is equal to

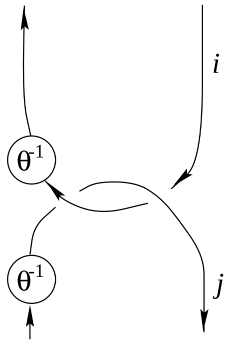
$\frac{d_i}{D^2}$ 

$= \frac{d_i}{D^2}$ 

$$= \frac{d_i}{D^2}p^+$$



$$= \sqrt{\frac{p^+}{p^-}}\frac{d_i}{D}$$



which equals $\sqrt{p^+/p^-}(T^{-1}ST^{-1})_{ij}$; now using Corollary 3.1.6 instead of Corollary 3.1.10. This proves that $(ST)^3 = \sqrt{p^+/p^-}S^2$.

Finally, using (2.3.17), it is easy to see that $(C^2)_{ij} = \delta_{ij}\theta^{-1}_{V_i \otimes V_i^*} = (\theta_H^{-1})_{ij}$. □

We cannot say that S, T give a projective representation of the modular group in H, since θ_H is not a constant. However, θ_H becomes a constant after restriction to an isotypic component of H. Equivalently, let us fix a simple object U in our category and consider the space

$$\operatorname{Hom}(U, H) = \bigoplus_{i \in I} \operatorname{Hom}(U, V_i \otimes V_i^*).$$

This is a vector space over k, and $\theta_H|_{\operatorname{Hom}(U,H)} = \theta_U \operatorname{id}_{\operatorname{Hom}(U,H)}$, $\theta_U \in k$.

THEOREM 3.1.17. *Define the maps* $S_U, T_U \colon \operatorname{Hom}(U, H) \to \operatorname{Hom}(U, H)$ *by*

$$\begin{aligned} S_U &\colon \Phi \mapsto S\Phi, \\ T_U &\colon \Phi \mapsto T\Phi. \end{aligned}$$

Then S_U, T_U *satisfy the following relations*

$$\begin{aligned} S_U^4 &= \theta_U^{-1}, \\ T_U S_U^2 &= S_U^2 T_U, \\ (S_U T_U)^3 &= \sqrt{\frac{p^+}{p^-}} S_U^2, \end{aligned}$$

and thus give a projective representation of the group $\mathrm{SL}_2(\mathbb{Z})$ *in* $\operatorname{Hom}(U, H)$.

EXAMPLE 3.1.18. Let $U = \mathbf{1}$ be the unit object in $\mathcal{C}$. Then we have a canonical identification $\operatorname{Hom}(\mathbf{1}, V_i \otimes V_i^*) \simeq k$, and thus we have a canonical basis $\{\chi_i\}$ of $\operatorname{Hom}(\mathbf{1}, H)$. In this case, the action of the modular group defined in Theorem 3.1.17 in the basis $\{\chi_i\}$ is given by s, t defined by (3.1.16) and (3.1.8).

The next theorem was proved by Vafa in the context of Conformal Field Theory.

THEOREM 3.1.19 (Vafa [**V2**]). *In any modular tensor category the numbers* θ_i *and* $\zeta = (p^+/p^-)^{1/6}$ *are roots of unity (regardless of the base field* k*).*

PROOF. We will use the following observation: if

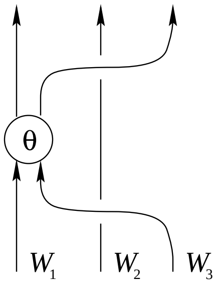
$$\prod_{j \in I} \theta_j^{M_{ij}} = 1, \qquad i \in I,$$

with a non-singular integer matrix M_{ij}, then all θ_j are roots of unity. Indeed, we can diagonalize the matrix M_{ij} by rows and columns operations.

For fixed objects W_1, W_2, W_3 in $\mathcal{C}$, define the following endomorphisms of $W_1 \otimes W_2 \otimes W_3$:

$$\theta_1 := \theta_{W_1} \otimes \mathrm{id} \otimes \mathrm{id}, \quad \theta_2 := \mathrm{id} \otimes \theta_{W_2} \otimes \mathrm{id}, \quad \theta_3 := \mathrm{id} \otimes \mathrm{id} \otimes \theta_{W_3},$$

$$\theta_{12} := \theta_{W_1 \otimes W_2} \otimes \mathrm{id}, \quad \theta_{23} := \mathrm{id} \otimes \theta_{W_2 \otimes W_3}, \quad \theta_{13} :=$$



$$\theta_{123} := \theta_{W_1 \otimes W_2 \otimes W_3}.$$

Then it is easy to check that

$$\theta_{12}\theta_{13}\theta_{23} = \theta_{123}\theta_1\theta_2\theta_3 \tag{3.1.35}$$

(this identity is sometimes called the *lantern identity*). Consider this identity for $W_1 = V_i$, $W_2 = V_i^*$, $W_3 = V_i$. It gives rise to an identity of operators in the vector space

$$U_i = \mathrm{Hom}(V_i, V_i \otimes V_i^* \otimes V_i)$$

which is non-zero since it contains $i_{V_i} \otimes \mathrm{id}_{V_i}$. We take determinant of both sides of this identity.

To compute $\det \theta_{12}|_{U_i}$, we use the decompositions of $V_i \otimes V_i^*$ and $V_j \otimes V_i$ as direct sums of simple objects:

$$V_i \otimes V_i^* = \sum_j N_{ii^*}^j V_j, \quad V_j \otimes V_i = \sum_k N_{ji}^k V_k,$$

and (2.4.4, 1.1.2). We obtain

$$\det \theta_{12}|_{U_i} = \prod_j \theta_j^{N_{ii^*}^j N_{ji}^i}.$$

Similarly, we compute the determinants of other θ's and get the identity

$$\prod_j \theta_j^{A_{ij}} = \theta_i^{4 \dim U_i},$$

where $A_{ij} = 2N_{ii^*}^j N_{ij}^i + N_{ii}^j N_{ji^*}^i$. Using that $\dim U_i = (1/3)\sum_j A_{ij} > 0$, it is easy to see that the matrix $A_{ij} - 4\delta_{ij} \dim U_i$ is nonsingular. It follows that all θ_i are roots of unity.

Since $\det t = \prod_i \theta_i$, $\det t$ is a root of unity. On the other hand, $s^4 = 1$ implies that $\det s$ is a 4th root of unity. Therefore, it follows from $(st)^3 = \zeta^3 s^2$ that ζ is a root of unity. $\square$

REMARK 3.1.20. In MTCs coming from Conformal Field Theory (CFT), when the base field is $\mathbb{C}$, one usually writes

$$\theta_i = e^{2\pi \mathrm{i} \Delta_i}, \quad \zeta = e^{2\pi \mathrm{i} c/24}. \tag{3.1.36}$$

The numbers Δ_i are called the *conformal dimensions* and c is called the (*Virasoro*) *central charge* of the theory. In this language Vafa's theorem asserts that the conformal dimensions and the central charge of the theory are rational numbers; this is one of the reasons why such CFTs are called *rational*.

One can also easily prove the following result.

THEOREM 3.1.21. *All the numbers* $s_{ij}/s_{0j} = \tilde{s}_{ij}/d_j$ *are algebraic integers.*

PROOF. By Verlinde formula (3.1.26), these numbers are the eigenvalues of the matrix N_i with integer entries. □

3.2. Example: Quantum double of a finite group

We will give the simplest example of a modular tensor category—the category of finite dimensional representations of the Hopf algebra $D(G)$, which is the quantum double of the group algebra $k[G]$ of a finite group G. It is interesting that this example appeared in two seemingly unrelated areas—the theory of characters of reductive groups over finite fields [**L5, L6**] and the orbifold constructions in Conformal Field Theory [**DVVV, KT**].

Let us first fix the notation. Let G be a finite group. Recall that its *group algebra* $k[G]$ over a field k is a Hopf algebra with a k-basis $\{x\}_{x\in G}$ and

multiplication	$x \otimes y \mapsto xy, \qquad x, y \in G,$
unit	$e \qquad$ (the unit element of G),
comultiplication	$\Delta(x) = x \otimes x, \qquad x \in G,$
counit	$\varepsilon(x) = 1,$
antipode	$\gamma(x) = x^{-1}.$

This Hopf algebra is cocommutative. A representation of $k[G]$ is the same as a representation of G. By Maschke's theorem, the category $\mathcal{R}ep_f\, k[G]$ of finite dimensional representations is semisimple.

The Hopf algebra dual to $k[G]$ is isomorphic to the function algebra $F(G)$ of the group G. It has a k-basis $\{\delta_g\}_{g\in G}$ consisting of delta functions:

$$\delta_g(x) = \delta_{g,x} = \begin{cases} 1 & \text{for } g = x, \\ 0 & \text{for } g \neq x. \end{cases}$$

It has

multiplication	$\delta_g \delta_h = \delta_{g,h}\delta_g, \qquad g, h \in G,$
unit	$1 = \sum_{g\in G} \delta_g,$
comultiplication	$\Delta(\delta_g) = \sum_{g_1 g_2 = g} \delta_{g_1} \otimes \delta_{g_2}, \qquad g \in G,$
counit	$\varepsilon(\delta_g) = \delta_{g,e},$
antipode	$\gamma(\delta_g) = \delta_{g^{-1}}.$

A representation of $F(G)$ is the same as a G-graded vector space (since $\{\delta_g\}_{g\in G}$ are projectors).

Applying Drinfeld's quantum double construction [**Dr3**] it is easy to describe explicitly the quantum double $D(G)$ of $k[G]$. As a vector space, $D(G) = F(G) \otimes_k$

$k[G]$. It is a Hopf algebra with

multiplication	$(\delta_g \otimes x)(\delta_h \otimes y) = \delta_{gx,xh}(\delta_g \otimes xy),$	$x, y, g, h \in G,$
unit	$1 = \sum_{g\in G} \delta_g \otimes e,$	
comultiplication	$\Delta(\delta_g \otimes x) = \sum_{g_1 g_2 = g}(\delta_{g_1} \otimes x) \otimes (\delta_{g_2} \otimes x),$	$g, x \in G,$
counit	$\varepsilon(\delta_g \otimes x) = \delta_{g,e},$	
antipode	$\gamma(\delta_g \otimes x) = \delta_{x^{-1}g^{-1}x} \otimes x^{-1}.$	

The Hopf algebra $D(G)$ is quasitriangular with

R-matrix $\qquad R = \sum_{g\in G}(\delta_g \otimes e) \otimes (1 \otimes g).$

(Of course, once we know the above formulas, they can be easily checked directly.)

Note that $F(G)$ and $k[G]$ embed in $D(G)$ as k-algebras and $D(G)$ is their semidirect product:

$$D(G) = F(G) \rtimes k[G], \tag{3.2.1}$$

with

$$x\delta_g x^{-1} = \delta_{xgx^{-1}} \quad \text{for } \ g, x \in G. \tag{3.2.2}$$

Let $\mathcal{R}ep_f D(G)$ be the category of finite dimensional representations of $D(G)$ as a k-algebra. By the above remarks, a representation V of $D(G)$ is the same as a G-module with a G-grading $V = \bigoplus_{g\in G} V_g$ satisfying $xV_g \subset V_{xgx^{-1}}$, $x, g \in G$. In other words, objects of $\mathcal{R}ep_f D(G)$ are finite dimensional G-equivariant vector bundles over G. We will show that the category $\mathcal{R}ep_f D(G)$ is semisimple and will describe its simple objects.

For $V \in \operatorname{Ob} \mathcal{R}ep_f D(G)$ and $v \in V$ the submodule generated by v is

$$D(G)v = \sum_{g\in G} k[G]\delta_g v = \sum_{g\in G} \bigoplus_{xgx^{-1}\in\overline{g}} xZ(g)\delta_g v,$$

where $\overline{g}$ denotes the conjugasy class and $Z(g)$ the centralizer of g in G. Note that $k[Z(g)]\delta_g v$ is an irreducible representation π of $Z(g)$. Hence

$$V_{\overline{g},\pi} := k[G]\delta_g v = \bigoplus_{xgx^{-1}\in\overline{g}} x\pi, \tag{3.2.3}$$

is an irreducible $D(G)$-module which depends only on the conjugacy class $\overline{g}$ and the isomorphism class of the irreducible representation π of $Z(g)$. The action of $D(G)$ on $V_{\overline{g},\pi}$ is given explicitly by:

$$(\delta_f \otimes h)(xv) = \delta_{f,hxgh^{-1}x^{-1}}\, hxv \quad \text{for } \ f, h, x \in G, \ v \in \pi. \tag{3.2.4}$$

This shows that the category $\mathcal{R}ep_f D(G)$ is semisimple with simple objects $V_{\overline{g},\pi}$ labeled by pairs $(\overline{g}, \pi)$, where $\overline{g} \in \overline{G}$ is a conjugacy class in G and $\pi \in \widehat{Z(g)}$ is an isomorphism class of irreducible representation of the centralizer $Z(g)$ of some element $g \in \overline{g}$ (π is independent of the choice of g).

In what follows we will use the orthogonality relations of irreducible characters of a finite group G:

$$\frac{1}{|G|}\sum_{h\in G}\operatorname{tr}_{\pi^*}(h)\operatorname{tr}_{\pi'}(hg)=\frac{\operatorname{tr}_\pi(g)}{\operatorname{tr}_\pi(e)}\delta_{\pi,\pi'},\qquad \pi,\pi'\in\widehat{G},\ g\in G, \tag{3.2.5}$$

$$\frac{1}{|Z(g)|}\sum_{\pi\in\widehat{G}}\operatorname{tr}_{\pi^*}(g)\operatorname{tr}_\pi(h)=\delta_{\overline{g},\overline{h}},\qquad h,g\in G. \tag{3.2.6}$$

Also recall that $|\overline{g}||Z(g)|=|G|$.

THEOREM 3.2.1. *$\mathcal{R}ep_f D(G)$ is a modular tensor category with simple objects $V_{\overline{g},\pi}$ labeled by $(\overline{g},\pi)$, $\overline{g}\in\overline{G}$, $\pi\in\widehat{Z(g)}$ $(g\in\overline{g})$. We have*:

$$V^*_{\overline{g},\pi}\simeq V_{\overline{g^{-1}},\pi^*}, \tag{3.2.7}$$

$$t_{(\overline{g},\pi),(\overline{g'},\pi')}=\delta_{(\overline{g},\pi),(\overline{g'},\pi')}\frac{\operatorname{tr}_\pi(g)}{\operatorname{tr}_\pi(e)}, \tag{3.2.8}$$

$$s_{(\overline{g},\pi),(\overline{g'},\pi')}=\frac{1}{|Z(g)||Z(g')|}\sum_{\substack{h\in G\\ hg'h^{-1}\in Z(g)}}\operatorname{tr}_\pi(hg'^{-1}h^{-1})\operatorname{tr}_{\pi'}(h^{-1}g^{-1}h). \tag{3.2.9}$$

The numbers $p^{\pm}$ *from* (3.1.7) *are equal to the order of* G.

The s-matrix (3.2.9) was first introduced by Lusztig [**L5**] (see also [**L6, L7**]) under the names "non-abelian Fourier transform" and "exotic Fourier transform". Then it appeared in [**DVVV**] and [**KT**] in connection with "orbifolds". Dijkgraaf, Pasquier and Roche [**DPR**] considered a generalization of the above construction which is also related to orbifolds. They introduced a quasi-Hopf algebra $D^c(G)$, depending on a cohomology class $c\in \mathrm{H}^3(G,\mathrm{U}(1))$, which reduces to $D(G)$ when $c=1$.

PROOF OF THEOREM 3.2.1. Eq. (3.2.7) follows easily from the definitions (note that $Z(g^{-1})=Z(g)$ and $\operatorname{tr}_{\pi^*}(h)=\operatorname{tr}_\pi(h^{-1})$).

To prove (3.2.8), we compute the twists θ using the results of Proposition 2.2.4 and Lemma 2.2.5. Since $\gamma^2=\mathrm{id}$, it follows that $\delta_V=\mathrm{id}$, cf. (2.2.11). Hence,

$$\theta=u^{-1}=\sum_{h\in G}\delta_h\otimes h. \tag{3.2.10}$$

As g is central in $Z(g)$, it acts as a constant $=\operatorname{tr}_\pi(g)/\operatorname{tr}_\pi(e)$ on the representation π; hence by (3.2.4), $\theta_{\overline{g},\pi}=\operatorname{tr}_\pi(g)/\operatorname{tr}_\pi(e)$.

To prove (3.2.9), we will use (3.1.2). We compute for $x,x'\in G$, $v\in\pi^*$, $v'\in\pi'$:

$$\begin{aligned}\theta_{V^*_{\overline{g},\pi}\otimes V_{\overline{g'},\pi'}}(xv\otimes x'v')&=\Delta(u^{-1})(xv\otimes x'v')\\ &=\sum_{\substack{h\in G\\ h_1h_2=h}}(\delta_{h_1}\otimes h)(xv)\otimes(\delta_{h_2}\otimes h)(x'v')\\ &=\sum_{\substack{h\in G\\ h_1h_2=h}}\delta_{h_1,hxg^{-1}x^{-1}h^{-1}}hxv\otimes\delta_{h_2,hx'g'x'^{-1}h^{-1}}hx'v'\\ &=(fxv\otimes fx'v'),\qquad \text{where } f=xg^{-1}x^{-1}x'g'x'^{-1}.\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{tr}\theta_{V^*_{\overline{g},\pi}\otimes V_{\overline{g'},\pi'}} &= \sum_{\substack{xg^{-1}x^{-1}\in\overline{g^{-1}}\\ x'g'x'^{-1}\in\overline{g'}\\ x^{-1}x'g'x'^{-1}x\in Z(g^{-1})}} \operatorname{tr}_{\pi^*}(g^{-1}x^{-1}x'g'x'^{-1}x)\operatorname{tr}_{\pi'}(x'^{-1}xg^{-1}x^{-1}x'g')\\ &= \frac{\operatorname{tr}_{\pi^*}(g^{-1})}{\operatorname{tr}_{\pi^*}(e)}\frac{\operatorname{tr}_{\pi'}(g')}{\operatorname{tr}_{\pi'}(e)}\frac{1}{|Z(g)||Z(g')|}\sum_{\substack{h\in G\\ hg'h^{-1}\in Z(g)}} \operatorname{tr}_{\pi^*}(hg'h^{-1})\operatorname{tr}_{\pi'}(h^{-1}g^{-1}h),\end{aligned}$$

which proves (3.2.9).

The computation of $p^{\pm}$ is straightforward (using (3.2.5, 3.2.6)), and is left to the reader. □

3.3. Quantum groups at roots of unity

We will show that the category of representations of a quantum group at root of unity is a modular tensor category.

We will use the notation and definitions from Section 1.3. Recall that the quantum group $U_q(\mathfrak{g})$ was defined over the field $\mathbb{C}_q$ where q is a formal variable (Definition 1.3.1). We also defined a version of the quantum group ("the quantum group with divided powers") which makes sense for $q \in \mathbb{C}$ (see (1.3.18)).

In this section we will consider the case $q = e^{\pi \mathrm{i}/m\varkappa}$ ($\varkappa \in \mathbb{Z}_+$ and m is from (1.3.17)), and we will abbreviate $U_q(\mathfrak{g})|_{q=e^{\pi\mathrm{i}/m\varkappa}}$ to $U_q(\mathfrak{g})$. As usual, we let $q^a = e^{a\pi\mathrm{i}/m\varkappa}$ for any $a \in \mathbb{Q}$. Let $\mathcal{C}(\mathfrak{g},\varkappa)$ be the category of finite dimensional representations of $U_q(\mathfrak{g})$ over $\mathbb{C}$ with weight decomposition:

$$V = \bigoplus_{\lambda\in P} V^{\lambda}, \qquad q^h|_{V^\lambda} = q^{(h,\lambda)}\,\mathrm{id}_{V^\lambda},$$
$$e_i^{(n)}(V^\lambda) \subset V^{\lambda+n\alpha_i}, \quad f_i^{(n)}(V^\lambda) \subset V^{\lambda-n\alpha_i}.$$

Note that our definition of weight decomposition is stronger than just requiring that all q^h be diagonalizable: the action of q^h does not allow one to distinguish between V^λ and $V^{\lambda+2m\varkappa\mu}$, $\mu \in P$.

THEOREM 3.3.1. *$\mathcal{C}(\mathfrak{g},\varkappa)$ is a ribbon category over $\mathbb{C}$.*

PROOF. The associatity, unit, etc., follow from the fact that $U_q(\mathfrak{g})$ is a Hopf algebra (cf. Examples 1.2.8(iii), 2.1.4). For the commutativity we need that the R-matrix can be defined over $U_q(\mathfrak{g})_{\mathbb{Z}}$, which was proved by Lusztig, see [**L2**]. □

DEFINITION 3.3.2. Let $\lambda \in P_+$ be a dominant integer weight of $\mathfrak{g}$. The *Weyl module* V_λ of $U_q(\mathfrak{g})$ is defined by

$$V_\lambda = (V_\lambda)_{\mathbb{Z}} \otimes_{\mathcal{A}} \mathbb{C},$$

where $\mathcal{A} = \mathbb{Z}[q^{\pm 1/|P/Q|}]$ and $(V_\lambda)_{\mathbb{Z}} = U_q(\mathfrak{g})_{\mathbb{Z}} v_\lambda \subset (V_\lambda)_{\mathbb{C}_q}$ is the $U_q(\mathfrak{g})_{\mathbb{Z}}$-submodule of $(V_\lambda)_{\mathbb{C}_q}$ generated by the highest weight vector.

This means that we choose a basis of $(V_\lambda)_{\mathbb{C}_q}$ such that the action of $U_q(\mathfrak{g})_{\mathbb{Z}}$ has coefficients from $\mathbb{Z}[q^{\pm 1/|P/Q|}]$ and then we can put q a complex number. This description shows that the weight subspaces of V_λ are the same as those of $(V_\lambda)_{\mathbb{C}_q}$.

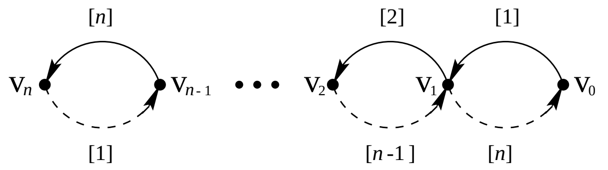
For example, let us consider first the case when $\mathfrak{g} = \mathfrak{sl}_2$. The weight lattice of $\mathfrak{sl}_2$ can be identified with $\mathbb{Z}$, so the Weyl modules are

$$V_n = \sum_{i=0}^{n} \mathbb{C} v_i\,, \qquad n \in \mathbb{Z}_+.$$

Here v_0 is the highest weight vector and $v_i = f^{(i)} v_0$. The action of $U_q(\mathfrak{sl}_2)$ is given by (recall that $[k] := (q^k - q^{-k})/(q - q^{-1})$):

$$q^h v_i = q^{n-2i} v_i, \quad e v_i = [n-i+1] v_{i-1}, \quad f v_i = [i+1] v_{i+1},$$

see the figure (f is represented by solid lines and e by dashed ones).



The coefficients of the above action are in $\mathbb{Z}[q^{\pm 1}]$, so it makes sense for $q \in \mathbb{C}^\times$. We will assume that $q \neq \pm 1$.

EXERCISE 3.3.3. Write the action of $e^{(k)}$ and $f^{(k)}$ in this basis.

Let $q = e^{\pi \mathrm{i}/\varkappa}$, $\varkappa \in \mathbb{Z}_+$. Then the module V_n may be reducible since $[k] = 0$ when $\varkappa$ divides k. For example, for $n = 3$, $\varkappa = 3$, the basis elements v_1 and v_2 span a submodule V_3'. This claim does not follow simply from the fact that V_3' is invariant under the operators e and f, because for example $e^{(3)}$ is a new operator different from $e^3/[3]!$ (since $[3] = 0$). We leave the proof as an exercise (not too difficult). The submodule V_3' is not a direct summand, hence V_3 is not semisimple.

THEOREM 3.3.4. (i) *The module V_n is irreducible for $n < \varkappa$.*
(ii) $\dim_q V_n = [n+1] = 0$ *if and only if $\varkappa$ divides $n+1$.*

The proof of this theorem is straightforward. In particular, this theorem implies that

(3.3.1) For $0 \le n \le \varkappa - 2$, V_n is irreducible and $\dim_q V_n \neq 0$,

which is obvious because in this case all q-factorials are non-zero. (In fact, one has a stronger statement: V_n is irreducible iff $n < \varkappa$ or $n = l\varkappa - 1$, $l \in \mathbb{Z}_+$, see [**AP**].)

We will need a similar result for an arbitrary semisimple finite dimensional Lie algebra $\mathfrak{g}$. Recall the number m from (1.3.17). We let $q = e^{\pi \mathrm{i}/m\varkappa}$, $\varkappa \in \mathbb{Z}$, and assume that $\varkappa \ge h^\vee$, where $h^\vee = \langle \rho, \theta \rangle + 1$ is the dual Coxeter number, ρ is the half sum of positive roots, and θ is the highest root of $\mathfrak{g}$.

THEOREM 3.3.5. $\dim_q V_\lambda = 0$ *if and only if* $\lambda + \rho \in H_{\alpha,l}$ *for some* $\alpha \in \Delta_+$, $l \in \mathbb{Z}$, *where $H_{\alpha,l}$ is the hyperplane*

$$H_{\alpha,l} := \{x \in \mathfrak{h}^* \mid \langle x, \alpha \rangle = l\varkappa\}.$$

PROOF. By (2.3.13) we have an explicit formula for $\dim_q$:

$$\dim_q V_\lambda = \operatorname{tr}_{V_\lambda} q^{2\rho} = \chi_\lambda(q^{2\rho}), \tag{3.3.2}$$

where χ_λ is the character of the representation V_λ. Here and below we use the notation $e^\lambda(q^\mu) = q^{\langle\langle \lambda, \mu \rangle\rangle}$ and extend it to $f(q^\mu)$ for $f \in \mathbb{C}[P]$, where P is the weight lattice of $\mathfrak{g}$.

We have the *Weyl formula* for χ_λ:

$$\chi_\lambda(q^{2\rho}) = \frac{1}{\delta(q^{2\rho})} \sum_{w\in W} (-1)^{l(w)} q^{\langle\langle w(\lambda+\rho),2\rho\rangle\rangle}, \tag{3.3.3}$$

where $l(w)$ is the length of w, and δ is the *Weyl denominator*

$$\delta = \prod_{\alpha\in\Delta_+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w\in W} (-1)^{l(w)} e^{w(\rho)}. \tag{3.3.4}$$

(This equality is the Weyl denominator formula.)

We can rewrite (3.3.3) as

(3.3.5)

$$\chi_\lambda(q^{2\rho}) = \frac{1}{\delta(q^{2\rho})} \sum_{w\in W} (-1)^{l(w)} q^{2\langle\langle \lambda+\rho, w(\rho)\rangle\rangle} = \frac{\delta(q^{2(\lambda+\rho)})}{\delta(q^{2\rho})} = \prod_{\alpha\in\Delta_+} \frac{[\langle\langle\alpha,\lambda+\rho\rangle\rangle]}{[\langle\langle\alpha,\rho\rangle\rangle]},$$

where, as usual, $[n]$ denotes the q-number.

Note that $\langle\langle\alpha,\rho\rangle\rangle \le \langle\langle\theta,\rho\rangle\rangle = m(h^\vee - 1) < m\varkappa$, thus the denominator is non-zero. The numerator is 0 exactly when $\lambda+\rho$ belongs to some $H_{\alpha,l}$. □

Let us define the *affine Weyl group* W^a to be the group generated by reflections with respect to the hyperplanes $H_{\alpha,l}$. It contains the Weyl group W of $\mathfrak{g}$ which is generated by reflections with respect to the hyperplanes $H_{\alpha,0}$. Recall the following standard facts (see e.g. [**K1**]).

THEOREM 3.3.6. (i) *W^a is a Coxeter group generated by the simple reflections s_i $(i = 1,\dots,\operatorname{rank}\mathfrak{g})$ and the reflection s_0 with respect to the hyperplane $H_{\theta,1}$.*

(ii) *$W^a = W \ltimes \varkappa Q^\vee$ where $Q^\vee$ is the coroot lattice embedded in $\mathfrak{h}^*$ using the form $\langle,\rangle$; $\varkappa Q^\vee$ acts on $\mathfrak{h}^*$ by translations.*

(iii) *A fundamental domain for the shifted action $w.\lambda := w(\lambda+\rho)-\rho$ of W on $\mathfrak{h}^*$ is the Weyl chamber*

$$\overline{C} = \{\lambda \in \mathfrak{h}^* \mid (\lambda+\rho,\alpha_i^\vee) \ge 0,\ (\lambda+\rho,\theta^\vee) \le \varkappa\}. \tag{3.3.6}$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h}^*$ is a line and $\overline{C}$ is the closed interval $[-1,\varkappa-1]$.

We will need a simple technical lemma.

LEMMA 3.3.7. (i) *Let $f \in \mathbb{C}[P]^{\pm W}$ be W invariant (respectively anti-invariant). Then $f(q^{2\mu})$ is (anti)symmetric with respect to the action of W^a on μ.*

(ii) *Conversely, if $f(q^{2\mu}) = f(q^{2\mu'})$ for all $f \in \mathbb{C}[P]^W$ then $\mu' = w(\mu)$ for some $w \in W^a$.*

PROOF. (i) The (anti)symmetry with respect to W is obvious. It suffices to check that $f(q^{2\mu})$ is symmetric with respect to translations from $\varkappa Q^\vee$, i.e.,

$$f(q^{2(\mu+\varkappa\alpha^\vee)}) = f(q^{2\mu}), \qquad \alpha^\vee \in Q^\vee.$$

This follows from the equation

$$e^\lambda(q^{2(\mu+\varkappa\alpha^\vee)}) = q^{2\langle\langle\lambda,\mu\rangle\rangle} q^{2\varkappa\langle\langle\mu,\alpha^\vee\rangle\rangle}$$

and the fact that $2\varkappa\langle\langle\mu,\alpha^\vee\rangle\rangle = 2\varkappa m\langle\mu,\alpha^\vee\rangle \in 2\varkappa m\mathbb{Z}$.

(ii) The proof of the converse statement is left to the reader as an exercise; the crucial step is proving that certain matrices are non-singular. We will give an example of a calculation of this sort later (see the proof of Theorem 3.3.20). □

COROLLARY 3.3.8. *If we define "$\dim_q V_\lambda$" for all $\lambda \in P$ as $\delta(q^{2(\lambda+\rho)})/\delta(q^{2\rho})$, then it is W^a-antisymmetric with respect to the shifted action on λ.*

PROOF. Follows from Lemma 3.3.7 and the fact that δ is a W-antisymmetric element in $\mathbb{C}[P]$ (see (3.3.4)). $\square$

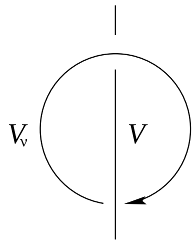
THEOREM 3.3.9. *Let $C = \{\lambda \in P_+ \mid (\lambda+\rho, \theta^\vee) < \varkappa\}$. Then for $\lambda \in C$ we have $\dim_q V_\lambda > 0$ and V_λ is irreducible.*

(In fact, one can describe exactly when V_λ is irreducible (see [**APW**]) but we will not need it.)

PROOF. The fact that $\dim_q V_\lambda > 0$ follows from Eq. (3.3.5). The irreducibility of V_λ follows from the so-called "linkage principle" (in a weak form):

> V_λ can have a subquotient with highest weight λ' only if $\lambda' = w(\lambda)$ for some $w \in W^a$.

To prove it, introduce operators $K_\nu \colon V \to V$ (where $\nu \in P_+$, V is any module) by the picture



Since K_ν is a morphism in the category $\mathcal{C}(\mathfrak{g}, \varkappa)$, it commutes with the action of $U_q(\mathfrak{g})$ on V. If v_λ is a highest weight vector in V, it is easy to see that $K_\nu(v_\lambda) = \chi_\nu(q^{2(\lambda+\rho)})v_\lambda$. Indeed, let $\{v_i\}$ and $\{v^i\}$ be dual bases in V_ν and V_ν^*. Using 1.2.8(iii), 2.3.4 and 2.2.4, we compute:

$$\begin{aligned}
K_\nu \colon v_\lambda &\overset{i}{\mapsto} \sum_i v_\lambda \otimes v_i \otimes v^i \\
&\overset{\sigma}{\mapsto} \sum_i q^{\langle\langle \lambda, \mathrm{wt}\, v_i \rangle\rangle}(v_i + \cdots) \otimes v_\lambda \otimes v^i \\
&\overset{\sigma}{\mapsto} \sum_i q^{2\langle\langle \lambda, \mathrm{wt}\, v_i \rangle\rangle} v_\lambda \otimes (v_i + \cdots) \otimes v^i \\
&\overset{\delta}{\mapsto} \sum_i q^{2\langle\langle \lambda+\rho, \mathrm{wt}\, v_i \rangle\rangle} v_\lambda \otimes (v_i + \cdots) \otimes v^i \\
&\overset{e}{\mapsto} \Big(\sum_i q^{2\langle\langle \lambda+\rho, \mathrm{wt}\, v_i \rangle\rangle}\Big) v_\lambda = \chi_\nu(q^{2(\lambda+\rho)}) v_\lambda,
\end{aligned}$$

where "$+\cdots$" denotes terms with lower weight than v_i.

The operators K_ν are central and act by constant on v_λ, therefore for subquotients we have

$$\chi_\nu(q^{2(\lambda+\rho)}) = \chi_\nu(q^{2(\lambda'+\rho)}).$$

Because all χ_ν, $\nu \in P_+$, span $\mathbb{C}[P]^W$, it follows from Lemma 3.3.7(ii) that $\lambda' = w(\lambda)$ for some $w \in W^a$.

This completes the proof of the theorem. $\square$

Note that $\mathcal{C}(\mathfrak{g}, \varkappa)$ is a very complicated category; in particular, it is not semisimple. We want to extract a semisimple part with simple objects V_λ, $\lambda \in C$. As an indication that this is possible, we give without proof the following fact (see [**AP**] and references therein).

PROPOSITION 3.3.10. *For $\lambda, \mu \in C$ we have*

$$V_\lambda \otimes V_\mu \simeq \Big(\bigoplus_{\nu \in C} N_{\lambda\mu}^{\nu} V_\nu\Big) \oplus Z$$

for some module Z with $\dim_q Z = 0$.

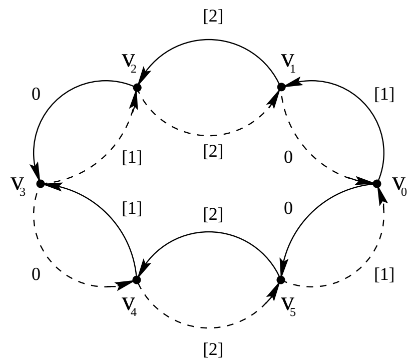
However, it is not possible to declare all modules of $\dim_q = 0$ to be 0. For example, for $\mathfrak{g} = \mathfrak{sl}_2$ we have $\dim_q(V_{\varkappa-2} \oplus V_\varkappa) = 0$, while both $V_{\varkappa-2}$ and $V_\varkappa$ are modules with non-zero q-dimension and $V_{\varkappa-2}$ is simple.

The correct construction was found by Andersen and Paradowski [**AP**] and is based on the use of an auxiliary category of tilting modules, which is interesting in its own right.

DEFINITION 3.3.11. A module T over $U_q(\mathfrak{g})$ is called *tilting* if both T and T^* have composition series with factors V_λ, $\lambda \in P_+$. Let $\mathcal{T}$ be the full subcategory of $\mathcal{C}(\mathfrak{g}, \varkappa)$ consisting of all tilting modules.

EXAMPLE 3.3.12. (i) If $\lambda \in C$ then $V_\lambda \simeq V_{\lambda^*}$ for $\lambda^* = -w_0(\lambda)$, where w_0 is the longest element in W. Therefore the module V_λ is tilting. However, for a general $\lambda \in P_+$, V_λ may not be tilting.

(ii) Let $\mathfrak{g} = \mathfrak{sl}_2$, $q = e^{\pi i/3}$, so $[3] = 0$. Consider the Weyl module V_3 over $U_q\mathfrak{sl}_2$. We add two more vectors to it and extend the action of $\mathfrak{sl}_2$ as shown in the figure for the elements e and f (f is represented by solid lines and e by dashed ones).



(The reader can define as an exercise the action of $e^{(k)}, f^{(k)}$ for $k > 0$.) We obtain a module $T = \sum_{i=0}^{5} \mathbb{C}v_i$. It is easy to see that the vectors v_0, v_1, v_2, v_3 generate a submodule isomorphic to V_3 and the factor by it is isomorphic to V_1. It can be easily shown that $T^* \simeq T$, hence the module T is tilting. Note that T is not a direct sum of V_3 and V_1.

The following important theorem was proved by Andersen and Paradowski (see [**AP**] and references therein).

THEOREM 3.3.13 ([**AP**]). (i) *The category of tilting modules $\mathcal{T}$ is closed under $*$, $\oplus$, $\otimes$ and direct summands.*

(ii) *For every $\lambda \in P_+$ there exists a unique indecomposable tilting module T_λ such that its weight subspace $(T_\lambda)^\mu$ is 0 unless $\mu \leq \lambda$ and $(T_\lambda)^\lambda = \mathbb{C}$.*

(iii) *For* $\lambda \in C$ *we have* $T_\lambda = V_\lambda$, *while for* $\lambda \notin C$ *we have* $\dim_q T_\lambda = 0$. *Hence* $\dim_q T \geq 0$ *for all* $T \in \operatorname{Ob} \mathcal{T}$.

We will not give a proof of the theorem. We only note that, for example, it is rather difficult to show that $\mathcal{T}$ is closed under $\otimes$.

COROLLARY 3.3.14. $\mathcal{T}$ *is a ribbon category.*

Note that $\mathcal{T}$ is not an abelian category since it is not closed under quotients.

DEFINITION 3.3.15. A tilting module T is called *negligible* if $\operatorname{tr}_q f = 0$ for any $f \in \operatorname{End} T$. (In particular, $\dim_q T = 0$.)

LEMMA 3.3.16. T *is negligible iff* $T = \bigoplus_{\lambda \notin C} n_\lambda T_\lambda$ *for some* $n_\lambda \in \mathbb{Z}_+$.

PROOF. Follows easily from Theorem 3.3.13. Indeed, it is enough to show that T_λ is negligible iff $\lambda \notin C$. Since T_λ is indecomposable and $\dim_{\mathbb{C}} T_\lambda < \infty$, every endomorphism f of T_λ in some homogeneous basis has the form $f = c \operatorname{id} +$upper triangular. Then $\operatorname{tr}_q f = c \dim_q T_\lambda$. $\square$

DEFINITION 3.3.17. A morphism $f\colon T_1 \to T_2$ is called *negligible* if $\operatorname{tr}_q(fg) = 0$ for all $g\colon T_2 \to T_1$.

Note that if T_1 or T_2 is negligible then any morphism $f\colon T_1 \to T_2$ is negligible.

LEMMA 3.3.18. (i) *If* T *is negligible, then so are* T^*, $T \otimes T'$ *for any* T', *and direct summands of* T.
(ii) *If* f *is negligible, then so are* f^*, $f \otimes g$, fg *and* gf *for any* g.

The proof being obvious is omitted.

DEFINITION 3.3.19. Let $\mathcal{C}^{\text{int}} \equiv \mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$ ($\varkappa \in \mathbb{Z}, \varkappa \geq h^\vee$) be the category with objects tilting modules and morphisms

$$\operatorname{Hom}_{\mathcal{C}^{\text{int}}}(V, W) = \operatorname{Hom}_{\mathcal{T}}(V, W)/\text{negligible morphisms}.$$

We list some properties of the category $\mathcal{C}^{\text{int}} \equiv \mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$:
1. $T \in \operatorname{Ob} \mathcal{T}$ is negligible iff it is isomorphic to 0 in $\mathcal{C}^{\text{int}}$.
2. $\mathcal{C}^{\text{int}}$ is a ribbon category.
3. Any object V in $\mathcal{C}^{\text{int}}$ is isomorphic to $\bigoplus_{\lambda \in C} n_\lambda V_\lambda$.
4. $\mathcal{C}^{\text{int}}$ is a semisimple abelian category and $\dim_{\mathcal{C}^{\text{int}}} V > 0$ if $V \not\simeq 0$.

These properties show that $\mathcal{C}^{\text{int}}$ is the category we wanted. It is a semisimple ribbon category with a finite number of simple objects. A natural question is whether this category is modular. We will show that the answer is positive.

THEOREM 3.3.20. $\mathcal{C}^{\text{int}}$ *is a modular tensor category with simple objects* V_λ ($\lambda \in C$),

$$s_{\lambda\mu} = |P/\varkappa Q^\vee|^{-1/2}\, \mathrm{i}^{|\Delta_+|} \sum_{w \in W} (-1)^{l(w)} q^{2\langle\langle w(\lambda+\rho), \mu+\rho\rangle\rangle}, \tag{3.3.7}$$

$$t_{\lambda\mu} = \delta_{\lambda\mu} q^{\langle\langle \lambda, \lambda+2\rho\rangle\rangle}, \tag{3.3.8}$$

and

$$D = \sqrt{|P/\varkappa Q^\vee|} \prod_{\alpha \in \Delta_+} \bigl(2 \sin(\pi\langle\alpha, \rho\rangle/\varkappa)\bigr)^{-1}, \tag{3.3.9}$$

$$\zeta = e^{2\pi \mathrm{i} c/24}, \quad c = (\varkappa - h^\vee) \dim \mathfrak{g}/\varkappa. \tag{3.3.10}$$

PROOF. The calculations in the proof of Theorem 3.3.9 and Eq. (3.1.5) give

$$\tilde{s}_{\lambda\mu} = \chi_\mu(q^{2(\lambda+\rho)}) \dim_q V_\lambda = \frac{1}{\delta(q^{2\rho})} \sum_{w\in W} (-1)^{l(w)} q^{2\langle\langle w(\lambda+\rho),\mu+\rho\rangle\rangle}.$$

To show that $\det \tilde{s} \neq 0$, we will calculate the matrix $\tilde{s}^2$. First note that if we use the formula above to extend $\tilde{s}_{\lambda\mu}$ for $\lambda, \mu \in P$, this extended matrix will be antisymmetric with respect to the shifted action of the affine Weyl group W^a:

$$\tilde{s}_{w.\lambda,\mu} = (-1)^{l(w)} \tilde{s}_{\lambda,\mu}, \qquad w \in W^a. \tag{3.3.11}$$

In particular, $\tilde{s}_{\lambda\mu} = 0$ when λ or μ are on the walls of C.

Since $\sum_{\mu\in C} \tilde{s}_{\lambda\mu}\tilde{s}_{\mu\nu}$ is symmetric with respect to the shifted action of W^a on μ and C is the fundamental domain for the action of W^a on P, we can replace the range of summation with P/W^a. Since $W^a \simeq W \ltimes \varkappa Q^\vee$, this sum equals

$$\begin{aligned} &\frac{1}{|W|} \sum_{\mu\in P/\varkappa Q^\vee} \tilde{s}_{\lambda\mu}\tilde{s}_{\mu\nu} \\ &\qquad = \frac{1}{|W|} \sum_{w,w'\in W} \sum_{\mu\in P/\varkappa Q^\vee} \delta(q^{2\rho})^{-2} (-1)^{l(w)+l(w')} q^{2\langle\langle \mu+\rho, w(\lambda+\rho)+w'(\nu+\rho)\rangle\rangle}. \end{aligned}$$

Now we need an obvious lemma.

LEMMA 3.3.21. $\displaystyle\sum_{\mu\in P/\varkappa Q^\vee} q^{2\langle\langle\mu,a\rangle\rangle} = \begin{cases} 0 & \textit{for } a \notin \varkappa Q^\vee, \\ |P/\varkappa Q^\vee| & \textit{for } a \in \varkappa Q^\vee. \end{cases}$

Note that $w(\lambda+\rho) + w'(\nu+\rho) = w(\lambda+\rho) - w'w_0(\nu^*+\rho) \in \varkappa Q^\vee$ iff $\lambda+\rho \in w^{-1}w'w_0(\nu^*+\rho) + \varkappa Q^\vee$ where w_0 is the longest elment in W. But since both λ and ν^* are in C, which is a fundamental domain of W^a, this is only possible if $\lambda+\rho = \nu^*+\rho$, $w^{-1}w' = w_0$. Therefore

$$\sum_{\mu\in C} \tilde{s}_{\lambda\mu}\tilde{s}_{\mu\nu} = \frac{|P/\varkappa Q^\vee|}{\delta(q^{2\rho})^2} (-1)^{l(w_0)} \delta_{\lambda,\nu^*}.$$

This number is non-zero, hence $\det \tilde{s} \neq 0$.

This also gives D since $(\tilde{s}^2)_{\lambda\nu} = D^2\delta_{\lambda,\nu^*}$. Formula (3.3.8) for the twist follows directly from Example 2.2.6. The rest of the proof is straightforward and is left to the reader. □

EXAMPLE 3.3.22. When $\mathfrak{g} = \mathfrak{sl}_2$, we have:

$$s_{\lambda\mu} = \sqrt{\frac{2}{\varkappa}} \sin\left(\pi \frac{(\lambda+1)(\mu+1)}{\varkappa}\right), \qquad 0 \le \lambda, \mu \le \varkappa - 2.$$

The arguments of Theorem 3.3.20 can be repeated for $q = e^{\pi \mathrm{i}/m\varkappa}$, $\varkappa \in \mathbb{Q}$, but in this case the matrix $\tilde{s}$ may be degenerate.

Note that the formulas for the matrices s, t coincide with the Kac–Peterson formula [**KP**] for the modular transformations of characters of the affine Lie algebra $\hat{\mathfrak{g}}$ when $q = e^{\pi \mathrm{i}/m\varkappa}$ (their matrix T corresponds to the matrix t/ζ in our notations). This fact will be explained later.

Finally, let us discuss the Verlinde algebra for $\mathcal{C}^{\mathrm{int}}$. Let $\mathcal{V} = K(\mathcal{R}ep_f(\mathfrak{g}))\otimes\mathbb{C}$ be the complexified Grothendieck ring of $\mathcal{R}ep_f(\mathfrak{g})$; similarly, denote $\mathcal{V}_k = K(\mathcal{C}^{\mathrm{int}})\otimes\mathbb{C}$ (where, as before, $\varkappa = k + h^\vee$).

PROPOSITION 3.3.23. *The Verlinde algebra* $\mathcal{V}_k$ *is the quotient of* $\mathcal{V}$*, namely,* $\mathcal{V}_k = \mathcal{V}/\mathcal{I}_k$*, where* $\mathcal{I}_k \subset \mathcal{V}$ *is the linear span of* $\langle V_\lambda \rangle - (-1)^{l(w)} \langle V_{w.\lambda} \rangle$ *for* $\lambda \in P_+, w \in W^a, w.\lambda \in P_+$.

PROOF. The construction given in Theorem 3.1.11 defines a surjective map $\mu \colon \mathcal{V} \to \mathcal{V}_k$. It follows from Weyl character formula that $\mathcal{I}_k \subset \ker \mu$. On the other hand, it follows from Theorem 3.3.6(iii) that $\dim \mathcal{V}/\mathcal{I}_k = |C| = \dim \mathcal{V}_k$. □

EXERCISE 3.3.24. (i) Show that for $\mathfrak{g} = A_n$, the ideal $\mathcal{I}_k$ is the linear span of $\langle V_\lambda \rangle$ for $\lambda \in P_+, (\lambda + \rho, \theta^\vee) = \varkappa$.

(ii) Show that for $\mathfrak{g} = E_8$ this is not so.

(iii) Show that the fusion rules for $U_q(\mathfrak{sl}_2)$ for $q = e^{\pi \mathrm{i}/(k+2)}$ are given by

$$\langle V_m \rangle \langle V_n \rangle = \sum_l N_{mn}^l \langle V_l \rangle,$$

where

$$N_{mn}^l = \begin{cases} 1 & \text{for } |m-n| \le l \le m+n,\ l \le 2k-(m+n),\ l+m+n \in 2\mathbb{Z}, \\ 0 & \text{otherwise} \end{cases}$$

(cf. Example 2.1.10).

CHAPTER 4

3-dimensional Topological Quantum Field Theory

In this chapter, following the ideas of [**RT2**] and [**T**], we will show that the algebraic formalism of modular tensor categories is closely related with the topology of 3-manifolds. In particular, we will show that every MTC gives rise to an invariant of compact 3-manifolds. Historically, this was one of the main motivations for the theory of modular tensor categories. We will quote without proofs several important results about 3-manifolds which are used in this construction. The reader is referred to an excellent introductory text [**PS**] and references therein for proofs and detailed discussion.

In this chapter, by "manifold" we mean an oriented compact topological manifold, possibly with boundary; by a "closed manifold" we will mean a manifold without boundary. All maps between manifolds will be continuous and preserving orientation, unless stated otherwise. By $\simeq$ we denote homeomorphism of manifolds, and by $\overline{M}$ we denote the manifold M with reversed orientation. Finally, for an oriented manifold M we endow its boundary ∂M with an orientation in the standard way: $(v_1, \dots, v_k)$ is a positive reper for ∂M if $(v_1, \dots, v_k, n)$ is a positive reper for M, where n is the outward normal vector to ∂M.

4.1. Invariants of 3-manifolds

In this section we construct invariants of closed 3-manifolds. This construction is based on the notion of surgery, which itself is a special case of the operation of gluing, described in the lemma below.

LEMMA 4.1.1. *Let M_1 and M_2 be two manifolds of the same dimension. Let N_1 be a connected component of ∂M_1, N_2 a connected component of ∂M_2 and $f\colon N_1 \xrightarrow{\sim} N_2$ be an orientation reversing homeomorphism.*

(i) *Define $M_1 \cup_f M_2$ by*

$$M_1 \cup_f M_2 := (M_1 \sqcup M_2)/\{(x,y) \mid y = f(x), x \in N_1\}.$$

Then $M_f \equiv M_1 \cup_f M_2$ is again a manifold. We will say that M_f is obtained by gluing M_1 and M_2 using the identification f of their boundary components.

(ii) *If $f' = f \circ \varphi$ for some $\varphi\colon N_1 \xrightarrow{\sim} N_1$ which extends to $M_1 \xrightarrow{\sim} M_1$, then $M_{f'} \simeq M_f$.*

(iii) *M_f depends only on the isotopy class of f, i.e., it does not change when we continuously deform f.*

PROOF. Only (iii) is not immediately obvious. It follows from the next claim: If $f' \sim f$ then one can write $f' = f \circ \varphi$ for some $\varphi\colon M_1 \xrightarrow{\sim} M_1$ such that $\varphi \neq \mathrm{id}$ only in a neighborhood of N_1. Indeed, it suffices to prove this in the case where M_1 is the cylinder $[0,1] \times N_1$, in which case it is obvious. $\square$

EXAMPLES 4.1.2. Let T_1 and T_2 be two solid tori. The boundary of a solid torus is a 2-dimensional torus which can be thought of as a rectangle with identified opposite sides. The sides of the rectangle give two 1-cycles which form a basis in homology (see Figure 4.1).

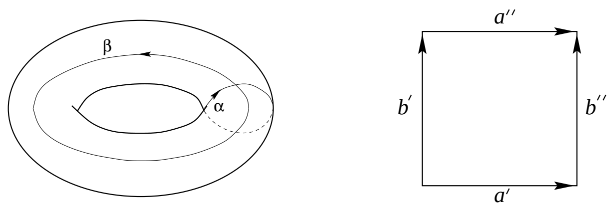
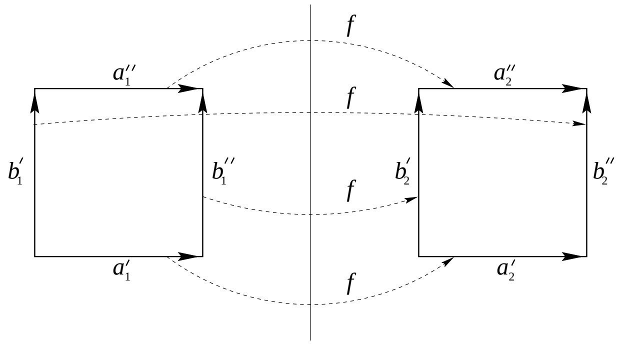


FIGURE 4.1. A torus obtained by gluing opposite sides of a rectangle.

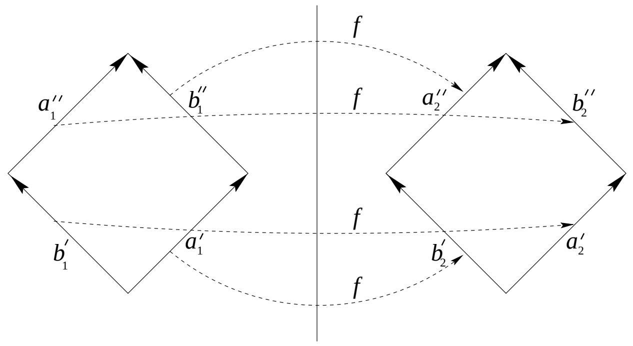
Below we will consider orientation reversing homeomorphisms $f\colon T_1 \xrightarrow{\sim} T_2$ as acting on the corresponding rectangles. We will consider two examples.

(i) f identifies a_1' with a_2', a_1'' with a_2'', b_1' with b_2'' and b_1'' with b_2', i.e., it is a reflection in the vertical line:



Then $M_f = T_1 \cup_f T_2 \simeq S^2 \times S^1$. Indeed, M_f has a well-defined projection on the circle β which is a common parallel of both tori (on the figure above, β is represented by any of the intervals $b_1, \dots, b_2''$). The fiber of this projection is S^2, obtained by gluing two copies of a disk along the boundary. Finally, it can be shown that this is indeed a direct product.

(ii) f identifies a_1' with b_2', a_1'' with b_2'', b_1' with a_2' and b_1'' with a_2'', i.e., it is a reflection in the diagonal:



Then we claim that $T_1 \cup_f T_2 \simeq S^3$. The easiest way to visualize this is to note that the complement of a solid torus in $S^3 = \mathbb{R}^3 \cup \infty$ is again a solid torus. Hence S^3 is a union of two solid tori with boundaries identified by the map f.

In general, the spaces $T_1 \cup_f T_2$ obtained by gluing two solid tori are called *lens spaces*, see e.g. [**PS**]. Since such a lens space is defined by the isotopy class of f, it is natural to study the group of isotopy classes of homeomorphisms of a 2-dimensional torus.

THEOREM 4.1.3. *Let T_0 be a solid torus. Fix a standard basis α, β of $\mathrm{H}_1(\partial T_0, \mathbb{Z})$, as shown in Figure* 4.1. (*Given a solid torus T_0, α is defined uniquely up to a sign, but β is not.*) *Then*:

(i) *For a map $f\colon \partial T_0 \to \partial T_0$, denote by f_* its action in $\mathrm{H}_1(\partial T_0, \mathbb{Z}) \simeq \mathbb{Z}^2$. Then $f \mapsto f_*$ is an isomorphism of the group $\Gamma_{1,0}$ of isotopy classes of homeomorphisms $\partial T_0 \xrightarrow{\sim} \partial T_0$ with $\mathrm{SL}_2(\mathbb{Z})$.* [1]

(ii) *The homeomorphisms of ∂T_0 corresponding to the matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ from $\mathrm{SL}_2(\mathbb{Z})$ $(k \in \mathbb{Z})$ can be extended to the whole T_0.*

PROOF. (i) For every f, the induced automorphism f_* of $H_1(\partial T_0, \mathbb{Z})$ preserves the intersection form. Hence the matrix of f_* in the basis α, β belongs to $\mathrm{SL}_2(\mathbb{Z})$. Surjectivity of this map is obvious if we write $\partial T_0 = \mathbb{R}^2/\mathbb{Z}^2$ and note that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$; injectivity can be proved by standard topological arguments, see details in [**PS**].

(ii) The homeomorphism of T_0 that extends the transformation $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is the so-called *Dehn twist*: cut the solid torus T_0 along the disk with boundary α, twist it k times and glue it back (see Figure 4.2 for $k = 1$). $\square$

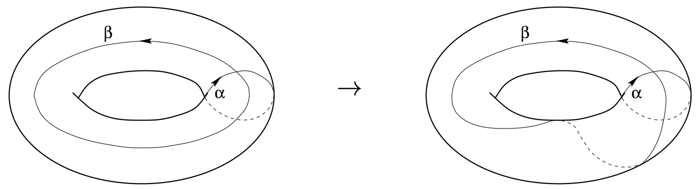


FIGURE 4.2. Dehn twist.

Let us also remind the following standard fact:

THEOREM 4.1.4. *The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements*

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

with the defining relations $s^4 = 1, (st)^3 = s^2$.

[1]The notation $\Gamma_{1,0}$ is chosen because later we will introduce more general groups $\Gamma_{g,n}$, corresponding to surfaces of genus g with n boundary components.

Let us restrict our attention to 3-manifolds. In this case, there are two classical constructions allowing to get any manifold by gluing of simpler pieces: Heegaard splitting and surgery along links. The approach of this chapter is based on surgery; the Heegaard splitting will be discussed later.

DEFINITION 4.1.5. Let L be a link in S^3. Denote by T_i some small tubular neighborhood of the i-th component of L. Each T_i is a solid torus. Let T_0 be a fixed standard solid torus. Suppose we are given (orientation preserving) homeomorphisms $f_i\colon \partial T_i \xrightarrow{\sim} \partial T_0$. Define a new 3-manifold $M_{L,f}$, called a *surgery* of S^3 along the link L, by the formula

$$M_{L,f} := \left(S^3 \setminus (\cup_i T_i^{\mathrm{int}})\right) \cup_{f_i} (T_0)^N, \tag{4.1.1}$$

where N is the number of components of L and "int" denotes interior.

In other words, we are cutting from S^3 a number of solid tori (possibly linked) and pasting instead new solid tori, but possibly in some twisted way.

THEOREM 4.1.6. *Any connected closed 3-manifold can be obtained as a surgery of S^3 along a link.*

Note that the definition above requires that we specify not only the link but also the attachment maps f_i. Our next observation is that there is a canonical way to construct f_i if the link is framed (= ribbon). Let L be a framed link in S^3 which is directed, i.e., each component of L has an arrow. Let T_i be a tubular neighborhood of the ith component, as before. Then the link L determines cycles α_i, β_i in ∂T_i. Instead of giving the formal definition, we draw a picture — see Figure 4.3, where the cycle β_i winds the same way as the ith component of L.

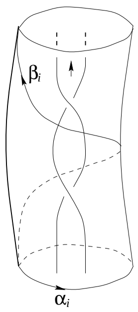


FIGURE 4.3. A ribbon link and the corresponding cycles α_i, β_i in its tubular neighborhood.

Let $f_i\colon \partial T_i \xrightarrow{\sim} \partial T_0$ be the homeomorphism which sends the cycle α_i to $-\beta$ and β_i to α. (By Theorem 4.1.3, such f_i exists and is unique up to isotopy.)

DEFINITION 4.1.7. Let L be a directed framed link in S^3. Then we define $M_L = M_{L,f_i}$ to be the surgery of S^3 along the link L with the attachment maps f_i described above.

LEMMA 4.1.8. *M_L does not depend on the choice of directions of the components of L.*

PROOF. Follows from the fact that $\alpha \mapsto -\alpha$, $\beta \mapsto -\beta$ can be extended to the whole T_0 (see Theorem 4.1.3(ii)). $\square$

This construction gives a 3-manifold M_L for any framed link L in S^3.

THEOREM 4.1.9. *Any connected compact oriented 3-manifold M without boundary can be obtained as M_L for some framed link L in S^3.*

EXAMPLES 4.1.10. (i) When $L = \emptyset$ is the empty link, then $M = S^3$.

(ii) Let $L = \bigcirc$ (no twisting). Then, by Example 4.1.2(ii), $S^3 \setminus T_1 = T_2$ is a solid torus. Then gluing T_0 and T_2 we obtain $M_L = S^2 \times S^1$ as in Example 4.1.2(i).

(iii) When $L = \bigcirc^1 =$  (framing 1), then $M_L = S^3$. This is by no means obvious; we leave it as an exercise to the reader to deduce it from Example 4.1.2(ii) and Theorem 4.1.3(ii).

The next question is when two different links L and L' in S^3 give the same 3-manifold. The answer (highly non-trivial) was found by Kirby. We present it here in a form due to Fenn and Rourke (see [**PS**] and references therein).

THEOREM 4.1.11 (Kirby calculus). *$M_L \simeq M_{L'}$ iff L' can be obtained from L by a sequence of* Kirby–Fenn–Rourke moves *shown in Figure* 4.4 *below and the same with overcrossing and undercrossing interchanged.* (*The number of strands can be arbitrary, including zero.*)

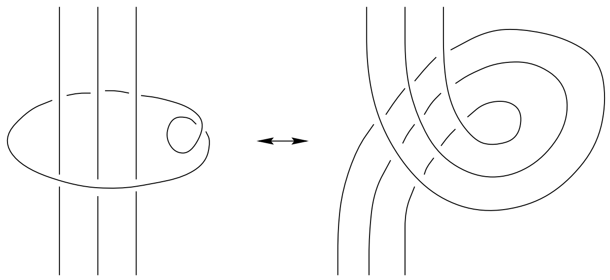


FIGURE 4.4. Kirby–Fenn–Rourke moves.

The proof of this theorem is quite difficult and will not be given here.

THEOREM 4.1.12 (Reshetikhin–Turaev [**RT2**]). *Let $\mathcal{C}$ be a MTC and L be a framed link in $\mathbb{R}^3 \subset S^3$. Define the number $\tau(M_L)$ by the formula*

$$\tau(M_L) := D^{-|L|-1} F^{-1}(L) \left(\frac{p^+}{p^-}\right)^{\sigma(L)/2}, \tag{4.1.2}$$

where $|L|$ is the number of components of L, D is from (3.1.15), *$p^{\pm}$ from* (3.1.7), *$\sigma(L)$ is the so-called wreath number of L* (*see* [**RT2**] *for its definition*), *and $F^{-1}(L)$ is the Reshetikhin–Turaev invariant of the link L from Theorem* 2.3.11 (*we use the convention* (3.1.4): *we take sum over all possible labelings of uncolored strands, with labeling V_i taken with weight d_i*).

Then $\tau(M_L)$ is an invariant of the 3-manifold M_L, i.e., it does not depend on the link L:

$$\tau(M_L) = \tau(M_{L'}) \quad \textit{if } M_L \simeq M_{L'}.$$

This invariant can be defined for an arbitrary 3-manifold M and is called *Reshetikhin–Turaev invariant*.

PROOF. If we assume that $p^+/p^- = 1$, then the proof immediately follows from Lemma 3.1.5. The general case is not much more difficult; we refer the reader to the original papers for details. □

EXAMPLES 4.1.13. (i) $\tau(S^3) = D^{-1}$, cf. 4.1.10(i).
(ii) $\tau(S^2 \times S^1) = \tau(M_{\bigcirc}) = F^{-1}(\bigcirc)D^{-2} = 1$, cf. 4.1.10(ii) and (3.1.20).

REMARK 4.1.14. One easily sees that if L_1, L_2 are two links in $\mathbb{R}^3$ which are not linked with each other (i.e., they can be separated by a plane), then $M_{L_1 \sqcup L_2} = M_{L_1} \# M_{L_2}$, where we denote by $\#$ the connected sum. Thus, RT invariants satisfy the following multiplicativity property:

$$\tau(M_1 \# M_2) = D\,\tau(M_1)\tau(M_2). \tag{4.1.3}$$

Reshetikhin–Turaev invariants can be generalized to manifolds with ribbon links inside. Let us assume that we have a partially $\mathcal{C}$-colored ribbon link in $\mathbb{R}^3$ which is presented as a union $\Omega \sqcup L$; here Ω is a $\mathcal{C}$-colored ribbon link (which may contain coupons), and L is an uncolored framed link without coupons. Performing surgery along L, we get the manifold M_L with a ribbon link Ω_L inside. Then Theorem 4.1.11 can be generalized to this situation as follows.

THEOREM 4.1.15 (Reshetikhin–Turaev [**RT2**]). (i) *Any connected closed 3-manifold M with a ribbon link Ω inside can be obtained as (M_L, Ω_L).*

(ii) $(M_L, \Omega_L) \simeq (M_{L'}, \Omega'_{L'})$ *iff the link $\Omega' \cup L'$ can be obtained from $\Omega \cup L$ by a sequence of Kirby–Fenn–Rourke moves, where the annulus in Figure* 4.4 *is a part of L.*

THEOREM 4.1.16 (Reshetikhin–Turaev [**RT2**]). *Let $\mathcal{C}$ be a MTC and let $\Omega \cup L$ be a partially colored framed link as above. Then*

$$\tau_{\mathcal{C}}(M_L, \Omega_L) := D^{-|L|-1} F^{-1}(L \cup \Omega) \left(\frac{p^+}{p^-}\right)^{\sigma(L)/2} \tag{4.1.4}$$

is an invariant, i.e., depends only on (M_L, Ω_L).

The proof is similar to the proof of the previous theorem. The explicit computation of Reshetikhin–Turaev invariants is not difficult for lens spaces (cf. 4.1.2(iii)) but in general is very complicated.

One may wonder is there a reason why MTCs give invariants of 3-manifolds. The reason is that MTCs give 3-dimensional Topological Quantum Field Theories.

4.2. Topological Quantum Field Theory

In this section, we introduce the second main hero of our lectures—topological quantum field theory (TQFT). (The first hero was the modular tensor category.) This notion was introduced in [**W1**], [**At**] and studied extensively in many papers, such as [**Q**]. As before, "manifold" = "compact topological oriented manifold with boundary". We also fix a base field k of characteristic zero; all vector spaces considered in this chapter will be vector spaces over k.

DEFINITION 4.2.1. A $(d+1)$-*dimensional Topological Quantum Field Theory* ($d+1$ D TQFT) is the following collection of **data**:

(a) To any d-manifold N without boundary assigned a finite dimensional vector space $\tau(N)$.

(b) To any $(d+1)$-manifold M (possibly with boundary) assigned a vector $\tau(M)$ in the vector space $\tau(\partial M)$.

(c) To any homeomorphism of d-manifolds $f\colon N \xrightarrow{\sim} N'$ assigned an isomorphism of vector spaces $f_*\colon \tau(N) \xrightarrow{\sim} \tau(N')$.

(d) Functorial isomorphisms

$$\tau(\overline{N}) \xrightarrow{\sim} \tau(N)^*, \tag{4.2.1}$$

$$\tau(\emptyset) \xrightarrow{\sim} k, \tag{4.2.2}$$

$$\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2), \tag{4.2.3}$$

where $\overline{N}$ is the manifold N with the opposite orientation, which are compatible in an obvious sense with each other and with the commutativity, associativity and unit morphisms.

These data are required to satisfy the following **axioms**:

(i) *Functoriality.* If $f\colon M \xrightarrow{\sim} M'$ is a homeomorphism of $(d+1)$-manifolds then $(f|_{\partial M})_*(\tau(M)) = \tau(M')$.

(ii) *Gluing axiom.* Let M be a $(d+1)$-manifold, $\partial M = N_1 \sqcup N_2 \sqcup N_3$, and $f\colon N_1 \xrightarrow{\sim} \overline{N}_2$ be a homeomorphism. Let $M' = M/f$ be the $(d+1)$-manifold obtained from M by identifying N_1 with $\overline{N}_2$ using f, i.e., by gluing N_1 to N_2. Then $\tau(M')$ is equal to the image of $\tau(M)$ via the map $\tau(N_1) \otimes \tau(N_2) \otimes \tau(N_3) \to \tau(N_2)^* \otimes \tau(N_2) \otimes \tau(N_3) \to \tau(N_3)$.

(iii) *Normalization axiom.* Let I be an interval and N be a d-manifold. Then $\partial(I \times N) = N \sqcup \overline{N}$ and we require that $\tau(I \times N)$ equals the image of $\mathrm{id}_{\tau(N)}$ in $\tau(\overline{N}) \otimes \tau(N) \simeq \tau(N)^* \otimes \tau(N)$.

(iv) *Normalization axiom.* $\tau(S^d) = k$ and $\tau(B^{d+1}) = 1 \in k$, where B^{d+1} is the unit ball in $\mathbb{R}^{d+1}$, and $S^d = \partial B^{d+1}$ is the d-sphere.

This completes the definition.

REMARK 4.2.2. For a more pedantic reader, we list here all the compatibility conditions mentioned in part (d) above. Functoriality of the morphisms (4.2.1)–(4.2.3) means that

$$(f \sqcup g)_* = f_* \otimes g_*, \quad \overline{f}_* = (f_*^{-1})^*,$$

where for $f\colon N_1 \xrightarrow{\sim} N_2$ we denote by $\overline{f}$ the same map considered as a homeomorphism $\overline{N_1} \xrightarrow{\sim} \overline{N_2}$, and for a map of vector spaces $\varphi\colon V_1 \to V_2$ we denote by φ^* the adjoint map $\varphi^*\colon V_2^* \to V_1^*$.

The compatibility conditions are as follows: to the canonical homeomorphisms

$$N \sqcup \emptyset \xrightarrow{\sim} N,$$

$$N_1 \sqcup N_2 \xrightarrow{\sim} N_2 \sqcup N_1,$$

$$(N_1 \sqcup N_2) \sqcup N_3 \xrightarrow{\sim} N_1 \sqcup (N_2 \sqcup N_3)$$

are assigned the usual isomorphisms of vector spaces

$$\tau(N) \otimes k \xrightarrow{\sim} \tau(N),$$

$$\tau(N_1) \otimes \tau(N_2) \xrightarrow{\sim} \tau(N_2) \otimes \tau(N_1),$$

$$(\tau(N_1) \otimes \tau(N_2)) \otimes \tau(N_3) \xrightarrow{\sim} \tau(N_1) \otimes (\tau(N_2) \otimes \tau(N_3)).$$

Also, to the canonical homeomorphisms

$$\emptyset \xrightarrow{\sim} \overline{\emptyset}, \quad \overline{N_1 \sqcup N_2} \xrightarrow{\sim} \overline{N_1} \sqcup \overline{N_2}$$

are assigned the usual isomorphisms

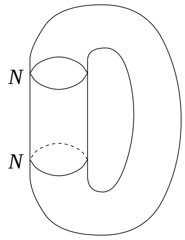
$$k^* \xrightarrow{\sim} k, \quad (\tau(N_1) \otimes \tau(N_2))^* \xrightarrow{\sim} \tau(N_1)^* \otimes \tau(N_2)^*.$$

This completes the list of compatibility conditions.

The axioms of TQFT show in particular that for every $(d+1)$-dimensional manifold M without boundary, the number $\tau(M) \in \tau(\emptyset) = k$ is a topological invariant of M. However, not every invariant can be extended to a TQFT (see [**T**], [**Fu**]).

Note that, if M is a $(d+1)$-manifold with $\partial M = \overline{N_1} \sqcup N_2$, then $\tau(M) \in \tau(N_1)^* \otimes \tau(N_2) \simeq \mathrm{Hom}_k(\tau(N_1), \tau(N_2))$. The gluing axiom says that gluing of two such manifolds M gives rise to multiplying the corresponding operators.

Another example: $M = S^1 \times N$ is obtained by gluing the bases of the cylinder $I \times N$ and it is easy to see that $\tau(S^1 \times N) = \mathrm{tr}\, \tau(I \times N) = \dim \tau(N)$. Compare this with (2.3.12):



Next, we have the following important result.

THEOREM 4.2.3. *In any TQFT, we have*:

(i) $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$.

(ii) *For* $f\colon N_1 \xrightarrow{\sim} N_2$ *the isomorphism* $f_*\colon \tau(N_1) \xrightarrow{\sim} \tau(N_2)$ *depends only on the isotopy class of* f.

PROOF. For a homeomorphism $f\colon N_1 \xrightarrow{\sim} N_2$, let M_f be a cylinder $N_1 \times I$ with boundary ∂M_f identified with $\overline{N_1} \sqcup N_2$ using f. Then it follows from the normalization axiom that $\tau(M_f) = f_*\colon \tau(N_1) \to \tau(N_2)$. To prove (i), it suffices to notice that the cylinder M_{fg} is homeomorphic to the cylinder obtained by gluing M_f with M_g. To prove (ii), note that if f is isotopic to identity then M_f is homeomorphic to the trivial cylinder $M_{\mathrm{id}} = N_1 \times I$; thus, by functoriality, $\tau(M_f) = \mathrm{id}$. We leave the details to the reader. □

COROLLARY 4.2.4. *For every d-manifold* N, *a TQFT gives a representation of the group of isotopy classes of homeomorphisms* $f\colon N \xrightarrow{\sim} N$ *in the space* $\tau(N)$.

This group is called the *mapping class group*. We will return to this observation later.

4.3. 1+1 dimensional TQFT

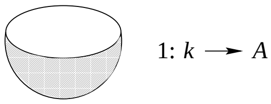
In this section, we consider a toy model of a TQFT: a (1+1) D TQFT. This case is rather trivial; however, the understanding of this example will be crucial for some of our future constructions.

THEOREM 4.3.1. *1+1 D TQFTs are in one-to-one correspondence with finite dimensional* Frobenius algebras, *i.e., commutative associative algebras A with unit and with a linear map* $\mathrm{tr}\colon A \to k$ *such that the bilinear form* $\mathrm{tr}(ab)$ *is non-degenerate.*

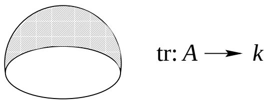
PROOF. 1. *Every 1+1 D TQFT gives a Frobenius algebra.*
There is only one 1-dimensional connected closed manifold: the circle S^1 and $\overline{S^1} = S^1$. Let A be the vector space $\tau(S^1)$:

$$A = \tau(\bigcirc).$$

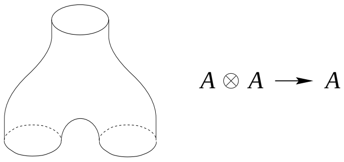
The disk D^1 has a boundary $\partial D^1 = S^1$, hence it gives a vector $\tau(D^1) \in A$ which we denote by 1. Since $\tau(\emptyset) := k$, we can consider $\tau(D^1)$ as a map from k to A giving the unit:



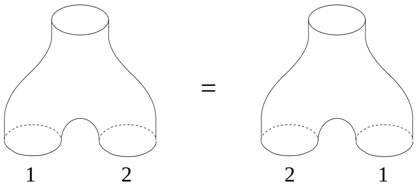
On the other hand, since $\overline{S^1} = S^1$, we can also write $\partial D^1 = \overline{S^1} \sqcup \emptyset$ which gives a map $\mathrm{tr}\colon A \to k$:



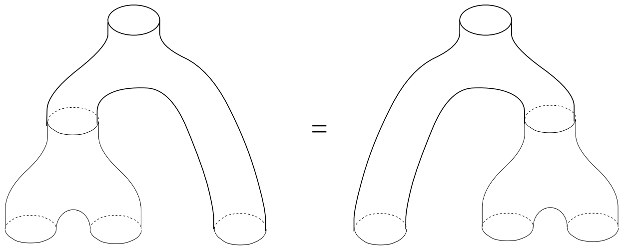
The pair of pants (also called a *trinion*) gives a map $A \otimes A \to A$ which is the multiplication $a \otimes b \mapsto ab$:



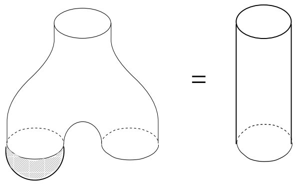
Now the commutativity of multiplication follows from the fact that the flipping of the legs is a homeomorphism:



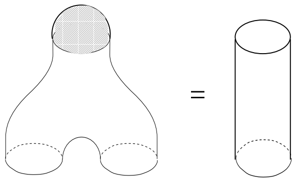
Similarly, associativity follows from the picture



and the gluing axiom. The unit property of 1 is a consequence of the gluing and normalization axioms:



Similarly, the non-degeneracy of the bilinear form $\mathrm{tr}(ab)$ follows from

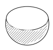


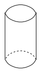
This shows that A is a Frobenius algebra.

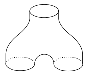
2. *Every Frobenius algebra gives a 1+1 D TQFT.*
Let A be a Frobenius algebra. To the circle S^1 we assign $\tau(S^1) := A$ and to a disjoint union of circles a tensor product of A with itself: $\tau(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n}) := A^{\otimes n}$. For $f\colon S^1 \xrightarrow{\sim} S^1$ we let $f_* = \mathrm{id}$ and for $g\colon S^1 \xrightarrow{\sim} \overline{S^1}$ let g_* be the isomorphism $A \xrightarrow{\sim} A^*$ given by the non-degenerate bilinear form $\mathrm{tr}(ab)$.

It is clear how to define $\tau(\text{disk}) \in A$, $\tau(\text{cylinder}) \in A^* \otimes A$ and $\tau(\text{trinion}) \in A^* \otimes A^* \otimes A$. The next lemma allows to extend this to any 2-manifold using the gluing axiom. Let us say that a *cutting* of a 2-dimensional manifold Σ is a finite collection of simple non-intersecting curves on Σ, which are not allowed to intersect with the boundary. Equivalently, we will say that these curves cut the manifold into a union of "pieces", i.e., the connected components of the complement to the curves.

LEMMA 4.3.2. *Every 2-manifold with a boundary can be cut into a union of:*

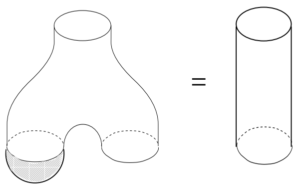
(a) *disks* 

(b) *cylinders* 

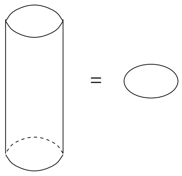
(c) *trinions* 

However, a 2-manifold M can be cut in several different ways. To check that $\tau(M)$ is well-defined, we need the next result.

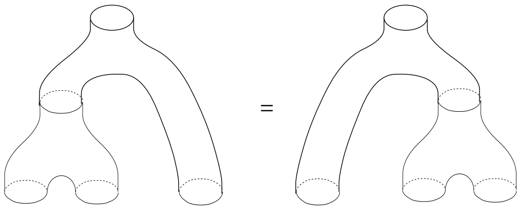
LEMMA 4.3.3 ([**HT**]). *Any two ways to cut a 2-manifold M into disks, cylinders and trinions can be related by isotopy of M and a sequence of "simple moves"*:

(i) 

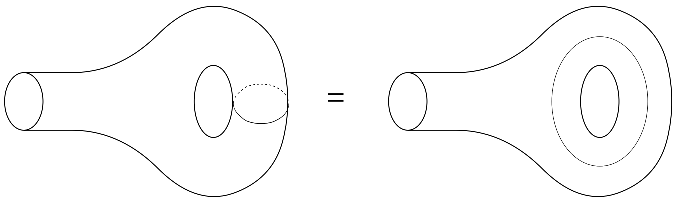
(ii)



(iii)



(iv)



It is easy to check that τ gives the same result on both sides of (i–iv), therefore it is well defined on any 2-manifold M. For example, both sides of (iv) correspond to the vector $\sum_i v_i v^i \in A$ where $\{v_i\}$ and $\{v^i\}$ are dual bases in A with respect to the bilinear form $\mathrm{tr}(ab)$.

This completes the proof of the theorem. □

REMARK 4.3.4. Note that every Frobenius algebra is also a coalgebra: one can define comultiplication $\Delta\colon A \to A \otimes A$ as the adjoint of the multiplication with respect to the bilinear form $\mathrm{tr}(ab)$. However, the relation between comultiplication and multiplication in a Frobenius algebra is different than in a Hopf algebra: instead of $\Delta(ab) = \Delta(a)\Delta(b)$, one has

$$\Delta(ab) = (a \otimes 1)\Delta(b) = (1 \otimes b)\Delta(a).$$

EXAMPLE 4.3.5. Let X be a finite set, $A = \mathcal{F}(X)$ the algebra of k-valued functions on X (with respect to multiplication), and $\mathrm{tr}\colon A \to k$ given by $\mathrm{tr}(f) = \sum_{x\in X} f(x)$. This obviously is a Frobenius algebra. Moreover, one easily sees that in this case the comultiplication is given by $\Delta(\delta_x) = \delta_x \otimes \delta_x$ (where δ_x is the delta function at x: $\delta_x(y) = \delta_{x=y}$), and for a connected surface with n boundary components, independent of genus, one has

$$\tau(\Sigma) = \Delta^{n-1}(1) = \sum_{x\in X} \delta_x \otimes \cdots \otimes \delta_x.$$

4.4. 3D TQFT from MTC

In this section we will show that every modular tensor category gives rise to a 3D TQFT, which generalizes the invariant of 3-manifolds without boundary constructed in Section 4.1. These ideas were first suggested by Witten [**W2**] (for the modular category arising from representations of $U_q(\mathfrak{sl}_2)$ at roots of unity). Our exposition is based on the construction in [**T**], with some modifications.

Before giving the precise definitions, let us note that we could have replaced in the definition of a TQFT topological manifolds by, say, smooth manifolds, or by manifolds with some additional structure. The only things we need are the notions of boundary, orientation reversing, gluing, and disjoint unions. One can formalize the requirements, but it is hardly necessary. Note that in dimensions 2 and 3 smooth and topological theories are equivalent: every topological manifold can be endowed with a smooth structure, and any two smooth structures are equivalent. Therefore, in this section we will not distinguish between smooth and topological structures.

In this section, we will construct an "extended" TQFT in the sense that we will consider manifolds with some additional structure. Let $\mathcal{C}$ be an abelian category.

DEFINITION 4.4.1. (i) A *$\mathcal{C}$-marked surface* is an oriented compact surface Σ with a finite number of points $p_1, \dots, p_m$ on it, and the following data attached to every point p_i: a non-zero tangent vector v_i, an object $W_i \in \operatorname{Ob}\mathcal{C}$ and a sign $\varepsilon_i = \pm$.

We define a change of orientation of a $\mathcal{C}$-marked surface Σ to be the surface $\overline{\Sigma}$ with the same data as Σ but with ε_i, v_i replaced by $-\varepsilon_i, -v_i$.

(ii) A *$\mathcal{C}$-marked 3-manifold* is a pair (M, T), where M is an oriented 3-manifold with a boundary and T is a partially $\mathcal{C}$-colored ribbon tangle in M such that the only uncolored components are annuli, and the colored components may end on the boundary of M. Orientation reversal is defined by $\overline{(M,T)} = (\overline{M}, \overline{T})$, where, as usual, $\overline{M}$ is M with reversed orientation, and $\overline{T}$ is T with reversed directions of all strands.

For a $\mathcal{C}$-marked 3-manifold M we will denote by ∂M its topological boundary provided with the following structure of a $\mathcal{C}$-marked surface (see Figure 4.5 below):

— the points p_i are the ends of the ribbon tangle T,

— W_i is the color of the corresponding strand of T,

— the sign ε_i is $+$ if the tangle goes outward and $\varepsilon_i = -$ if the tangle goes inward,

— the tangent vector v_i at the point p_i is determined (up to a positive real factor) by the condition that it is tangent to the base of T, and the direction is chosen so that the reper (n_i, v_i) has positive orientation (with respect ot the orientation in ∂M), where n_i is the unit normal vector to the ribbon (on the "face side").

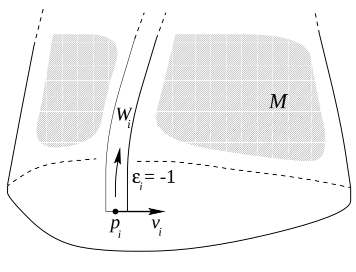


FIGURE 4.5. A $\mathcal{C}$-marked 3-manifold.

It is clear that $\overline{\partial M} = \partial\overline{M}$, and $\overline{\overline{M}} = M$. With this extended notions of 3-manifolds, boundaries, etc., one can also define the notion of 3D TQFT. We will call such TQFT's *"$\mathcal{C}$-extended"*, or for brevity, simply "extended" when there is no ambiguity.

REMARK 4.4.2. Instead of considering manifolds with ribbon tangles inside, we could have considered manifolds with tubular neighborhoods of these tangles removed, and a certain framing and coloring of these tubes. Such a manifold is not even a manifold with a boundary, but a manifold with corners. For this reason, extended TQFT's are sometimes called "TQFT's with corners".

THEOREM 4.4.3 (Turaev [**T**]). *To any MTC $\mathcal{C}$ such that $p^+/p^- = 1$ one can associate a $\mathcal{C}$-extended 3D TQFT which generalizes the Reshetikhin–Turaev invariants $\tau(M,\Omega)$ of closed 3-manifolds defined in Section* 4.1.

REMARK 4.4.4. It was shown in [**Fu**] that every complex-valued invariant of closed 3-manifolds satisfying the multiplicativity $\tau(M_1\#M_2) = \tau(M_1)\tau(M_2)$ and reality $\tau(\overline{M}) = \overline{\tau(M)}$ properties can be obtained from some (non-extended) TQFT, which is essentially uniquely defined by this invariant. Since Reshetikhin–Turaev invariants, up to a constant, satisfy the multiplicativity property (see (4.1.3)), it is not surprising that they come from some TQFT. The problem is to construct this TQFT explicitly.

The remaining part of this section is devoted to the construction of the TQFT, and thus, the proof of Theorem 4.4.3. The main idea of the proof is to reduce everything to manifolds without boundary (but with some kind of a ribbon link inside) and then use the results of Section 4.1.

Step 1. Parameterized manifolds.

Let us start with constructing some supply of "standard" $\mathcal{C}$-marked surfaces. Let us call a *type* t a finite sequence of the form

$$t = ((W_1,\varepsilon_1),(W_2,\varepsilon_2),H,(W_3,\varepsilon_3),H,\dots) \tag{4.4.1}$$

where W_i are objects of $\mathcal{C}$, $\varepsilon_i = \pm$, and H is some formal symbol. We will denote by $g = g(t)$ the number of occurrences of H.

For every type t as above, we define some "standard" $\mathcal{C}$-marked surface Σ_t. First of all, let us define the "standard sphere" to be $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$, which we consider as the boundary of the unit ball in $\mathbb{R}^3$. We will also use the "equator" of this sphere, which we define to be the circle given by equation $y = 0$ (see Figure 4.6). The clockwise direction (in the xz-plane) of this circle will be referred to as the positive direction. Equivalently, one can view the standard sphere as $S^2 = \mathbb{CP}^1$, with the equator being the completed real axis, and the positive direction given by the positive direction on the real axis. We will always identify these two realizations by stereographic projection, so that the the south pole $(0,0,-1) \in \mathbb{R}^3$ is identified with $\infty \in \mathbb{CP}^1$.

Now, given a type t, we construct Σ_t as follows. Let us take the standard sphere. Choose points $p_1 < \cdots < p_n$ on the equator (n is the number of terms in t). For every term (W_i,ε_i) of t, assign to the corresponding point the object W_i, the sign ε_i, and the vector v_i which goes along the positive direction of the equator. For every occurrence of H, glue a handle to the corresponding p_i. This gives a $\mathcal{C}$-marked surface Σ_t which is defined uniquely up to a unique homeomorphism.

More formally, Σ_t can be defined as follows. Define for every t the ribbon tangle T_t which consists of:

(a) One (uncolored) coupon (placed in the bottom of T_t).

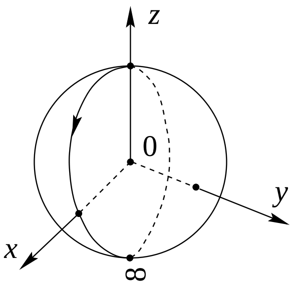


FIGURE 4.6. Standard sphere.

(b) A strand of color W for each occurrence of (W,ε), which connects the coupon with the top of the tangle. This strand is directed upward if $\varepsilon = +$ and downward if $\varepsilon = -$.

(c) An arc (uncolored and non-directed) for each occurrence of H.

Define the handlebody M_t as a neighborhood of the coupon and arcs of T_t in $\mathbb{R}^3$, and let $\Sigma_t = \partial M_t$. One easily sees that this defines Σ_t uniquely up to a homeomorphism, and the homeomorphism is unique up to isotopy.

For example, for $t = ((W_1,+),H,(W_2,-))$, the tangle T_t and the surface Σ_t are shown in Figure 4.7 below (for technical resons, the tangent vectors v_i are not shown in the figure).

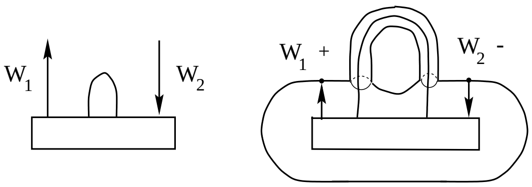


FIGURE 4.7. The tangle T_t and the surface Σ_t.

Note that some of the surfaces Σ_t are homeomorphic. What is more important is that every $\mathcal{C}$-marked surface is (not canonically) isomorphic to at least one of Σ_t.

For a type t, we define $\bar{t}$ to be obtained by reversing the order of terms in the sequence and replacing every ε_i by $-\varepsilon_i$. For example, for $t = ((W_1,+),H,(W_2,-))$, we have $\bar{t} = ((W_2,+),H,(W_1,-))$. Then we have a canonical homeomorphism $rev_t\colon \overline{\Sigma_t} \xrightarrow{\sim} \Sigma_{\bar{t}}$ given by reflection in the vertical plane $x=0$, and $rev_t \circ rev_{\bar{t}} = \mathrm{id}$.

In order to construct a TQFT, we will construct first an auxiliary TQFT with "parameterized manifolds". The 2-manifolds in this TQFT will be $\mathcal{C}$-marked surfaces Σ together with a parameterization φ, i.e., an isomorphism

$$\varphi\colon \Sigma \xrightarrow{\sim} \Sigma_{t_1} \sqcup \cdots \sqcup \Sigma_{t_l}. \tag{4.4.2}$$

Homeomorphisms are homeomorphisms preserving these parameterizations (which essentially kills all non-trivial homeomorphisms: the only automorphisms of a parameterized surface are permutations of components of the same type). The parameterized 3-manifolds will be $\mathcal{C}$-marked 3-manifolds equipped with parameterizations of their boundaries.

We claim that if we construct a TQFT for the parameterized manifolds then we automatically get a TQFT for non-parameterized manifolds. Indeed, let us assume that we have constructed a "parameterized" TQFT; in particular, for every pair (Σ, φ), where Σ is a $\mathcal{C}$-marked surface and φ is a homeomorphism $\Sigma \xrightarrow{\sim} \Sigma_t$, we have a vector space $\tau(\Sigma, \varphi)$. Let f be a homeomorphism $f\colon \Sigma_t \xrightarrow{\sim} \Sigma_t$. Consider the cylinder $M = \Sigma \times [0,1]$ with the parameterization of its boundary chosen as follows:

$$\partial M = \overline{\Sigma} \sqcup \Sigma \xrightarrow{\varphi \sqcup (f \circ \varphi)} \overline{\Sigma_t} \sqcup \Sigma_t.$$

This gives us an operator

$$f_* = \tau(M)\colon \tau(\Sigma, \varphi) \xrightarrow{\sim} \tau(\Sigma, f \circ \varphi)$$

and it follows from the gluing axiom that $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$ (compare with the proof of Theorem 4.2.3). Taking tensor product, we can define f_* for a homeomorphism of disjoint union of Σ_t's.

This allows us to construct canonical isomorphisms $f_{\varphi,\psi}\colon \tau(\Sigma, \psi) \xrightarrow{\sim} \tau(\Sigma, \varphi)$, which satisfy the compatibility condition $f_{\varphi_1,\varphi_2} f_{\varphi_2,\varphi_3} = f_{\varphi_1,\varphi_3}$. In this case, we can identify all these spaces with each other, thus forming a space $\tau(\Sigma)$ which is canonically isomorphic to each of $\tau(\Sigma, \varphi)$ (compare with the construction in Definition 1.1.11).

Now let M be a $\mathcal{C}$-marked 3-manifold. Choose a parameterization φ of its boundary $\Sigma = \partial M$ (see (4.4.2)). Then $\tau(M, \varphi)$ is a vector in the vector space $\tau(\partial M, \varphi)$. Identifying $\tau(\Sigma) = \tau(\Sigma, \varphi)$, we get a vector $\tau(M) \in \tau(\Sigma)$; it is easy to see that this vector does not depend on the choice of φ.

It is straightforward to check all the axioms of a 3D TQFT. Thus, from every "parameterized" TQFT one can automatically construct a "non-parameterized" TQFT.

Step 2. Reducing to closed manifolds.

Now we are going to construct a TQFT based on parameterized manifolds. Let us start by defining the spaces $\tau(\Sigma_t) \equiv \tau(\Sigma_t, \mathrm{id})$. For t given by (4.4.1), let

$$W_t = W_1^{\varepsilon_1} \otimes W_2^{\varepsilon_2} \otimes H \otimes W_3^{\varepsilon_3} \otimes H \otimes \cdots, \tag{4.4.3}$$

where $W^\varepsilon = W$ if $\varepsilon = +$ and W^* if $\varepsilon = -$, and as before, $H = \bigoplus_{i \in I} V_i \otimes V_i^*$—see (2.4.9). Then we define

$$\tau(\Sigma_t) := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_t) =: \langle W_t \rangle, \tag{4.4.4}$$

For a $\mathcal{C}$-marked surface Σ along with a parameterization $\Sigma \xrightarrow{\sim} \Sigma_{t_1} \sqcup \cdots \sqcup \Sigma_{t_l}$, we let $\tau(\Sigma) = \tau(\Sigma_{t_1}) \otimes \cdots \otimes \tau(\Sigma_{t_l})$.

Next, let us construct an isomorphism $\tau(\Sigma_t)^* \xrightarrow{\sim} \tau(\Sigma_{\overline{t}})$. Let $\varphi\colon \mathbf{1} \to W_t$, $\psi\colon \mathbf{1} \to W_{\overline{t}}$. Define $(\varphi, \psi) \in k$ by

$$(\varphi, \psi) = D^{-g}(\mathbf{1} \xrightarrow{\varphi \otimes \psi} W_t \otimes W_{\overline{t}} \xrightarrow{e_t} \mathbf{1}) \tag{4.4.5}$$

where e_t is the tensor product of the evaluation maps $W_i^{\varepsilon_i} \otimes W_i^{-\varepsilon_i} \to \mathbf{1}$ and renormalized evaluation maps $e_H(\eta \otimes \mathrm{id})\colon H \otimes H \to \mathbf{1}$. Here e_H is the evaluation map induced by the identification $H \xrightarrow{\sim} H^*$ (see (2.4.9)), and $\eta|_{V_i \otimes V_i^*} = d_i^{-1}\,\mathrm{id}$. As before, we denote by d_i the (quantum) dimension of V_i, and D is given by (3.1.15).

One easily checks that the pairing (4.4.5) is nondegenerate and symmetric. Thus, we have identifications $\tau(\overline{\Sigma_t}) \xrightarrow{\sim} \tau(\Sigma_t)^*$. We extend them to disjoint unions of Σ_t in an obvious way.

Now, let us construct for every parameterized $\mathcal{C}$-marked 3-manifold M a vector $\tau(M) \in \tau(\partial M)$. The main idea is, of course, to reduce everything to invariants of closed manifolds. This can be done as follows.

Let us define a *special* link to be a ribbon link X (which may contain coupons), with some of the strands and coupons colored by objects, respectively, morphisms from $\mathcal{C}$ and such that the following condition holds:

— any uncolored strand is either an annulus, or has both ends at the same uncolored coupon, in which case these ends are next to each other,

— all uncolored coupons are of the form T_t.

Examples of special links can be found in the next section.

Let X be a special link, with uncolored coupons of types $t_1, \dots, t_k$. Define its Reshetikhin–Turaev invariant $F^{-1}(X) \in (\bigotimes_{i=1}^k \langle W_{t_i}\rangle)^*$ as follows. For any collection $\varphi_i \in \langle W_{t_i}\rangle$, let us denote by $T(\varphi_1, \dots, \varphi_k)$ the ribbon link obtained by coloring each uncolored coupon by the corresponding φ. Then we write

$$F^{-1}(X)(\varphi_1, \dots, \varphi_k) = \sum_c d_c F^{-1}(T(\varphi_1, \dots, \varphi_k)),$$

where the sum is taken over all colorings c of uncolored strands $c\colon U \to I$, and $d_c = \prod_{u\in U} d_{c(u)}$ where U is the set of all uncolored strands. This generalizes the conventions of Chapter 3.

The crucial observation is that every connected $\mathcal{C}$-marked parameterized 3-manifold M can be obtained from S^3 by a surgery along some special link X. More precisely, let X be a special link in $\mathbb{R}^3 \subset S^3$, with uncolored coupons of types $t_1, \dots, t_k$. Each such coupon determines an embedding of the standard handlebody M_t, defined in Step 1, into $\mathbb{R}^3$. Define

$$M_X = M_L \setminus \bigcup M_{t_i},$$

where L is the ribbon link formed by all uncolored annuli of X. In other words, we first do surgery along all uncolored annuli and then remove from the obtained closed manifold neighborhoods of uncolored coupons. This gives a $\mathcal{C}$-marked manifold with boundary which is canonically identified with $\sqcup \overline{\Sigma}_{t_i}$.

We let $\tau(M_X) = F^{-1}(X) \in (\bigotimes \langle W_{t_i}\rangle)^* \simeq \bigotimes \langle W_{\overline{t_i}}\rangle$, where we use the pairing (4.4.5) to identify $\langle W_{\overline{t}}\rangle \simeq \langle W_t\rangle^*$. Theorem 4.1.15 implies that this is well-defined. This completes the definition of the TQFT based on parameterized $\mathcal{C}$-marked 3-manifolds. All the compatibility conditions are easy to prove and are left to the reader (or can be found in [**T**]); functoriality is also obvious. Therefore, we have to prove the gluing and the normalization axioms.

Step 3. Proving the gluing axiom.

Now, let us prove the gluing axiom, assuming that $p^+/p^- = 1$. Looking at the definitions, we see that it is equivalent to the following statement: if X is a special link which contains two uncolored coupons T, T' of types $t, \overline{t}$ respectively, and $X' = \sqcup_{T,T'}(X)$ is the link obtained from X by "canceling" these two coupons as shown in Figure 4.8 below, then $M_X = M_{X'}$. This statement is of purely topological nature, and we omit its proof.

Finally, the proof of the normalization axiom will be given in the next section, along with other examples. This completes the proof of Theorem 4.4.3.

In the case where $p^+/p^- \neq 1$, the theorem above is incorrect as stated. For example, the gluing axiom holds only up to a multiplicative factor, and instead of the action of the mapping class group in $\tau(\Sigma)$, one would get a projective action.

FIGURE 4.8. "Canceling" of two coupons.

(In the physics language, this is referred to as "anomalies", so for $p^+/p^- \neq 1$, we get a TQFT with anomalies.) As in the case of a projective representation of a group, it is possible to get rid of these factors (anomalies) by a suitable "central extension" of the TQFT. We will discuss this generalization later (see Section 5.7).

4.5. Examples

In this section we give several examples of the calculation of the vector spaces and operators for the TQFT defined in the previous section. As before, we fix a MTC $\mathcal{C}$.

First of all, let us get some working experience with special links.

EXAMPLE 4.5.1. Let T be a ribbon tangle, in which all the strands are $\mathcal{C}$-colored except for some annuli (such tangles were discussed in Chapter 3). Such a tangle defines two types $t_{\text{top}}, t_{\text{bot}}$ and a linear map $F^{-1}(T)\colon W_{t_{\text{bot}}} \to W_{t_{\text{top}}}$ (see Theorem 2.3.9).

Let us form a special link X by adding to T two uncolored coupons of type $t_{\text{bot}}, \overline{t_{\text{top}}}$. An example of such ribbon tangle T and the corresponding link X is shown in Figure 4.9.

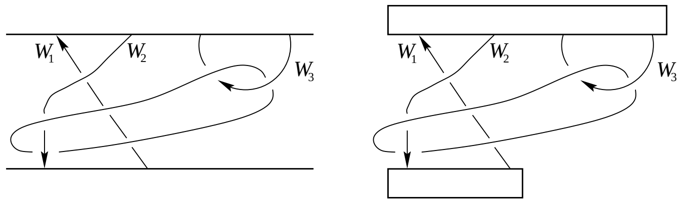


FIGURE 4.9. A ribbon tangle and the corresponding special link.

Then the definition of the previous section gives $F^{-1}(X) \in \langle W_{t_{\text{bot}}}\rangle^* \otimes \langle W_{\overline{t_{\text{top}}}}\rangle^* \simeq \operatorname{Hom}(\langle W_{t_{\text{bot}}}\rangle, \langle W_{t_{\text{top}}}\rangle)$. We claim that this operator is given by $\Phi \mapsto F^{-1}(T)\Phi$. Indeed, it suffices to prove that for $\Phi \in \langle W_{t_{\text{bot}}}\rangle, \Psi \in \langle W_{\overline{t_{\text{top}}}}\rangle$, we have

$$F^{-1}(X)(\Phi, \Psi) = (\mathbf{1} \xrightarrow{\Phi\otimes\Psi} W_{t_{\text{bot}}} \otimes W_{\overline{t_{\text{top}}}} \xrightarrow{F^{-1}(T)\otimes \mathrm{id}} W_{t_{\text{top}}} \otimes W_{\overline{t_{\text{top}}}} \xrightarrow{e_{t_{\text{top}}}} \mathbf{1}).$$

This is immediate from the definition.

The same statement holds if we allow T to be a partially colored ribbon tangle which is allowed to have uncolored strands ending at the top or bottom, as long as the two ends of any such strand are next to each other, and we define

$$F^{-1}(T) = \bigoplus_c d_c F^{-1}(T,c)\colon W_{t_{\mathrm{bot}}} \to W_{t_{\mathrm{top}}},$$

where $c\colon U \to I$ is a coloring (here U is the set of all uncolored strands), and $d_c = \prod d_{c(u)}$ over all annuli and the strands which end at the top (but not the bottom!). In this case, the statement is a little bit less obvious, since one has to check the normalizations.

Thus, we see that as a special case, the operators $F^{-1}(X)$ for special links contain the operators $F^{-1}(T)$ for any partially colored tangle T.

Next, let us see how the vectors $\tau(M)$ look in the simplest examples.

EXAMPLE 4.5.2. Let T be a $\mathcal{C}$-colored ribbon tangle such that *bottom*$(T) = \emptyset$, *top*$(T) = t$; thus, $F^{-1}(T)\colon \mathbf{1} \to W_t$ is a vector in $\langle W_t\rangle$. Let M be the unit ball in $\mathbb{R}^3$ with the tangle T placed inside so that the top of T is on the equator of the standard sphere $S^2 = \partial M$. Then it immediately follows from the definition and the previous example that $\tau(M) = F^{-1}(T) \in \langle W_t\rangle$.

More generally, let T be any $\mathcal{C}$-colored ribbon tangle, with *bottom*$(T) = t_1$, *top*$(T) = t_2$, and let M be the domain

$$\{\mathbf{x} \in \mathbb{R}^3 \mid 1 \le \|\mathbf{x}\| \le 2\} \simeq S^2 \times I,$$

with the tangle T placed inside so that the bottom of T is placed on the equator of the inner sphere $\|\mathbf{x}\| = 1$, and the top of T is placed on the equator of the outer sphere $\|\mathbf{x}\| = 2$. Then $F^{-1}(T)\colon \langle W_{t_1}\rangle \to \langle W_{t_2}\rangle$ and

$$\tau(M) \in \langle W_{t_1}\rangle^* \otimes \langle W_{t_2}\rangle \simeq \mathrm{Hom}(\langle W_{t_1}\rangle, \langle W_{t_2}\rangle)$$

is given by $\Phi \mapsto F^{-1}(T)\Phi$ for $\Phi\colon \mathbf{1} \to W_{t_1}$.

The next several examples deal with the case when $g(t) = 1$, so that Σ_t is a torus. We will make heavy use of the results of Examples 4.1.2, 4.1.10. We will identify our "standard torus" Σ_t with the torus considered in these examples so that the cycles $\alpha, \beta \in \mathrm{H}_1(\Sigma_t)$ shown below correspond to α, β shown in Figure 4.1.

EXAMPLE 4.5.3. Let $t = (H)$, so that the handlebody M_t is the solid torus T, and $\partial T = \Sigma_t$ is a torus with no marked points. By (4.4.4), $\tau(\Sigma_t) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, H) = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_i \otimes V_i^*)$. We claim that the vector $\tau(T) \in \tau(\partial T)$ is exactly the image of the identity morphism $\mathrm{id}\colon \mathbf{1} \to \mathbf{1} \otimes \mathbf{1}$ $(\in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_0 \otimes V_0^*))$.

PROOF. The proof is based on the fact that the standard torus can be obtained as a result of surgery of S^3 along the special link shown in Figure 4.10.

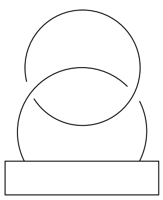
$X =$ 

FIGURE 4.10

Indeed, performing the surgery along the annulus, we get $S^2 \times S^1$ (see Example 4.1.10(ii)); after this, we cut a neighborhood of the coupon, which is isomorphic to a solid torus T_0. But by Example 4.1.2(i), the complement of a torus in $S^2 \times S^1$ is again a torus (of course, one also has to check that the attachment maps given by the link X are the same as in Example 4.1.2(i)—we leave it to the reader).

By Example 4.5.1, the corresponding RT invariant $F^{-1}(X) \in \langle H \rangle$ coincides with $F^{-1}(L)$, where L is obtained from the tangle X by removing the coupon. By (3.1.19, 3.1.20), $F^{-1}(L)$ is equal to $\mathrm{id}\colon \mathbf{1} \to V_0 \otimes V_0^* \subset H$. □

REMARK 4.5.4. In the same way one can prove that for the solid handlebody T_g of genus g one has

$$\tau(T_g) = (\mathrm{id}\colon \mathbf{1} \to (\mathbf{1} \otimes \mathbf{1})^{\otimes g}) \in \tau(\partial T_g) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, H^{\otimes g}). \tag{4.5.1}$$

EXAMPLE 4.5.5. Let $t = ((W, -), H)$, so that Σ_t is the 2-dimensional torus with one marked point, and $\tau(\Sigma_t) = \langle W^*, H \rangle = \mathrm{Hom}_{\mathcal{C}}(W, H)$. Let $S\colon \Sigma_t \to \Sigma_t$ be the homeomorphism corresponding to the matrix $S \in \mathrm{SL}_2(\mathbb{Z})$ defined in Theorem 4.1.4. Then we claim that $S_*\colon \mathrm{Hom}_{\mathcal{C}}(W, H) \to \mathrm{Hom}_{\mathcal{C}}(W, H)$ coincides with the operator S_W defined in Theorem 3.1.17.

PROOF. Let us consider the manifold $M = \Sigma_t \times I$, so that $\partial M = \Sigma_t \sqcup \overline{\Sigma_t}$, with the parameterization of the boundaries given by $\mathrm{id} \sqcup S$. We claim that M can be obtained from S^3 by a surgery along the partially colored link X shown in Figure 4.11.

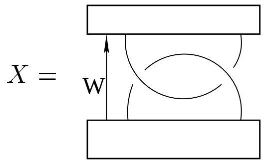


FIGURE 4.11

Indeed, in this case we do not have to do any surgery at all but just to remove from S^3 the two linked solid tori—the neighborhoods of the two coupons. By Example 4.1.2(ii), this gives a cylinder $\Sigma_t \times I$. As before, we leave it to the reader to check that parameterization of the boundaries given by X coincides with the one given in Example 4.1.2(ii). After this, the result follows form Example 4.5.1. □

This explains why we defined the operator $S\colon H \to H$ by this picture—cf. (3.1.32).

EXERCISE 4.5.6. In a similar way, prove that $T_*\colon \mathrm{Hom}_{\mathcal{C}}(W, H) \to \mathrm{Hom}_{\mathcal{C}}(W, H)$ coincides with the operator T_W defined in Theorem 3.1.17.

EXAMPLE 4.5.7. Let us check the normalization axiom. For simplicity, let us consider $t = ((W, -), H)$, so that Σ_t is the torus with 1 marked point, $\tau(\Sigma_t) = \mathrm{Hom}_{\mathcal{C}}(W, H)$. Let M be the cylinder $\Sigma_t \times I$, $I = [0, 1]$. Let us check that $\tau(M) = \mathrm{id} \in \mathrm{Hom}(\tau(\Sigma_t), \tau(\Sigma_t))$.

One can represent M as M_X with X from Figure 4.12 below. Indeed, a surgery of S^3 along $\bigcirc$ gives $S^2 \times S^1$, as we already discussed in Example 4.1.10(ii). Then we cut two solid tori from $S^2 \times S^1$. But $S^2 \times S^1$ can be also obtained by gluing two solid tori along their boundaries, see Example 4.1.2(ii). Now removing two solid tori from $S^2 \times S^1$, we get $T^2 \times I$.

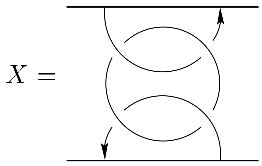


FIGURE 4.12

Note the similarity of the picture for X and the one which defines the matrix S^2 in $\mathcal{C}$ (cf. the proof of Theorem 3.1.16).

EXAMPLE 4.5.8. Let

$$t = ((W_1, +), \dots, (W_n, +)),$$
$$t' = ((W_1, +), \dots, (W_{i+1}, +), (W_i, +), \dots, (W_n, +)),$$

so that $\Sigma_t, \Sigma_{t'}$ are spheres with n marked points, and

$$\tau(\Sigma_t) = \langle W_t \rangle = \operatorname{Hom}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n),$$
$$\tau(\Sigma_{t'}) = \operatorname{Hom}(\mathbf{1}, W_1 \otimes \cdots \otimes W_{i+1} \otimes W_i \otimes \cdots \otimes W_n).$$

Let $b_i\colon \Sigma_t \xrightarrow{\sim} \Sigma_{t'}$ be the homeomorphism which exchanges i-th, $(i+1)$-st marked points as shown in Figure 4.13 below (for convenience, we are not showing the tangent vectors). Then we claim that $(b_i)_*\colon \tau(\Sigma_t) \to \tau(\Sigma_{t'})$ is given by $\Phi \mapsto \sigma_{W_i, W_{i+1}} \Phi$.

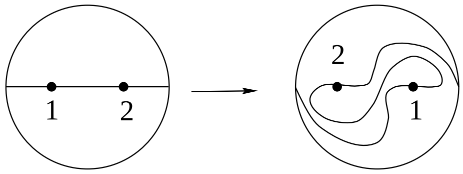


FIGURE 4.13. Braiding homeomorphism.

Indeed, let M be the cylinder $\Sigma_t \times I$, with the parameterization of the boundary given by

$$\varphi\colon \partial M = \overline{\Sigma_t} \sqcup \Sigma_t \xrightarrow{\operatorname{id} \times b_i} \overline{\Sigma_t} \sqcup \Sigma_{t'}.$$

By definition, $(b_i)_*$ coincides with $\tau(M) \in \operatorname{Hom}(\langle W_t \rangle, \langle W_{t'} \rangle)$. To compute $\tau(M)$, note that M is homeomorphic (as a parameterized manifold) to the cylinder $S^2 \times I$ with the trivial parameterization of the boundaries, and with the ribbon tangle shown in Figure 4.14 placed inside (cf. Example 4.5.2). Therefore, by Example 4.5.2, $\tau(M) = \sigma_{W_i, W_{i+1}}$. (This example was used without proof in the Preface.)

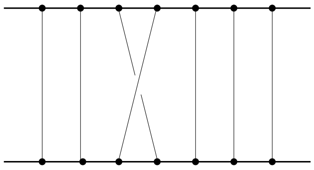


FIGURE 4.14

CHAPTER 5

Modular Functor

Given a modular tensor category $\mathcal{C}$, in the previous chapter we constructed a 3-dimensional Topological Quantum Field Theory (3D TQFT). Moreover, this 3D TQFT was based on an extended notion of a manifold (a usual manifold with additional data). In this chapter, we will show that the notion of a modular tensor category (MTC) is essentially equivalent to some geometric construction in dimension 2. The right notion here is that of a modular functor, which was introduced by Segal (see [**S**]). Our exposition mostly follows the papers [**S, MS1, MS2, T**] and folklore of mathematical physicists.

5.1. Modular functor

DEFINITION 5.1.1. A (*topological*) *d-dimensional modular functor* (MF for short) is the following collection of data:

(i) A vector space $\tau(N)$ assigned to any oriented compact d-manifold N without boundary.

(ii) An isomorphism $f_*: \tau(N_1) \xrightarrow{\sim} \tau(N_2)$ of vector spaces assigned to every homeomorphism $f: N_1 \xrightarrow{\sim} N_2$, which depends only on the isotopy class of f.

(iii) Isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2)$, where k is the base field.

These data have to satisfy the following axioms:

Multiplicativity: $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$.

Functoriality: the isomorphisms (iii) are functorial.

Compatibility: the isomorphisms of part (iii) are compatible with the canonical isomorphisms $N \sqcup \emptyset = N$, $N_1 \sqcup N_2 = N_2 \sqcup N_1$, $(N_1 \sqcup N_2) \sqcup N_3 = N_1 \sqcup (N_2 \sqcup N_3)$.

Normalization: We have an isomorphism $\tau(S^d) = k$, where S^d is the d-dimensional sphere.

Detailed statement of the functoriality and compatibility axioms can be found in Remark 4.2.2, where the same conditions appear in the definition of TQFT.

REMARK 5.1.2. Any $(d+1)$D TQFT (see Definition 4.2.1) gives a d-dimensional MF, because the axioms of a MF, except for the requirement that f_* depends only on the isotopy class of f, are contained in the axioms of a TQFT, and this last condition is satisfied by Theorem 4.2.3.

This modular functor is *unitary*: in addition to the data above, there are functorial isomorphisms $\tau(\overline{\Sigma}) \xrightarrow{\sim} \tau(\Sigma)^*$, where $\overline{\Sigma}$ is the manifold Σ with opposite orientation, which are compatible with the isomorphisms of part (iii).

DEFINITION 5.1.3. (i) We define a category Γ with:

Objects: d-manifolds.

Morphisms: $\mathrm{Mor}_\Gamma(N_1, N_2)$ = isotopy classes of orientation-preserving homeomorphisms $N_1 \xrightarrow{\sim} N_2$.

This is a symmetric tensor category with the "tensor product" given by disjoint union, and the unit given by $\emptyset$. (Note that this category is not additive: one can not add homeomorphisms!)

(ii) For a manifold N, its *mapping class group* $\Gamma(N)$ is the group of isotopy classes of homeomorphisms $N \xrightarrow{\sim} N$. In other words, $\Gamma(N) := \mathrm{Mor}_\Gamma(N, N)$.

The category Γ is a *groupoid*, i.e., a category in which every morphism is invertible. One easily sees that d-dimensional modular functor is the same as a representation of the groupoid Γ, i.e., a tensor functor $\Gamma \to \mathcal{V}ec_f(k)$. This explains the origin of the term "modular functor".

In particular, by 5.1.1(ii), every MF defines a representation of the mapping class group $\Gamma(N)$ of any d-manifold N on the vector space $\tau(N)$.

From now on, let us assume that $d = 2$. Then every connected compact oriented surface is determined up to homeomorphism by its genus g, and defining a modular functor is equivalent to defining for every $g \geq 0$ a representation of the mapping class group Γ_g. We quote here some classical results regarding the mapping class groups.

THEOREM 5.1.4 (Dehn). *Let Σ be a compact oriented surface, and let c be a simple closed curve on Σ. Define the* Dehn twist $t_c \in \Gamma(\Sigma)$ *by Figure* 5.1.[1] *Then the elements t_c generate the mapping class group $\Gamma(\Sigma)$.*

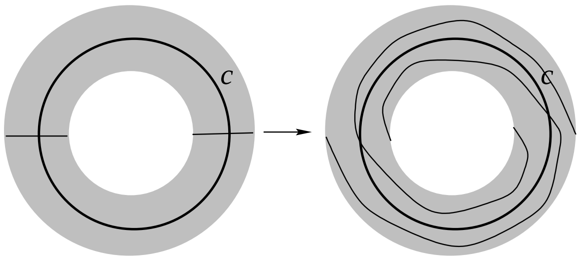


FIGURE 5.1. Dehn twist.

This theorem was later refined by Lickorish [**Li**], who suggested a finite set of Dehn twists generating $\Gamma(\Sigma)$. Finally, an approach allowing one to describe the generators and relations in $\Gamma(\Sigma)$ was given in [**HT**]. For surfaces of genus g with 0 or 1 boundary components (or marked points), the ideas of [**HT**] were fully developed in [**Waj**], where a complete set of generators and relations for $\Gamma_g \equiv \Gamma_{g,0}$ and $\Gamma_{g,1}$ is written.

EXAMPLE 5.1.5. Let $g = 1$, i.e., let Σ be a two-dimensional torus. Then, by Theorem 4.1.3, $\Gamma_1 \simeq \mathrm{SL}_2(\mathbb{Z})$, which can be described as the group with generators

[1]Here we put some auxiliary lines on the surface to demonstrate the action of the homeomorphisms. These lines are for illustration purposes only. Note that c is not required to be oriented.

s, t and relations $(st)^3 = s^2, s^4 = 1$ (which implies $s^2t = ts^2$). It can also be generated by the elements

$$t_a = t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t_b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which correspond to Dehn twists around the meridian and the parallel of the torus.

It turns out that for $d = 2$ the notion of modular functor can be generalized by allowing surfaces with "holes", i.e., with boundary.

DEFINITION 5.1.6. An *extended surface* is a compact oriented surface Σ, possibly with boundary, together with an orientation-preserving parameterization $\pi_i : (\partial\Sigma)_i \xrightarrow{\sim} S^1$ of every boundary circle. Here $(\partial\Sigma)_i$ is considered with the orientation induced from Σ, and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with the counterclockwise orientation.

By a *genus* of an extended surface, we will mean the genus of the closed surface $cl(\Sigma)$ obtained by "patching the holes of Σ", i.e., gluing a disk to every boundary circle.

A *homeomorphism* of extended surfaces $f : \Sigma \xrightarrow{\sim} \Sigma'$ is an orientation-preserving homeomorphism which also preserves parameterizations.

Finally, for an extended surface $(\Sigma, \pi_i : (\partial\Sigma)_i \xrightarrow{\sim} S^1)$ we define the operation of *orientation reversal* by $(\overline{\Sigma}, -\overline{\pi_i})$ (**note the minus sign!**).

The notion of isotopy of homeomorphisms is trivially generalized to this case, as well as the notion of disjoint union. Thus, we can define the extended groupoid $\mathcal{T}eich$ similarly to Definition 5.1.3(i).

DEFINITION 5.1.7. (i) The (*extended*) *Teichmüller groupoid* $\mathcal{T}eich$ is the category with objects extended surfaces, and morphisms isotopy classes of homeomorphisms of extended surfaces (see Definition 5.1.6).

(ii) For any extended surface Σ, its *mapping class group* $\Gamma(\Sigma)$ is the group of all isotopy classes of homeomorphisms $\Sigma \xrightarrow{\sim} \Sigma$. (Sometimes the name "mapping class group" is used for the smaller group $\Gamma'(\Sigma)$ of all isotopy classes of homeomorphisms $\Sigma \xrightarrow{\sim} \Sigma$ which act trivially on the set of connected components of the boundary.) If Σ is a surface of genus g with n boundary components, we will denote $\Gamma(\Sigma) \equiv \Gamma_{g,n}$.

Again, it can be shown that $\Gamma'(\Sigma)$ is generated by Dehn twists (a complete set of relations for $\Gamma'_{g,n}$ is given in [**Ge1**], [**Luo**], [**Ge2**]), and $\Gamma_{g,n}$ is generated by Dehn twists and the "braiding operation" shown in Figure 5.2.[2]

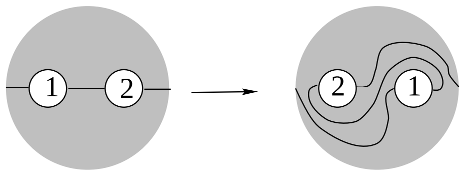


FIGURE 5.2. Braiding.

It will be useful in the future to give an alternative definition of an extended surface. We give below two such definitions. Both of them are equivalent to Definition 5.1.6 in the following sense:

[2]See the footnote on page 94.

PROPOSITION 5.1.8. *The extended groupoids $\mathcal{T}eich$, defined by Definitions* 5.1.6, 5.1.9 *and* 5.1.10, *are equivalent as categories, and this equivalence preserves the operation of orientation reversal.*

DEFINITION 5.1.9. An *extended surface* is an oriented compact surface with boundary and with a specified point p_i on every component of the boundary.

A *homeomorphism* of extended surfaces is an orientation-preserving homeomorphism $\Sigma \to \Sigma'$ which maps marked points to marked points.

Orientation reversal is defined in the obvious way, by reversing the orientation of Σ while leaving the points p_i unchanged.

DEFINITION 5.1.10. An *extended surface* is an oriented compact surface Σ without boundary, with marked points z_i, and with non-zero tangent vectors v_i attached to each marked point.

A *homeomorphism* of extended surfaces is an orientation-preserving homeomorphism $\Sigma \to \Sigma'$ which maps marked points to marked points, and marked tangent vectors to marked tangent vectors.

Orientation reversal is defined by $\overline{(\Sigma, z_i, v_i)} = (\overline{\Sigma}, z_i, -v_i)$.

This definition is analogous to Definition 4.4.1.

PROOF OF PROPOSITION 5.1.8. To establish the equivalence of Definitions 5.1.6 and 5.1.9, note that a parameterization of a boundary circle gives a distinguished point $p_i = \pi_i^{-1}(\mathrm{i})$. Since the set of all homeomorphisms $S^1 \xrightarrow{\sim} S^1$ preserving orientation and the distinguished point $\mathrm{i} \in S^1$ is contractible, this is an equivalence of categories. Similarly, to establish the equivalence of Definitions 5.1.6 and 5.1.10, note that given Σ as in Definition 5.1.6, we can glue to Σ n copies of the standard disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ (with reversed orientation), using the identifications of the boundary circles of Σ with S^1. This gives a new surface $cl(\Sigma)$ without boundary, with marked points images of $0 \in D$, and tangent vectors images of the unit vector going along the real axis in D. As before, it is easy to check that this gives an equivalence of categories. □

EXAMPLES 5.1.11. (i) Let Σ be a two-dimensional torus "with one puncture": $\partial\Sigma \simeq S^1$ and Σ has genus 1. Then the mapping class group $\Gamma_{1,1} = \Gamma(\Sigma)$ is generated by the elements s, t with the relations $(st)^3 = s^2, s^2$ is central (compare with Example 5.1.5). Moreover, s^4 is the inverse of the Dehn twist around the puncture. The easiest way to check this is to use the realization of the torus with one puncture as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ with a non-zero tangent vector at the origin.

(ii) Let $\Sigma_n = \mathbb{R}^2$, with n marked points on the x-axis and with the tangent vector v_i going along this axis in positive direction (all such surfaces are canonically isomorphic). This surface is not compact, so it does not formally satisfy our definition, but let us ignore this. Then the group $\Gamma(\Sigma)$ is isomorphic to the group FB_n of all framed braids with n strands. This group is a semidirect product of the usual braid group B_n and $\mathbb{Z}^n$ (see Definition 1.2.1). In general, there is indeed a relationship between the group $\Gamma(\Sigma)$, where Σ is an extended surface with n holes, and the framed braid group $FB_n(cl(\Sigma))$, where $cl(\Sigma)$ is the closed surface obtained by patching the holes of Σ. This relationship is studied in detail in [**B2**].

The most important difference between extended surfaces and usual surfaces is that extended surfaces can be glued (or sewed) together along the boundary circles. Therefore, if we additionally require a modular functor to behave nicely under this operation, we could define $\tau(\Sigma)$ by gluing Σ from simpler pieces. This motivates the following definition.

DEFINITION 5.1.12. Let $\mathcal{C}$ be an abelian category over a field k, and let R be a symmetric object in ind$-\mathcal{C}^{\boxtimes 2}$ (see Section 2.4). Then a *$\mathcal{C}$-extended modular functor* is the following collection of data:

(i) To every extended surface Σ is assigned a polylinear functor $\tau(\Sigma)\colon \mathcal{C}^{\boxtimes \pi_0(\partial\Sigma)} \to \mathcal{V}ec_f$, where $\pi_0(\partial\Sigma)$ is the set of boundary components (or punctures, depending on the point of view) of Σ. In other words, for every choice of objects $W_a \in \mathcal{C}$ attached to every boundary component of Σ (so, a runs through the set of connected components of $\partial\Sigma$) is assigned a finite-dimensional vector space $\tau(\Sigma; \{W_a\})$, and this assignment is functorial in W_a.

(ii) To every homeomorphism $f\colon \Sigma \xrightarrow{\sim} \Sigma'$ is assigned a functorial isomorphism $f_*\colon \tau(\Sigma) \xrightarrow{\sim} \tau(\Sigma')$.

(iii) Functorial isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2)$.

(iv) **Gluing isomorphism:** Let $c \subset \Sigma$ be a closed curve without self-intersections and p be a marked point on c. Cutting Σ along c, we obtain a new surface Σ' (which may be connected or not). Σ' has a natural structure of an extended surface in the sense of Definition 5.1.9 which has the same boundary components as Σ plus two more components c_1, c_2, which come from the circle c (with marked points p_1, p_2 coming from p).

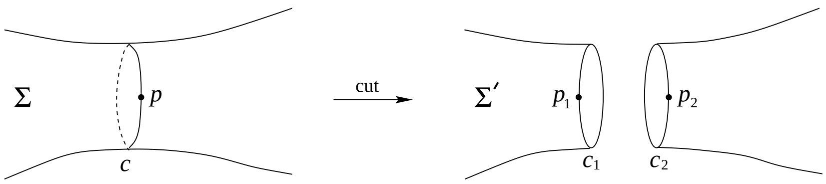


FIGURE 5.3. Cutting of a surface.

Then we are given a functorial isomorphism

$$\tau(\Sigma'; \{W_a\}, R^{(1)}, R^{(2)}) \xrightarrow{\sim} \tau(\Sigma; \{W_a\}), \tag{5.1.1}$$

where we use the notation of Section 2.4.

The above data have to satisfy the following axioms:

Multiplicativity: $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$.

Functoriality: all isomorphisms in parts (iii), (iv) above are functorial in Σ.

Compatibility: all isomorphisms in parts (iii), (iv) above are compatible with each other.

Normalization: $\tau(S^2) = k$.

As before, we leave it to the reader to write the explicit statements of the functoriality and compatibility axioms, taking as an example the definitions in Section 4.2. From now on, we will always work with extended modular functors (unless otherwise specified).

DEFINITION 5.1.13. A $\mathcal{C}$-extended MF is called *non-degenerate* if for every object $V \in \operatorname{Ob}\mathcal{C}$ there exists an extended surface Σ and $\{W_a\} \subset \operatorname{Ob}\mathcal{C}$ such that $\tau(\Sigma; V, \{W_a\}) \neq 0$.

The main goal of this chapter is to show that *for a given* ***semisimple*** *abelian category $\mathcal{C}$ defining a non-degenerate $\mathcal{C}$-extended MF is essentially equivalent to defining a structure of a modular tensor category on $\mathcal{C}$*, with the object $R = \bigoplus V_i \boxtimes V_i^*$, where $\{V_i\}$ are representatives of the equivalence classes of simple objects in $\mathcal{C}$. The precise statements are given in Theorems 5.4.1 and 5.5.1.

Finally, let us introduce the notion of a unitary MF.

Definition 5.1.14. An extended modular functor is called *unitary*, if in addition to the data above, we are also given functorial isomorphisms $\tau(\overline{\Sigma}) \xrightarrow{\sim} \tau(\Sigma)^*$, where $\overline{\Sigma}$ is the manifold Σ with opposite orientation. These isomorphisms must be compatible with the isomorphisms f_* and the isomorphisms of part (iii) of Definition 5.1.12 in the natural way. Also, we require the following compatibility of the unitary structure with the gluing isomorphism. Let $\langle , \rangle_\Sigma : \tau(\Sigma) \otimes \tau(\overline{\Sigma}) \to k$ be the pairing induced by the isomorphism $\tau(\Sigma) \simeq \tau(\overline{\Sigma})^*$. Let Σ, Σ' be as in part (iv) of Definition 5.1.12, and for $f \in \tau(\Sigma), g \in \tau(\overline{\Sigma})$, write $f = \sum f_i$, $g = \sum g_i$ with $f_i \in \tau(\Sigma'; A_i, B_i)$, $g_i \in \tau(\overline{\Sigma}'; B_i, A_i)$, using (5.1.1). Then:

$$\langle f, g\rangle_\Sigma = \sum a_i \langle f_i, g_i\rangle_{\Sigma'} \tag{5.1.2}$$

?! for some non-zero constants a_i which do not depend on Σ.

5.2. The Lego game

Let us denote by $S_{0,n}$ "the standard sphere with n holes":

$$S_{0,n} = \mathbb{CP}^1 \setminus \{D_1, \dots, D_n\}, \quad D_j = \{z \mid |z - z_j| < \varepsilon\}, \quad z_1 < \dots < z_n, \tag{5.2.1}$$

where $\varepsilon > 0$ is small enough so that the disks D_j do not intersect, and let us mark on each boundary circle a point $p_j = z_j - \varepsilon \mathrm{i}$. This endows $S_{0,n}$ with the structure of an extended surface which is independent of the choice of z_j, ε (i.e., surfaces obtained for different choices of z_j, ε are canonically homeomorphic). Note that the set of boundary components of the standard sphere is naturally indexed by numbers $1, \dots, n$; we will use bold numbers for denoting these boundary components: $\pi_0(\partial S_{0,n}) = \{\mathbf{1}, \dots, \mathbf{n}\}$.

Obviously, every extended surface Σ can be obtained by gluing together standard spheres. Therefore, using the gluing axiom we can define the vector space $\tau(\Sigma)$ once we know $\tau(S_{0,n})$. However, the same surface Σ can be obtained by gluing the standard spheres in many ways, and in order for $\tau(\Sigma)$ to be correctly defined we need to construct canonical isomorphisms between the resulting vector spaces. This leads to the following problem.

Definition 5.2.1. Let Σ be an extended surface. A *parameterization* of Σ is the following collection of data, considered up to isotopy:

(i) A finite set $C = \{c_1, \dots\}$ of simple non-intersecting closed curves (cuts) on Σ, with one point marked on every cut (the cuts do not have to be ordered).

(ii) A collection of homeomorphisms $\psi_a : \Sigma_a \xrightarrow{\sim} S_{0,n_a}$, where Σ_a are the connected components of $\Sigma \setminus C$.

We denote the set of all parameterizations of Σ by $M(\Sigma)$.

Our goal is to construct some number of edges ("moves") and 2-cells ("relations among moves") which would turn $M(\Sigma)$ into a connected and simply-connected 2-complex. This problem was first considered by Moore and Seiberg [**MS1**], who conjectured a set of moves and relations. However, their paper contains certain gaps making it not rigorous even by the physicists standards. An accurate proof was recently found independently by the authors [**BK**], and by [**FG**]. Our exposition follows the paper [**BK**] with minor changes.

Define the homeomorphisms

$$\begin{aligned} z\colon S_{0,n} &\xrightarrow{\sim} S_{0,n}, \\ b\colon S_{0,3} &\xrightarrow{\sim} S_{0,3} \end{aligned} \tag{5.2.2}$$

as follows: z is rotation of the sphere which preserves the real axis and induces a cyclic permutation of the holes $\mathbf{1} \mapsto \mathbf{2} \mapsto \cdots \mapsto \mathbf{n} \mapsto \mathbf{1}$, and b is the braiding of the 2nd and 3rd punctures, as shown in Figure 5.2.

Also, for $k, l \geq 0$, denote by $S_{0,k+1} \sqcup_{k+1,1} S_{0,l+1}$ the surface obtained by identifying the $(k+1)$-st hole of $S_{0,k+1}$ with the first hole of $S_{0,l+1}$, and define the map

$$\alpha_{k,l} : S_{0,k+1} \sqcup_{k+1,1} S_{0,l+1} \to S_{0,k+l} \tag{5.2.3}$$

by the condition that it maps the first hole of $S_{0,k+1}$ to the first hole of $S_{0,k+l}$ and preserves the real axis (these properties define $\alpha_{k,l}$ uniquely up to isotopy).

Now, let us define the following edges ("simple moves") in $M(\Sigma)$. To avoid confusion, we will write $E\colon M_1 \rightsquigarrow M_2$ if the edge E connects parameterizations M_1, M_2.

Z-move (rotation): If $M = (C, \{\psi_a\}) \in M(\Sigma)$ and Σ_i is one of the connected components of $\Sigma \setminus C$, then we define an edge

$$Z \equiv Z_i\colon M \rightsquigarrow (C, \{\psi_a, z \circ \psi_i\}_{a \neq i}).$$

B-move (braiding): If $M = (C, \{\psi_a\}) \in M(\Sigma)$ and Σ_i is a connected component of $\Sigma \setminus C$ which has three holes, then we define an edge

$$B \equiv B_i\colon M \rightsquigarrow (C, \{\psi_a, b \circ \psi_i\}_{a \neq i}).$$

F-move (fusion): If $M = (C, \{\psi_a\}) \in M(\Sigma)$ and $c \in C$ separates two different components Σ_i, Σ_j, with $k+1$ and $l+1$ holes respectively, and $\psi_i(c) = \mathbf{k+1}, \psi_j(c) = \mathbf{1}$, then we define an edge

$$F \equiv F_c\colon M \rightsquigarrow (C \setminus \{c\}, \{\psi_a, \alpha_{kl} \circ (\psi_i \sqcup \psi_j)\}_{a \neq i,j}).$$

Before describing the relations, it is convenient to introduce some notation. First of all, let us place on each of the standard spheres $S_{0,n}$ the graph m_0 as shown in Figure 5.4 (for $n = 4$). This graph has one internal vertex, marked by a star; all other vertices are 1-valent and coincide with the marked points on the boundary components of $S_{0,n}$. The graph has a distinguished edge—the one which connects the vertex $*$ with the boundary component $\mathbf{1}$; in the figure, this edge is marked by an arrow. Also, this graph has a natural cyclic order on the set of all edges, given by $\mathbf{1} < \cdots < \mathbf{n} < \mathbf{1}$. Whenever we draw such a graph in the plane, we will always do it in such a way that this order coincides with the clockwise order.

Every parameterization M of a given surface Σ gives rise to a graph $m = \bigcup \psi_a^{-1}(m_0)$ on Σ, which we call the *marking graph* of M. It is easy to show that a parameterization is uniquely determined by C and m; therefore, these graphs give a way to visualize the parameterizations. In some cases, we will draw such graphs on Σ to illustrate a certain sequence of moves. However, in many cases it suffices just to draw the corresponding graphs on the plane, and then the moves can be reconstructed uniquely.

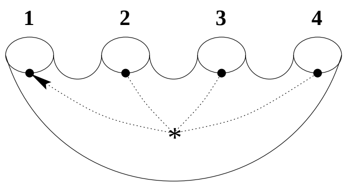


FIGURE 5.4. A standard sphere (with 4 holes).

EXERCISE 5.2.2. Show that the moves Z, B, F connect the parameterizations corresponding to the marking graphs shown in Figures 5.5, 5.6 and 5.7 below.

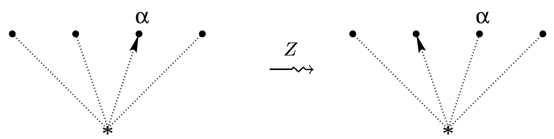


FIGURE 5.5. Z-move ("rotation").

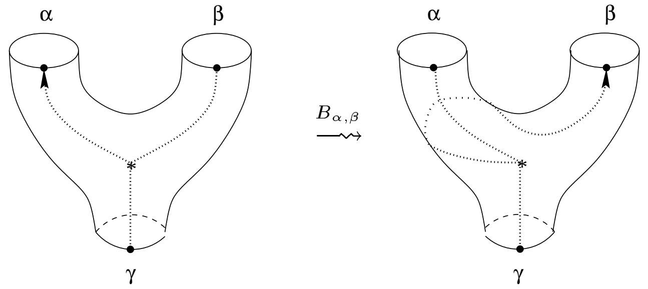


FIGURE 5.6. B-move ("braiding").

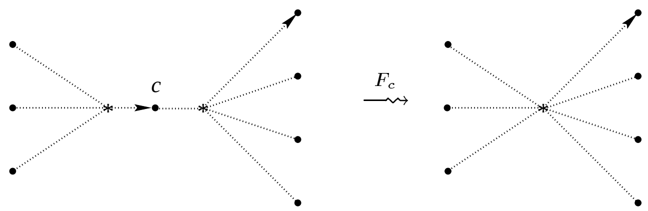


FIGURE 5.7. F-move ("fusion" or "cut removal").

Next, one often needs compositions of the form $Z^a F_c(Z_i^m \sqcup Z_j^n)$, where c is a cut separating components Σ_i and Σ_j (compare with the definition of the F-move). We will call any such composition a *generalized F-move*; for brevity, we will frequently denote it just by F_c. The Rotation axiom formulated below implies that such a composition is uniquely determined by the original parameterization M and by the choice of the distinguished edge for the resulting parameterization $F_c(M)$. Moreover, the Symmetry of F axiom along with the commutativity of disjoint union, also formulated below, imply that if we switch the roles of Σ_1 and Σ_2, then we get the same generalized F-move. Thus, the generalized F-move is completely determined by the marking graph of M and by the choice of the distinguished edge for the resulting marking graph of $F_c(M)$.

Finally, let $M \in M(\Sigma)$ and let Σ_i be one of the components of Σ. As discussed before, the parameterization ψ_i defines an order on the set of boundary components of Σ_i. Let us assume that we have a presentation of $\pi_0(\partial\Sigma_i)$ as a disjoint union, $\pi_0(\partial\Sigma_i) = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$, where the order is given by $I_1 < I_2 < I_3 < I_4$ (some of the I_k may be empty). Then we define the *generalized braiding move* B_{I_2,I_3} to be the product of simple moves shown in Figure 5.8 below (note that we are using generalized F-moves, see above). It is easy to show that this figure uniquely defines the cuts c_1, c_2, c_3 and thus, the generalized braiding move B.

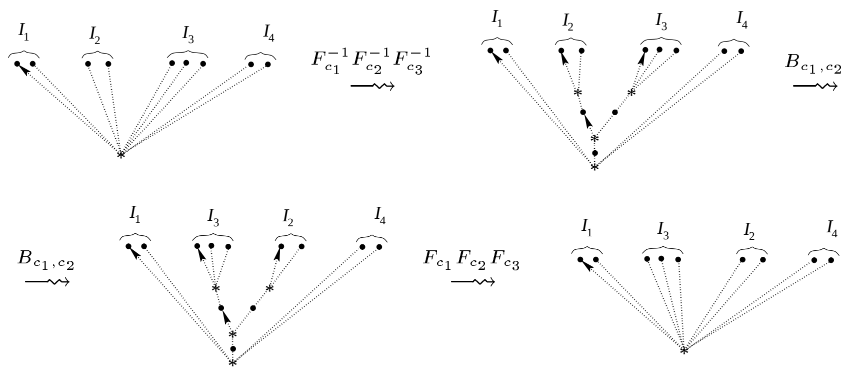


FIGURE 5.8. Generalized braiding move.

Now let us impose some relations among these moves:

MF1: Rotation axiom: If Σ_i is a component with n holes, then $Z_i^n = \mathrm{id}$.

MF2: Symmetry of F: If c, Σ_i, Σ_j are as in the definition of the F-move, then $Z^{k-1}F_c = F_c(Z_i^{-1} \sqcup Z_j)$.

MF3: Associativity of F: If Σ is a connected surface of genus zero, and $M = (C, m) \in M(\Sigma)$ is a parameterization with two cuts, $C = \{c_1, c_2\}$, then

$$F_{c_1}F_{c_2}(M) = F_{c_2}F_{c_1}(M) \tag{5.2.4}$$

(here F denotes generalized F-moves).

MF4: Commutativity of disjoint union: If E_1, E_2 are simple moves that involve non-intersecting sets of components, then $E_1E_2 = E_2E_1$.

MF5: Cylinder axiom: Let $S_{0,2}$ be a cylinder with boundary components α_0, α_1 and with the standard parameterization $M_0 = (\emptyset, \mathrm{id})$. Let Σ be an extended surface, $M \in M(\Sigma)$ be a parameterization, and α be a boundary component of Σ. Then, for every move $E\colon M \rightsquigarrow M'$ we require that the following square be commutative:

$$
\begin{tikzcd}
M \sqcup_{\alpha,\alpha_1} M_0 \arrow[r, "E\sqcup_{\alpha,\alpha_1}\mathrm{id}"] \arrow[d, "F_\alpha"'] & M' \sqcup_{\alpha,\alpha_1} M_0 \arrow[d, "F_\alpha"] \\
M \arrow[r, "E"'] & M'
\end{tikzcd}
\tag{5.2.5}
$$

see Figure 5.9 below.

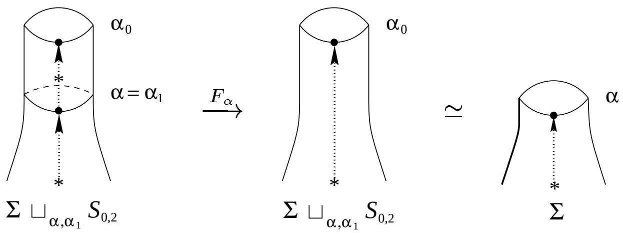


FIGURE 5.9. Cylinder Axiom.

MF6: Braiding axiom: Let Σ_i be a connected component of $\Sigma \setminus C$ which has 4 holes. Denote the boundary components $\psi_i^{-1}(\mathbf{1}), \dots, \psi_i^{-1}(\mathbf{4})$ of Σ_i by $\alpha, \dots, \delta$, respectively. Then:

$$B_{\alpha,\beta\gamma} = B_{\alpha,\gamma} B_{\alpha,\beta}, \tag{5.2.6}$$

$$B_{\alpha\beta,\gamma} = B_{\alpha,\gamma} B_{\beta,\gamma}. \tag{5.2.7}$$

For an illustration of Eq. (5.2.6), see Figure 5.10. Note that all braidings involved are generalized braidings as defined above.

MF7: Dehn twist axiom: Let Σ_i be a connected component of $\Sigma \setminus C$ which has 2 holes: $\alpha = \psi_i^{-1}(\mathbf{1}), \beta = \psi_i^{-1}(\mathbf{2})$. Then

$$Z_i B_{\alpha,\beta} = B_{\beta,\alpha} Z_i \tag{5.2.8}$$

(as before, B denotes the generalized braidings). This axiom is equivalent to the identity $T_\alpha = T_\beta$, where T_α is the Dehn twist defined in Example 5.2.4 below (see Figure 5.11).

THEOREM 5.2.3. *Let Σ be an extended surface of genus zero. Denote by $\mathcal{M}(\Sigma)$ the 2-complex with a set of vertices $M(\Sigma)$, edges given by the B-, Z-, and F-moves*

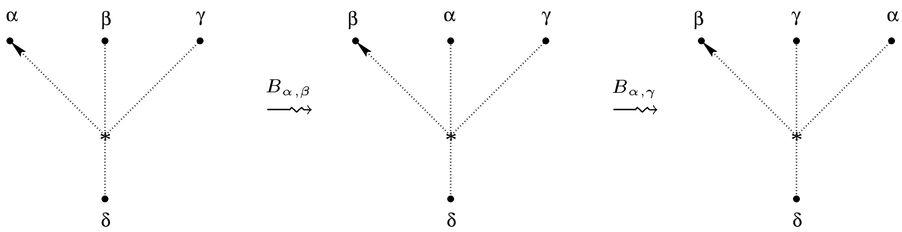


FIGURE 5.10. Braiding axiom (5.2.6).

defined above, and 2-cells given by relations MF1–MF7. *Then $\mathcal{M}(\Sigma)$ is connected and simply-connected.*

As mentioned above, this theorem was first proved (in a different form) in [**MS1**]; our exposition follows [**BK**].

EXAMPLE 5.2.4. Let Σ be an extended surface, $\psi\colon \Sigma \xrightarrow{\sim} S_{0,n}$ be a homeomorphism, and let α be one of the boundary components. Then one can connect the parameterization $(\emptyset, \psi)$ with $(\emptyset, t_\alpha \circ \psi)$, where $t_\alpha \in \Gamma(S_{0,n})$ is the Dehn twist around α (see Figure 5.1), by the following sequence of moves:

$$T_\alpha = F_c B_{\alpha,c} F_c^{-1},$$

where c is a small closed curve around the hole α (see Figure 5.11).

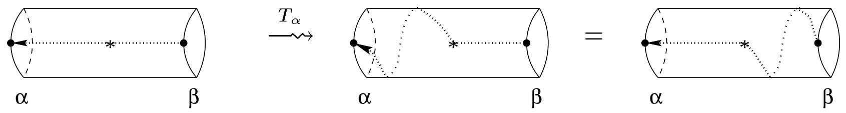


FIGURE 5.11. Dehn twist ($T_\alpha = T_\beta$).

EXERCISE 5.2.5. Let $S_{0,3}$ be the standard sphere with 3 holes, with the marking as shown in the left hand side of Figure 5.6. Deduce from the axioms MF1–MF7 the following relation in $\mathcal{M}(S_{0,3})$:

$$T_\gamma = B_{\beta,\alpha} B_{\alpha,\beta} T_\alpha T_\beta. \tag{5.2.9}$$

Hint: this is analogous to Step 7 in the proof of Theorem 5.3.8.

Now, let us consider extended surfaces of positive genus. In this case, we need to add to the complex $\mathcal{M}(\Sigma)$ one more simple move and several more relations.

S-move: Let $S_{1,1}$ be a "standard" torus with one boundary component and one cut, and with the parameterization M corresponding to the graph in the left hand side of Figure 5.12. Then we add the edge $S\colon M \rightsquigarrow M'$ where the parameterization M' corresponds to the graph shown in the right hand side of Figure 5.12.

More generally, let Σ_a be a component of an extended surface and ψ be a homeomorphism $\psi\colon \Sigma_a \xrightarrow{\sim} S_{1,1}$. Then we add the move $S\colon \psi^{-1}(M) \rightsquigarrow \psi^{-1}(M')$.

REMARK 5.2.6. If Σ is a surface of genus one with one hole, we can identify the set of all parameterizations with one cut on Σ with the set of all homeomorphisms $\psi\colon \Sigma \xrightarrow{\sim} S_{1,1}$. Then the S-move connects the marking ψ with $s \circ \psi$, where $s \in \Gamma(S_{1,1})$ is as in Example 5.1.11(i).

Now, let us formulate the new relations. In addition to relations MF1–MF7, let us also impose the following ones:

MF8: Relations for $g = 1, n = 1$: Let Σ be a marked torus with one hole α, isomorphic to the one shown in the left hand side of Figure 5.13. For any parameterization $M = (\{c\}, \psi)$ with one cut, we let T act on M as the edge

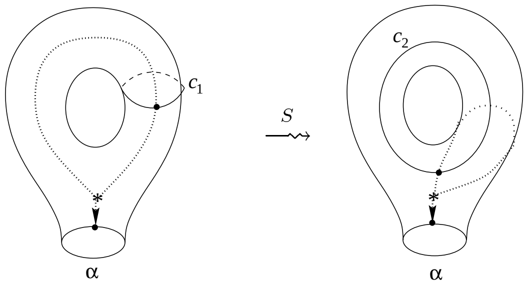


FIGURE 5.12. S-move.

Dehn twist T_c around c (see Example 5.2.4). Then we impose the following relations:

$$S^2 = Z^{-1}B_{\alpha,c_1}, \tag{5.2.10}$$

$$(ST)^3 = S^2. \tag{5.2.11}$$

The left hand side of relation (5.2.10) is shown in Figure 5.13. An illustration of (5.2.11) can be found in [**BK**, Appendix A].

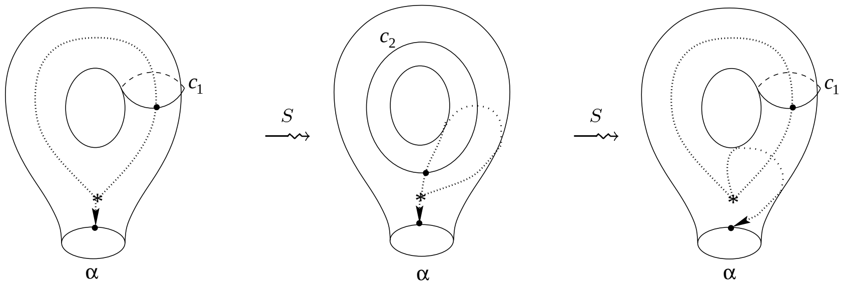


FIGURE 5.13. The relation $S^2 = Z^{-1}B_{\alpha,c_1}$.

MF9: Relation for $g = 1, n = 2$: Let Σ be a marked torus with two holes α, β, isomorphic to the one shown in Figure 5.14. Then we require

$$Z^{-1}B_{\alpha,\beta}F_{c_6}^{-1}F_{c_1} = S^{-1}F_{c_6}^{-1}F_{c_4}T_{c_3}T_{c_4}^{-1}F_{c_4}^{-1}F_{c_5}SF_{c_5}^{-1}F_{c_2} \tag{5.2.12}$$

— see Figure 5.15, where all unmarked arrows are compositions of the form FF^{-1} (see also [**BK**, Appendix B]).

Note that, by their construction, the above relations are invariant under the action of the mapping class group.

REMARK 5.2.7. It is not trivial that relations (5.2.11, 5.2.12) make sense, i.e., that they are indeed closed paths in $\mathcal{M}(\Sigma)$. This is equivalent to checking that the corresponding identities hold in the mapping class group $\Gamma(\Sigma)$. This is indeed so (see, e.g., [**B1, MS2**]). Of course, these relations can also be checked by explicitly drawing the corresponding sequence of cuts and graphs and checking that the final one coincides with the original one, as done in [**BK**].

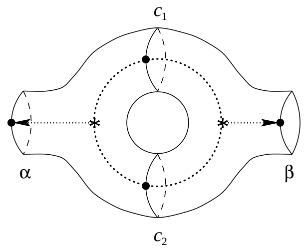


FIGURE 5.14. A marked torus with two holes.

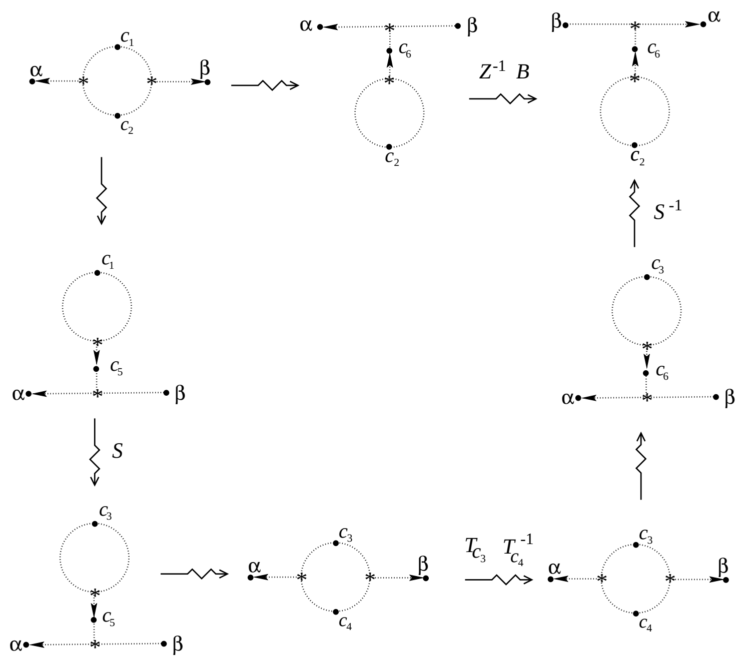


FIGURE 5.15. The relation for $g = 1, n = 2$.

EXAMPLE 5.2.8. Let Σ be a marked torus with one cut c_1 and one hole α (see the left hand side of Figure 5.12). Then we have:

$$(ST)^3 = S^2, \tag{5.2.13}$$

$$S^2T = TS^2, \tag{5.2.14}$$

$$S^4 = T_\alpha^{-1}. \tag{5.2.15}$$

Indeed, (5.2.13) is exactly (5.2.11). Equation (5.2.14) follows from (5.2.10), the Cylinder axiom, and the commutativity of disjoint union, and (5.2.15) easily follows from (5.2.10) and the braiding axiom.

In particular, this implies that the elements $t, s \in \Gamma_{1,1}$ (cf. Example 5.1.11) satisfy relations (5.2.13–5.2.15). In fact, it is known that these are the defining relations of the group $\Gamma_{1,1}$ (see [**B1**]).

Now we can formulate our main result for arbitrary genus.

THEOREM 5.2.9. *Let Σ be an extended surface. Let $\mathcal{M}(\Sigma)$ be the 2-complex with a set of vertices $M(\Sigma)$, edges given by the the Z-, F-, B-, and S-moves, and 2-cells given by relations* MF1–MF9. *Then $\mathcal{M}(\Sigma)$ is connected and simply-connected.*

Again, this theorem was stated (with minor inaccuracies) in [**MS1**], but the proof given there was seriously flawed. An accurate proof was found independently in [**BK**] and, in a different form, [**FG**]. The formulation above is taken from [**BK**].

5.3. Ribbon categories via the Hom spaces

In this section $\mathcal{C}$ will be a semisimple abelian category with representatives of the equivalence classes of simple objects V_i, $i \in I$. We use the notations and conventions of Section 2.4.

In a semisimple abelian category, any object $A \in \mathcal{C}$ is determined by the collection of vector spaces $\mathrm{Hom}(A, \cdot)$. More formally, we have the following well-known lemma.

LEMMA 5.3.1. (i) *Every functor $F: \mathcal{C} \to \mathcal{V}ec_f$ is exact (recall that we are considering only additive functors).*

(ii) *Let $F: \mathcal{C} \to \mathcal{V}ec_f$ be a functor satisfying the following finiteness condition:*

$$F(V_i) = 0 \quad \text{for all but a finite number of } i. \tag{5.3.1}$$

Then F is representable, i.e., there exists an object X_F, unique up to a unique isomorphism, such that $F(A) = \mathrm{Hom}_{\mathcal{C}}(X_F, A)$. Similarly, for a functor $G: \mathcal{C}^{\mathrm{op}} \to \mathcal{V}ec_f$ there exists a unique $Y_G \in \mathcal{C}$ such that $G(A) = \mathrm{Hom}_{\mathcal{C}}(A, Y_G)$.

(iii) *For two functors $F, F': \mathcal{C} \to \mathcal{V}ec_f$ satisfying the finiteness condition above, there is a bijection between the space of functor morphisms $F \to F'$ and $\mathrm{Hom}_{\mathcal{C}}(X_{F'}, X_F)$. A similar statement holds for $G, G': \mathcal{C}^{\mathrm{op}} \to \mathcal{V}ec_f$.*

Therefore, to construct, say, a functor $F: \mathcal{C} \to \mathcal{C}$, it suffices to define a bifunctor $A: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}ec_f$ satisfying suitable finiteness conditions, and then define $F(X)$ by the identity $\mathrm{Hom}(\cdot, F(X)) = A(\cdot, X)$; more formally, one would say "let $F(X)$ be the object representing the functor $A(\cdot, X)$". Similarly, all the functorial isomorphisms can be defined in terms of vector spaces.

Our goal in this section is to rewrite the axioms of a ribbon category in terms of the vector spaces

$$\langle W_1, \dots, W_n \rangle := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n). \tag{5.3.2}$$

This was first done in [**MS1**]. The following definition is essentially taken from [**MS1**]; for this reason, we think it is proper to commemorate their names.

DEFINITION 5.3.2. *Moore–Seiberg data* (MS data for short) for a semisimple abelian category $\mathcal{C}$ is the following collection of data:

- **Conformal blocks:** A collection of functors $\langle\, \rangle: \mathcal{C}^{\boxtimes n} \to \mathcal{V}ec_f$ ($n \geq 0$), which are locally finite in the first component: for every $A_1, \dots, A_{n-1} \in \mathcal{C}$, we have $\langle V_i, A_1, \dots, A_{n-1} \rangle = 0$ for all but a finite number of i. (Here $\mathcal{C}^{\boxtimes n}$ denotes the tensor product $\mathcal{C} \boxtimes \cdots \boxtimes \mathcal{C}$ defined in 1.1.15.)
- **Rotation isomorphisms:** Functorial isomorphisms
$$Z: \langle A_1, \dots, A_n \rangle \xrightarrow{\sim} \langle A_n, A_1, \dots, A_{n-1} \rangle.$$
- **R:** A symmetric object $R \in \text{ind}-\mathcal{C}^{\boxtimes 2}$ (see Section 2.4).

Gluing isomorphisms: For every $k, l \in \mathbb{Z}_+$ functorial isomorphisms

$$G: \langle A_1, \dots, A_k, R^{(1)} \rangle \otimes \langle R^{(2)}, B_1, \dots, B_l \rangle \xrightarrow{\sim} \langle A_1, \dots, A_k, B_1, \dots, B_l \rangle.$$

Commutativity isomorphism: A functorial isomorphism

$$\sigma: \langle X, A, B \rangle \xrightarrow{\sim} \langle X, B, A \rangle.$$

These data have to satisfy the axioms MS1–MS7 listed below.

MS1: Non-degeneracy: For every i, there exists an object X such that $\langle X, V_i \rangle \neq 0$.

MS2: Normalization: The functor $\langle\,\rangle: \mathcal{C}^0 \equiv \mathcal{V}ec_f \to \mathcal{V}ec_f$ is the identity functor.

MS3: Associativity of G: Let us consider two functorial isomorphisms

$$G'G'', G''G': \langle A_1, \dots, R'^{(1)} \rangle \otimes \langle R'^{(2)}, B_1, \dots, R''^{(1)} \rangle \otimes \langle R''^{(2)}, C_1, \dots, C_n \rangle$$
$$\xrightarrow{\sim} \langle A_1, \dots, B_1, \dots, C_1, \dots, C_n \rangle,$$

where R', R'' are two copies of R, and G', G'' are the corresponding gluing isomorphisms. Then $G'G'' = G''G'$.

MS4: Rotation axiom: $Z^n = \mathrm{id}: \langle A_1, \dots, A_n \rangle \xrightarrow{\sim} \langle A_1, \dots, A_n \rangle$.

MS5: Symmetry of G: For any $m, n \geq 0$ the following diagram is commutative:

$$\begin{tikzcd}
\langle A_1, \dots, A_n, R^{(1)} \rangle \otimes \langle R^{(2)}, B_1, \dots, B_m \rangle \arrow[r, "G"] \arrow[d, "P(Z \otimes Z^{-1})"'] & \langle A_1, \dots, A_n, B_1, \dots, B_m \rangle \arrow[d, "Z^m"'] \\
\langle B_1, \dots, B_m, R^{(2)} \rangle \otimes \langle R^{(1)}, A_1, \dots, A_n \rangle \arrow[r, "G \circ s"] & \langle B_1, \dots, B_m, A_1, \dots, A_n \rangle
\end{tikzcd} .$$

(Here P is the permutation of the two factors in the tensor product and $s: R^{\mathrm{op}} \xrightarrow{\sim} R$ is as in Section 2.4.)

MS6: Hexagon axioms: (i) The following diagram is commutative:

$$\begin{tikzcd}
\langle X, A, B, C \rangle \arrow[rr, "\sigma_{A,BC}"] \arrow[dr, "\sigma_{A,B}"'] & & \langle X, B, C, A \rangle \\
& \langle X, B, A, C \rangle \arrow[ur, "\sigma_{A,C}"'] &
\end{tikzcd}$$

where $\sigma_{A,BC}$ is defined as the composition

$$\langle X, A, B, C \rangle \xrightarrow{G^{-1}} \langle X, A, R^{(1)} \rangle \otimes \langle R^{(2)}, B, C \rangle$$
$$\xrightarrow{\sigma \otimes \mathrm{id}} \langle X, R^{(1)}, A \rangle \otimes \langle R^{(2)}, B, C \rangle \xrightarrow{Z^{-1} G (Z \otimes \mathrm{id})} \langle X, B, C, A \rangle,$$

and $\sigma_{A,B}$ is defined as the composition

$$\langle X, A, B, C \rangle \xrightarrow{G^{-1} Z} \langle C, X, R^{(1)} \rangle \otimes \langle R^{(2)}, A, B \rangle$$
$$\xrightarrow{\mathrm{id} \otimes \sigma} \langle C, X, R^{(1)} \rangle \otimes \langle R^{(2)}, B, A \rangle \xrightarrow{Z^{-1} G} \langle X, B, A, C \rangle.$$

(ii) The same, but with σ replaced by σ^{-1}.

MS7: Dehn twist axiom: $Z \sigma_{A,B} = \sigma_{B,A} Z: \langle A, B \rangle \xrightarrow{\sim} \langle A, B \rangle$, where $\sigma_{A,B} = G(\sigma \otimes \mathrm{id}) G^{-1}$ is defined similarly to MS6.

Now we describe how the MS data are related with the tensor structure on the category. Let $\mathcal{C}$ be a semisimple ribbon category. Define:

$$\langle A_1,\dots,A_n\rangle = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A_1\otimes\dots\otimes A_n), \tag{5.3.3}$$

$$R = \bigoplus V_i^*\otimes V_i, \qquad \text{cf. (2.4.7)}, \tag{5.3.4}$$

$$\begin{aligned} Z\colon \mathrm{Hom}(\mathbf{1}, A_1\otimes\dots\otimes A_n) &\xrightarrow{\sim} \mathrm{Hom}({}^*A_n, A_1\otimes\dots\otimes A_{n-1})\\ &\xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, {}^{**}A_n\otimes A_1\otimes\dots\otimes A_{n-1})\\ &\xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, A_n\otimes A_1\otimes\dots\otimes A_{n-1}),\end{aligned} \tag{5.3.5}$$

$$\begin{aligned} G\colon \bigoplus \mathrm{Hom}(\mathbf{1}, A_1\otimes\dots\otimes A_n\otimes V_i^*)\otimes\mathrm{Hom}(\mathbf{1}, V_i\otimes B_1\otimes\dots\otimes B_k)&\\ \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, A_1\otimes\dots\otimes A_n\otimes V_i^*)\otimes\mathrm{Hom}(V_i^*, B_1\otimes\dots\otimes B_k)&\\ \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, A_1\otimes\dots\otimes A_n\otimes B_1\otimes\dots\otimes B_k),&\end{aligned} \tag{5.3.6}$$

$$\sigma\colon \mathrm{Hom}(\mathbf{1}, X\otimes A\otimes B)\xrightarrow{\sim}\mathrm{Hom}(\mathbf{1}, X\otimes B\otimes A). \tag{5.3.7}$$

Here we used the rigidity isomorphisms (2.1.13, 2.1.14), the isomorphisms $\delta\colon V\xrightarrow{\sim} V^{**}$, and the fact that in a semisimple category, $\mathrm{Hom}(X,Y)\simeq\bigoplus\mathrm{Hom}(X,V_i)\otimes\mathrm{Hom}(V_i,Y)$.

PROPOSITION 5.3.3. *If $\mathcal{C}$ is a semisimple ribbon category, formulas* (5.3.3)–(5.3.7) *define MS data.*

The proof of this proposition is straightforward: if we use the technique of ribbon graphs developed in Chapter 1, then all the axioms are obvious. □

A natural question is whether this proposition can be reversed, i.e., is it true that every collection of MS data on a semisimple abelian category comes from a structure of a ribbon category. It turns out that it is almost true; to get a precise statement, we must somewhat weaken the rigidity axiom.

Let $\mathcal{C}$ be a monoidal category. We say that an object $V\in\mathrm{Ob}\,\mathcal{C}$ has a weak dual if the functor $\mathrm{Hom}(\mathbf{1}, V\otimes\cdot)$ is representable. In this case, we denote the corresponding representing object by V^*: $\mathrm{Hom}(\mathbf{1}, V\otimes X)=\mathrm{Hom}(V^*,X)$. Obviously, $*$ is functorial: every morphism $f\colon V\to W$ defines a morphism $f^*\colon W^*\to V^*$, provided that V^*, W^* exist.

DEFINITION 5.3.4. A monoidal category $\mathcal{C}$ is called *weakly rigid* if every object has a weak dual and $*\colon\mathcal{C}\to\mathcal{C}^{\mathrm{op}}$ is an equivalence of categories.

Of course, every rigid category is weakly rigid; the converse, however, is not true. It is also useful to note that in every weakly rigid category we have a canonical morphism $i_V\colon\mathbf{1}\to V\otimes V^*$, corresponding to $\mathrm{id}\in\mathrm{Hom}(V^*,V^*)=\mathrm{Hom}(\mathbf{1}, V\otimes V^*)$. If the category is rigid, then i_V defined in this way coincides with the one defined by the rigidity axioms.

DEFINITION 5.3.5. A *weakly ribbon* category is a weakly rigid braided tensor category $\mathcal{C}$ endowed with a family of functorial isomorphisms $\theta\colon V\xrightarrow{\sim}V$ satisfying (2.2.8)–(2.2.10).

As discussed in Section 2.2, for a rigid category defining θ satisfying (2.2.8)–(2.2.10) is equivalent to defining $\delta\colon V\xrightarrow{\sim}V^{**}$, so every ribbon category is also weakly ribbon.

EXERCISE 5.3.6. (i) Show that in every semisimple weakly ribbon category, the map $\phi\colon \mathrm{Hom}(V^*, X) \to \mathrm{Hom}(\mathbf{1}, X \otimes V^{**})$ given by $\psi \mapsto (\psi \otimes \mathrm{id})i_{V^*}$ is an isomorphism.

(ii) Show that in every semisimple weakly ribbon category one can define a family of functorial isomorphisms $\delta\colon V \xrightarrow{\sim} V^{**}$ by the condition that the following diagram be commutative:

$$\begin{array}{ccc} \langle V, X\rangle & \xrightarrow{\simeq} & \mathrm{Hom}(V^*, X) \\ \sigma\big\downarrow & & \big\downarrow\phi \\ \langle X, V\rangle & \xrightarrow{\mathrm{id}\otimes\delta} & \langle X, V^{**}\rangle \end{array}.$$

(iii) Show that in every semisimple weakly ribbon category, one has $(\theta_A \otimes \mathrm{id})f = (\mathrm{id} \otimes \theta_B)f$ for every $f\colon \mathbf{1} \to A \otimes B$. (Hint: use $\theta_V^* = \theta_{V^*}$.)

Note, however, that in general, $(V \otimes W)^* \not\simeq W^* \otimes V^*$, so the axiom $\delta_{V\otimes W} = \delta_V \otimes \delta_W$ does not make sense.

REMARK 5.3.7. The authors do not know any example of a semisimple abelian category which is weakly rigid but not rigid.

Now we can formulate the main theorem of this section.

THEOREM 5.3.8. *Let $\mathcal{C}$ be a semisimple weakly ribbon category with simple objects $V_i, i \in I$. Then formulas* (5.3.3)–(5.3.7), *with δ defined as in Exercise* 5.3.6, *define MS data for $\mathcal{C}$. Conversely, every collection of MS data for a semisimple abelian category $\mathcal{C}$ is obtained in this way.*

PROOF. The first statement of the theorem is parallel to Proposition 5.3.3. The proof is also quite parallel; we just have to check that all the arguments work in a weakly rigid category as well as in a rigid one. This is left to the reader as an exercise; part of it is contained in Exercise 5.3.6. In particular, the identity (2.2.8) $\theta_{V\otimes W} = \sigma_{WV}\sigma_{VW}(\theta_V \otimes \theta_W)$ will give the Rotation axiom, and the identity (2.2.10) $\theta_{V^*} = \theta_V^*$ will give the Dehn twist axiom.

The proof of the converse statement is more complicated. For convenience, we split it into several steps. To simplify the notation, we will write just $\langle \dots, R\rangle \otimes \langle R, \dots\rangle$, omitting the superscripts. Since R is symmetric, this causes no problems. The symmetry of G axiom MS5 implies that the order of the factors is not important for defining G. We will implicitly use this.

Let us start by constructing the duality and tensor product on $\mathcal{C}$ from the MS data.

LEMMA 5.3.9. *Given MS data for $\mathcal{C}$, there exists an involution $*\colon I \to I$ such that $\dim\langle V_i, V_j\rangle = \delta_{i,j^*}$. Also, R is isomorphic (non-canonically) to $\bigoplus V_i \boxtimes V_{i^*}$.*

PROOF. Define $A_{ij} = \dim\langle V_i, V_j\rangle$, and define B_{ij} by $R \simeq \bigoplus B_{ij} V_i \boxtimes V_j$. It follows from the non-degeneracy axiom and the existence of Z that A is a symmetric matrix with no zero rows or columns. From the symmetry of R, we get that B is a symmetric matrix.

Writing the identity $\langle V_i, V_j\rangle = \langle V_i, R^{(1)}\rangle \otimes \langle R^{(2)}, V_j\rangle$ we get the identity $A = ABA$. We leave it to the reader to show that if A, B are symmetric matrices with non-negative integer entries and A has no zero columns, then such an identity is possible only if $A = B$ is a permutation of order 2. (Hint: use $AB = (AB)^2$ to prove that AB either has a zero row or column, or it is the identity matrix.) $\square$

1. Defining the duality functor. Define the functor $*$ by

$$\operatorname{Hom}(V^*, X) = \langle V, X\rangle \tag{5.3.8}$$

(see Lemma 5.3.1). Then the previous lemma immediately implies $V_i^* \simeq V_{i^*}$ (not canonically!). It is easy to see from this that $*$ is an anti-equivalence of categories. In particular, this implies that every object $V \in \mathcal{C}$ is completely determined by the functor $\langle V, \cdot\rangle = \operatorname{Hom}(V^*, \cdot)$.

Note that if the MS data come from the structure of a weakly ribbon category on $\mathcal{C}$ (see Proposition 5.3.3), then the $*$ functor defined above coincides with the one given by the rigidity axioms.

2. $R = \bigoplus V_i^* \boxtimes V_i$**.** To prove this, let us write $R \simeq \sum X_i \boxtimes V_i$ for some $X_i \in \operatorname{ind}-\mathcal{C}$. The isomorphism G gives, in particular, an isomorphism

$$\langle A, V_i^*\rangle \simeq \bigoplus \langle A, X_i\rangle \otimes \langle V_i, V_i^*\rangle.$$

Since $\langle V_i, V_i^*\rangle = \operatorname{Hom}(V_i^*, V_i^*) = k$, we get canonical isomorphisms $\langle A, V_i^*\rangle = \langle A, X_i\rangle$. Thus, we have constructed an isomorphism $R \simeq \bigoplus V_i^* \boxtimes V_i$ such that the isomorphism $G\colon \langle X, Y\rangle \simeq \langle X, R\rangle \otimes \langle R, Y\rangle$ is given by (5.3.6).

3. Tensor product. Define the functor $\otimes\colon \mathcal{C}^{\boxtimes 2} \to \mathcal{C}$ by

$$\langle X, A\otimes B\rangle = \langle X, A, B\rangle, \tag{5.3.9}$$

(it is well defined by the results of Step 1). More generally, define the tensor product of n objects by the following formula:

$$\langle X, A_1 \otimes \cdots \otimes A_n\rangle = \langle X, A_1, \dots, A_n\rangle.$$

Next, define isomorphisms

$$\begin{aligned} A_1 \otimes \cdots \otimes A_i \otimes (B_1 \otimes \cdots \otimes B_k) \otimes A_{i+1} \otimes \cdots \otimes A_n \\ \simeq A_1 \otimes \cdots \otimes A_i \otimes B_1 \otimes \cdots \otimes B_k \otimes A_{i+1} \otimes \cdots \otimes A_n \end{aligned} \tag{5.3.10}$$

as the following composition:

$$\begin{aligned} &\langle X, A_1, \dots, A_i, B_1 \otimes \cdots \otimes B_k, A_{i+1}, \dots, A_n\rangle \\ &\quad \simeq \langle X, A_1, \dots, A_i, R, A_{i+1}, \dots, A_n\rangle \otimes \langle R, B_1 \otimes \cdots \otimes B_k\rangle \\ &\quad \simeq \langle X, A_1, \dots, A_i, R, A_{i+1}, \dots, A_n\rangle \otimes \langle R, B_1, \dots, B_k\rangle \\ &\quad \simeq \langle X, A_1, \dots, A_i, B_1, \dots, B_k, A_{i+1}, \dots, A_n\rangle, \end{aligned}$$

where the isomorphisms are, respectively, G^{-1}, the definition of tensor product, and G.

LEMMA 5.3.10. *Let X be an expression of the form*

$$X = (A_1 \otimes (A_2 \otimes \cdots)) \otimes A_n$$

with any grammatically correct parentheses arrangement (parentheses may enclose any number of factors). Then any two isomorphisms

$$\varphi\colon X \simeq A_1 \otimes \cdots \otimes A_n,$$

obtained as a composition of the morphisms of the form (5.3.10)*, are equal.*

PROOF. Easy induction arguments show that it suffices to prove this statement in the case when we have just two pairs of parentheses. Thus, we need to consider the arrangements of the form $\cdots(\cdots(\cdots)\cdots)\cdots$ and $\cdots(\cdots)\cdots(\cdots)\cdots$. For both of them the statement easily follows from the definitions and the associativity of G. □

This shows that $\otimes$ is indeed associative; in particular, we can define associativity constraint $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ which satisfies the pentagon axiom.

4. Unit. Define the object $\mathbf{1} \in \mathcal{C}$ by

$$\langle \mathbf{1}, X \rangle = \langle X \rangle \tag{5.3.11}$$

(as before, it is well defined due to the results of Step 1).

Define morphisms $\langle A_1, \dots, A_i, \mathbf{1}, A_{i+1}, \dots, A_n \rangle \simeq \langle A_1, \dots, A_i, A_{i+1}, \dots, A_n \rangle$ as the following composition

$$\begin{aligned} \langle A_1, \dots, A_i, \mathbf{1}, A_{i+1}, \dots, A_n \rangle &\simeq \langle A_1, \dots, A_i, R, A_{i+1}, \dots, A_n \rangle \otimes \langle \mathbf{1}, R \rangle \\ &\simeq \langle A_1, \dots, A_i, R, A_{i+1}, \dots, A_n \rangle \otimes \langle R \rangle \simeq \langle A_1, \dots, A_i, A_{i+1}, \dots, A_n \rangle. \end{aligned}$$

Note that this construction remains valid for $n = 0$, in which case, using the normalization axiom, we get

$$\langle \mathbf{1} \rangle = k. \tag{5.3.12}$$

Using the definition of tensor product, we see that the isomorphism

$$\langle X, A_1, \dots, A_i, \mathbf{1}, A_{i+1}, \dots, A_n \rangle \simeq \langle X, A_1, \dots, A_i, A_{i+1}, \dots, A_n \rangle$$

gives rise to an isomorphism

$$A_1 \otimes \cdots \otimes A_i \otimes \mathbf{1} \otimes A_{i+1} \otimes \cdots \otimes A_n \simeq A_1 \otimes \cdots \otimes A_i \otimes A_{i+1} \otimes \cdots \otimes A_n. \tag{5.3.13}$$

LEMMA 5.3.11. *The following diagram, with the horizontal map given by the associativity isomorphism and the two others by the unit isomorphisms* (5.3.13)*, is commutative*:

$$\begin{tikzcd} A \otimes (\mathbf{1} \otimes B) \arrow[rr] \arrow[dr] & & A \otimes \mathbf{1} \otimes B \arrow[dl] \\ & A \otimes B & \end{tikzcd} .$$

PROOF. Looking at the definitions, we see that the statement is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} \langle X, A, R' \rangle \otimes \langle R', \mathbf{1} \otimes B \rangle \arrow[r] \arrow[d] & \langle X, A, R' \rangle \otimes \langle R', B \rangle \arrow[r] & \langle X, A, B \rangle \\ \langle X, A, \mathbf{1}, B \rangle \arrow[r] & \langle X, A, R'', B \rangle \otimes \langle \mathbf{1}, R'' \rangle \arrow[r] & \langle X, A, R'', B \rangle \otimes \langle R'' \rangle \arrow[u] \end{tikzcd}$$

where, as before, R' and R'' are two copies of R. But this easily follows from the associativity of G applied to the space $\langle X, A, R'', R' \rangle \otimes \langle \mathbf{1}, R'' \rangle \otimes \langle R', B \rangle$. We leave the details to the reader. □

COROLLARY 5.3.12. *The isomorphisms* $\mathbf{1} \otimes X \xrightarrow{\sim} X$ *and* $X \otimes \mathbf{1} \xrightarrow{\sim} X$*, given by* (5.3.13)*, satisfy the triangle axiom.*

Combining this fact with the MacLane coherence theorem (Theorem 1.1.9), we see that the MS data indeed defines a structure of a monoidal category on $\mathcal{C}$.

5. Definition of $\langle\, \rangle$. Using the unit isomorphisms (5.3.13), we can identify

$$\langle A_1, \dots, A_n \rangle \xrightarrow{\sim} \langle \mathbf{1}, A_1, \dots, A_n \rangle \xrightarrow{\sim} \operatorname{Hom}(\mathbf{1}^*, A_1 \otimes \cdots \otimes A_n).$$

Next, let us construct an isomorphism $\mathbf{1} \xrightarrow{\sim} \mathbf{1}^*$. Using (5.3.12), we can write $\operatorname{Hom}(\mathbf{1}^*, \mathbf{1}) = \langle \mathbf{1} \rangle = k$. Thus, $1 \in k$ gives an isomorphism $\mathbf{1} \xrightarrow{\sim} \mathbf{1}^*$; combining this isomorphism with the previous identity, we can identify

$$\langle A_1, \dots, A_n \rangle \simeq \operatorname{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_n). \tag{5.3.14}$$

6. Commutativity isomorphism. Define the commutativity isomorphism $\sigma\colon A \otimes B \to B \otimes A$ using the following composition:

$$\langle X, A \otimes B \rangle = \langle X, A, B \rangle \xrightarrow{\sigma} \langle X, B, A \rangle = \langle X, B \otimes A \rangle.$$

Then one easily sees that the Hexagon axioms given in Theorem 1.2.5(iii) are immediate corollaries of the Hexagon axioms for MS data. Thus, the MS data defines a structure of a BTC on $\mathcal{C}$.

7. Balancing. Consider the functorial isomorphism

$$\langle V, X \rangle \xrightarrow{\sigma^{-1}} \langle X, V \rangle \xrightarrow{Z} \langle V, X \rangle. \tag{5.3.15}$$

By Lemma 5.3.1, there exists a functorial isomorphism $\theta_V\colon V \xrightarrow{\sim} V$ such that the above composition is given by $\theta_V \otimes \mathrm{id}_X$. One easily checks that $\theta_{\mathbf{1}} = \mathrm{id}$ and that

$$\theta_{W_1}^{-1} = Z\sigma_{W_1, W_2 \otimes \cdots \otimes W_n} = \sigma_{W_2 \otimes \cdots \otimes W_n, W_1} Z^{-1} \colon \langle W_1, \dots, W_n \rangle \xrightarrow{\sim} \langle W_1, \dots, W_n \rangle$$

(this is where we need the Dehn twist axiom MS7).

To prove the identity $\theta_{A\otimes B} = \sigma_{B,A}\sigma_{A,B}(\theta_A \otimes \theta_B)$, note that it is equivalent to

$$\sigma_{B,A}\sigma_{A,B}\theta_A\theta_C^{-1}\theta_B = \mathrm{id}\colon \langle A, B, C \rangle \xrightarrow{\sim} \langle A, B, C \rangle, \tag{5.3.16}$$

?!

which follows from the identities

$$\begin{aligned}
\theta_A^{-1} &= Z\sigma_{A,BC} = Z\sigma_{A,C}\sigma_{A,B},\\
\theta_B^{-1} &= \sigma_{B,A} Z \sigma_{B,C},\\
\theta_C^{-1} &= Z\sigma_{A,C} Z \sigma_{B,C}.
\end{aligned}$$

Finally, we leave it to the reader to show that the Dehn twist axiom MF7 is essentially equivalent to the identity $\theta_{V^*} = \theta_V^*$. Thus, the so defined θ satisfies the balancing axioms (2.2.8)–(2.2.10).

This completes the proof of Theorem 5.3.8. $\square$

It would be nice if we had some axiom for MS data which would automatically ensure that the corresponding BTC is rigid. However, the only way of doing it that we know of is explicitly imposing the rigidity condition. (It is claimed in [**MS2**] that rigidity follows from the other axioms; however, at some point, they say "we can check the universality property" without doing it explicitly—we were unable to reconstruct their arguments.)

In the semisimple case the rigidity condition is equivalent to the non-vanishing of certain coefficients, which shows that "almost all" weakly rigid semisimple categories are rigid.

Let $\mathcal{C}$ be a semisimple weakly rigid monoidal category such that $V^{**} \simeq V$ (as discussed above, this holds for any category obtained from MS data). Let $\varphi_i\colon V_i^* \to$

$V_i^* \otimes V_i \otimes V_i^*$ be given by $\varphi_i = \mathrm{id} \otimes i_{V_i}$. Using the associativity isomorphism, we can write

$$\varphi_i = a_i \otimes \mathrm{id} + \sum_{j \neq 0} \psi_j,$$

where a_i are certain morphisms $\mathbf{1} \to V_i^* \otimes V_i$, and ψ_j are some morphisms which are obtained as the composition

$$V_i^* \to V_j \otimes V_i^* \xrightarrow{\psi_j' \otimes \mathrm{id}} (V_i^* \otimes V_i) \otimes V_i^*.$$

Note that since $V_i^* \otimes V_i$ contains $\mathbf{1}$ with multiplicity one, the morphisms a_i lie in a one-dimensional space.

PROPOSITION 5.3.13. *Let $\mathcal{C}$ be a semisimple weakly rigid monoidal category such that $V^{**} \simeq V$, and let $a_i \colon \mathbf{1} \to V_i^* \otimes V_i$ be defined as above. Then $\mathcal{C}$ is rigid iff $a_i \neq 0$ for all $i \in I$.*

PROOF. If $\mathcal{C}$ is rigid, then $e_{V_i} a_i = 1$, which immediately follows from taking composition of φ_i with $e_{V_i} \otimes \mathrm{id}$. Thus, $a_i \neq 0$. Conversely, assume that $a_i \neq 0$. Then define $e_{V_i} \colon V_i^* \otimes V_i \to \mathbf{1}$ by the condition $e_{V_i} a_i = 1$; since $V_i^* \otimes V_i$ contains $\mathbf{1}$ with multiplicity one, this is possible. From this condition, we immediately see that the composition

$$V_i^* \xrightarrow{\mathrm{id} \otimes i_{V_i}} V_i^* \otimes V_i \otimes V_i^* \xrightarrow{e_{V_i} \otimes \mathrm{id}} V_i^*$$

is equal to identity; thus, the second rigidity axiom (2.1.6) is satisfied.

To check the first rigidity axiom, denote the composition

$$V_i \xrightarrow{i_{V_i} \otimes \mathrm{id}} V_i \otimes V_i^* \otimes V_i \xrightarrow{\mathrm{id} \otimes e_{V_i}} V_i$$

by c_i; since $\mathrm{End}(V_i) = k$, c_i is a number. We need to show that $c_i = 1$.

Consider the composition

$$\Phi \colon \mathbf{1} \xrightarrow{i \otimes i} V_i \otimes V_i^* \otimes V_i \otimes V_i^* \xrightarrow{\mathrm{id} \otimes e \otimes \mathrm{id}} V_i \otimes V_i^*.$$

From the second rigidity axiom (already proved), $\Phi = i_{V_i}$. On the other hand, form the definition of c_i, we have $\Phi = c_i i_{V_i}$. This proves $c_i = 1$ and thus, the first rigidity axiom for V_i.

Therefore, if $a_i \neq 0$, then V_i is rigid. But since a direct sum of rigid objects is again rigid, every object in $\mathcal{C}$ is rigid. □

5.4. Modular functor in genus zero and tensor categories

In this section we prove the first main theorem of this chapter, establishing that the axioms of a (weakly) ribbon category are essentially equivalent to the axioms of a modular functor in genus zero.

Let $\mathcal{C}$ be a semisimple abelian category with representatives of the equivalence classes of simple objects V_i, $i \in I$. Let us call a *$\mathcal{C}$-extended modular functor in genus zero* the same data as in Definition 5.1.12 but with the spaces $\tau(\Sigma)$ defined only for Σ of genus zero; therefore, the only gluing allowed is the gluing of two different connected components.

THEOREM 5.4.1 (Moore–Seiberg [**MS1**]). *Let $\mathcal{C}$ be a semisimple weakly ribbon category. Then there is a unique $\mathcal{C}$-extended genus zero modular functor satisfying the properties* (i)–(iii) *below.*

(i) *For the standard sphere* $S_{0,n}$ (*see* (5.2.1)):

$$\tau(S_{0,n}; W_1, \dots, W_n) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) =: \langle W_1, \dots, W_n \rangle. \tag{5.4.1}$$

(ii) $R = \bigoplus V_i^* \otimes V_i$, *and the isomorphism* $s\colon R \xrightarrow{\sim} R^{\mathrm{op}}$ *is given by* (2.4.8).
(iii) *We have*:

$$z_* = Z, \quad b_* = \sigma, \tag{5.4.2}$$

where the homeomorphisms z, b *are defined by* (5.2.2), *and the isomorphisms* Z, σ *are defined by* (5.3.5), (5.3.7). *Also, for every* $k, l \geq 0$, *the composition*

$$\tau(S_{0,k+1}; \dots, R^{(1)}) \otimes \tau(S_{0,l+1}; R^{(2)}, \dots) \to \tau(S_{0,k+1} \sqcup_{k+1,1} S_{0,l+1}) \xrightarrow{(\alpha_{kl})_*} \tau(S_{0,k+l}),$$

where the first arrow is the sewing isomorphism (5.1.1) *and* α_{kl} *is as in* (5.2.3), *coincides with the isomorphism* G *defined by* (5.3.6).

This modular functor is non-degenerate and has the following properties:

(iv) *Let* $t_i\colon S_{0,n} \to S_{0,n}$ *be the Dehn twist around* i^{th} *puncture. Then, under the isomorphism* (5.4.1), $(t_i)_*$ *is given by the twist*

$$\theta_{W_i}\colon \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n).$$

(v) *If* $\mathcal{C}$ *is rigid, then this modular functor is unitary, with the pairing* (5.1.2)

$$\langle , \rangle_{S_{0,n}}\colon \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_n^* \otimes \cdots \otimes W_1^*) \to k$$

given by

$$\langle \varphi, \psi \rangle\colon \mathbf{1} \to \mathbf{1} \otimes \mathbf{1} \to W_1 \otimes \cdots \otimes W_n \otimes W_n^* \otimes \cdots \otimes W_1^* \to \mathbf{1}.$$

Here we identify $k = \mathrm{End}(\mathbf{1})$ *and use the fact that for a standard sphere* $S_{0,n}$, *there is a canonical isomorphism* $\overline{S_{0,n}} \xrightarrow{\sim} S_{0,n}$, *which reverses the order of the punctures. This isomorphism is given by the reflection around the imaginary axis.*

Conversely, let τ *be a non-degenerate genus zero* $\mathcal{C}$*-extended MF. Then there is a unique structure of a weakly ribbon category on* $\mathcal{C}$ *such that the above properties* (i)–(iii) *hold.*

Proof. The proof is based on the comparison of the results of Sections 5.2 and 5.3. Since by Theorem 5.3.8 the structure of a weakly ribbon category on $\mathcal{C}$ is equivalent to what we called MS data, it suffices to show that a non-degenerate genus zero MF defines MS data and vice versa.

Let us assume we are given a collection of MS data. To construct a genus zero MF, let us first consider the pairs (Σ, M), where $M = (C, \{\psi_a\})$ is a parameterization of Σ (see Definition 5.2.1). For each such pair, define the vector space $\tau(\Sigma, M)$ as follows. For every cut c, take a copy R_c of the object R, and define

$$\tau(\Sigma, M) = \bigotimes_a \tau(S_{0,n_a}), \tag{5.4.3}$$

where the index a runs through the set of connected components of $\Sigma \setminus C$, and for each connected component Σ_a, we assign $R_c^{(\varepsilon)}$ to every boundary component of Σ_a which is a cut, where $\varepsilon \in \{1, 2\}$ is chosen so that for one of the occurrences of R_c we take $\varepsilon = 1$ and for the other we take $\varepsilon = 2$ (note that each R_c appears exactly twice in (5.4.3)). Since R is symmetric, it does not matter which occurrence is which.

More explicitly, the same formula can be written as follows. For each cut $c \in C$, choose one of its sides as "positive" and the other as "negative". Then we can define

$$\tau(\Sigma, M) = \bigoplus_{i_c \in I,\, c \in C} \bigotimes_a \tau(S_{0,n_a}), \tag{5.4.4}$$

where the sum is taken over all ways to assign an index $i_c \in I$ to every cut $c \in C$, and for each connected component Σ_a of $\Sigma \setminus C$ we assign V_{i_c} to its boundary component if it is the positive side of the cut c, and $V_{i_c}^*$ if it is the negative side of the cut c. This formula depends on the choice of "positive" side for each cut; to identify the formulas corresponding to different choices, one has to use the canonical isomorphism $V_i^* \boxtimes V_i \xrightarrow{\sim} V_{i^*} \boxtimes V_{i^*}^*$ defined in (2.4.8).

For example, if Σ is a sphere with 4 holes which we index by $\alpha, \beta, \gamma, \delta$, and φ is a parameterization with one cut c as in Figure 5.16, then the above formula gives

$$\begin{aligned}\tau(\Sigma, \varphi; W_\alpha, W_\beta, W_\gamma, W_\delta) &= \langle W_\alpha, W_\beta, R^{(1)}\rangle \otimes \langle R^{(2)}, W_\gamma, W_\delta\rangle \\ &= \bigoplus_{i \in I} \langle W_\alpha, W_\beta, V_i\rangle \otimes \langle V_i^*, W_\gamma, W_\delta\rangle.\end{aligned}$$

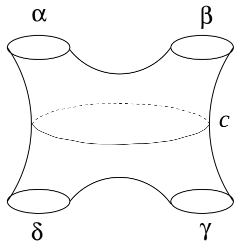


FIGURE 5.16

Of course, every extended surface Σ can be parametrized in many ways. However, if we construct a system of isomorphisms $f_{M,M'} : \tau(\Sigma, M') \xrightarrow{\sim} \tau(\Sigma, M)$, compatible in the following sense: $f_{M,M'} f_{M',M''} = f_{M,M''}$, then we can identify all of these spaces with each other and define the space $\tau(\Sigma)$, which is canonically isomorphic to each of $\tau(\Sigma, M)$ (see a formal definition in the proof of Theorem 4.4.3).

Moreover, such a system of isomorphisms would automatically give a representation of the extended mapping class groupoid $\mathcal{T}eich$, as follows. Let $f \colon \Sigma_1 \xrightarrow{\sim} \Sigma_2$ be a homeomorphism of extended surfaces, and let M_2 be a parameterization of Σ_2. Then f gives rise to a parameterization M_1 of Σ_1 in the obvious way. Moreover, f establishes a one-to-one correspondence between the cuts C_1 on Σ_1 and C_2 on Σ_2, and between the components $(\Sigma_1)_a$ and $(\Sigma_2)_a$. Thus, f gives rise to an identification $\tau(\Sigma_1, M_1) = \bigoplus_{i_c \in I,\, c \in C_1} \bigotimes_a \tau(S_{0,n_a}) = \tau(\Sigma_2, M_2)$. Combining this with the isomorphisms $\tau(\Sigma_1) = \tau(\Sigma_1, M_1)$, $\tau(\Sigma_2) = \tau(\Sigma_2, M_2)$, we get an isomorphism $f_* \colon \tau(\Sigma_1) \xrightarrow{\sim} \tau(\Sigma_2)$. We leave it to the reader to check that this isomorphism does not depend on the choice of M_2 and satisfies $(fg)_* = f_* g_*, \mathrm{id}_* = \mathrm{id}$. Also, it is immediately obvious from (5.4.3) that the so constructed modular functor will satisfy the gluing axiom.

Therefore, our goal is to construct a compatible system of isomorphisms $\tau(\Sigma, M') \xrightarrow{\sim} \tau(\Sigma, M)$. By Theorem 5.2.3, every two parameterizations can be connected by a sequence of simple moves Z, B, F; let us assign to these moves the isomorphisms

Z, σ, G given by the MS data. A comparison of the axioms MF1–MF7 and MS1–MS7 shows that all the relations among the moves Z, B, F also hold for their analogues Z, σ, G; the only relation which is not immediately obvious is the cylinder axiom MF5, but it follows from the functoriality of the morphisms Z, σ, G. Thus, every MS data defines a genus zero MF.

The construction in the opposite direction is quite similar. Assume that we have a genus zero MF. Define the functors $\langle\,\rangle$ and the isomorphisms Z, σ, G as in the statement of the theorem. Again, a comparison of the axioms MF1–MF7 and MS1–MS7 shows that these data satisfy the axioms of MS data. This completes the proof of Theorem 5.4.1. $\square$

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EXAMPLE 5.4.2. Consider the surface Σ and the "associativity move" $M \overset{F_c}{\rightsquigarrow} M_0 \overset{F_{c'}^{-1}}{\rightsquigarrow} M'$ shown in Figure 5.17. Assign to the boundary components $\alpha, \dots, \delta$ objects $A, \dots, D$. Then:

$$\begin{aligned}
\tau(\Sigma, M) &= \bigoplus_{i \in I} \langle A, B, V_i\rangle \otimes \langle V_i^*, C, D\rangle,\\
\tau(\Sigma, M_0) &= \langle A, B, C, D\rangle,\\
\tau(\Sigma, M') &= \bigoplus_{j \in I} \langle D, A, V_j\rangle \otimes \langle V_j^*, B, C\rangle.
\end{aligned}$$

Then the corresponding isomorphisms $\tau(\Sigma, M) \to \tau(\Sigma, M_0) \to \tau(\Sigma, M')$ are given by Figure 5.18 below.

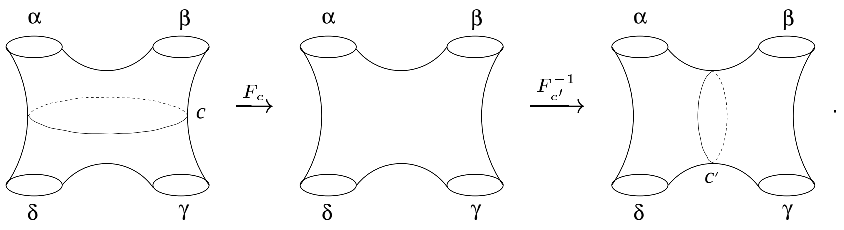


FIGURE 5.17. Associativity move.

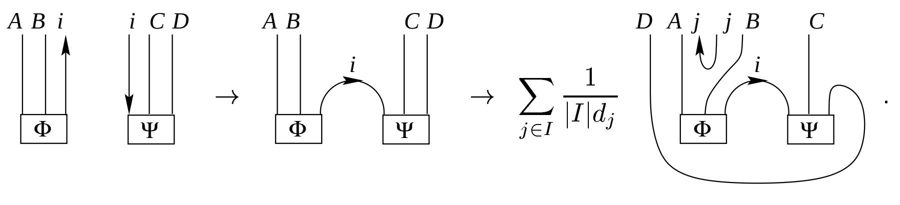


FIGURE 5.18. Associativity isomorphism.

5.5. Modular categories and modular functor for zero central charge

In this section, we will show, developing the ideas of the previous section, that the notion of a modular functor (for arbitrary genus) is equivalent to the notion of a modular tensor category. Recall that for every modular category we have defined the numbers $p^{\pm}$ by (3.1.7). In this section we consider the special case of modular categories with $p^+/p^- = 1$. (For the modular categories coming from conformal field theory this identity holds if the Virasoro central charge of the theory is equal to 0 (cf. Remark 3.1.20), hence the title of this section.)

THEOREM 5.5.1. *Let $\mathcal{C}$ be a modular tensor category with $p^+/p^- = 1$. Then there exists a unique $\mathcal{C}$-extended modular functor τ which satisfies conditions* (i)–(iii) *of Theorem* 5.4.1. *This MF is non-degenerate and satisfies conditions* (iv), (v) *of Theorem* 5.4.1 *and condition* (vi) *below.*

(vi) *Let $S_{1,1}$ be the torus with one hole. Identify*

$$\tau(S_{1,1}; A) = \bigoplus \langle A, V_i, V_i^* \rangle = \bigoplus \operatorname{Hom}(A^*, V_i \otimes V_i^*)$$

using the parameterization of $S_{1,1}$ shown in Figure 5.12. *Let $s\colon S_{1,1} \to S_{1,1}$ be as defined in* (5.1.5). *Then the corresponding*

$$s_* = S\colon \bigoplus \operatorname{Hom}(A^*, V_i \otimes V_i^*) \to \bigoplus \operatorname{Hom}(A^*, V_i \otimes V_i^*) \tag{5.5.1}$$

is given by Theorem 3.1.17.

Conversely, let $\mathcal{C}$ be a semisimple abelian category, and let τ be a non-degenerate $\mathcal{C}$-extended MF. Assume for simplicity that the corresponding structure of a monoidal category on $\mathcal{C}$ (see Theorem 5.4.1) *is rigid. Then $\mathcal{C}$ is a modular tensor category with $p^+ = p^-$; in particular, it has only a finite number of simple objects.*

PROOF. Assume that $\mathcal{C}$ is a modular category. By Theorem 5.4.1, the structure of a modular category on $\mathcal{C}$ defines a genus zero MF. Therefore, we only need to show that this MF can be extended to positive genus. In order to do this, by Theorem 5.2.9, we need to define an isomorphism $S\colon \tau(S_{1,1}, M) \xrightarrow{\sim} \tau(S_{1,1}, M')$, where $S_{1,1}$ is the standard torus and M, M' are the parameterizations shown in Figure 5.12, and then check that relations MF8, MF9 are satisfied.

Note that by definition

$$\tau(S_{1,1}, M; A) = \tau(S_{1,1}, M'; A) = \bigoplus_i \langle A, V_i, V_i^* \rangle = \operatorname{Hom}(A^*, H),$$

where, as before, $H = \bigoplus V_i \otimes V_i^*$. Thus, defining an isomorphism $S\colon \tau(S_{1,1}, M) \xrightarrow{\sim} \tau(S_{1,1}, M')$ is the same as defining a functorial system of isomorphisms $\operatorname{Hom}(A^*, H) \xrightarrow{\sim} \operatorname{Hom}(A^*, H)$ for every object A. By Lemma 5.3.1, this is the same as defining an isomorphism $S\colon H \to H$.

Let us first show that if we define S as in the statement of the theorem, then relations MF8, MF9 are satisfied. Relations MF8 immediately follow from Theorem 3.1.17 and the assumption $p^+ = p^-$.

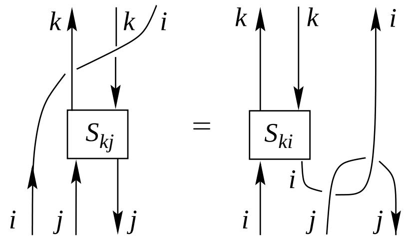
To check relation MF9 for a torus with two holes, let us rewrite it in terms of tensor categories. ?!

LEMMA 5.5.2. *Let $\mathcal{C}$ be a semisimple ribbon category with finite number of simple objects, and let S be an isomorphism*

$$S = \bigoplus S_{ji}\colon \bigoplus V_i \otimes V_i^* \to \bigoplus V_j \otimes V_j^*. \tag{5.5.2}$$

Then relation MF9 *for S is equivalent to the following condition*:

(5.5.3)



for every $i, j, k \in I$.

The proof of this lemma will be given after the proof of the theorem.

It is easy to check that the operator S defined by (3.1.32) satisfies (5.5.3).

Now, let us prove uniqueness. Assume that we have defined an operator S of the form (5.5.2) such that relations MF8, MF9 are satisfied. Rewrite relation MF9 in the form (5.5.3), put $j = 0$ and note that $S_{k0}\colon \mathbf{1} \to V_k \otimes V_k^*$ is a non-zero multiple of i_{V_k}. This immediately implies that $S_{ki} = a_k S^{\mathrm{st}}_{ki}$ for some non-zero constant a_k, where we temporarily denoted by S^{st} the operator defined by (3.1.32). Equivalently, we can write $S = AS^{\mathrm{st}}$, where the operator $A\colon H \to H$ is "diagonal": $A|_{V_i \otimes V_i^*} = a_i \,\mathrm{id}$. Now, let us use the axiom MF8. In particular, we have $TSTST = S$. Since $S = AS^{\mathrm{st}}$, and A commutes with T, we get $TS^{\mathrm{st}}TAS^{\mathrm{st}}T = S^{\mathrm{st}}$. On the other hand, the operator S^{st} itself satisfies the axiom MF8, and thus, $TS^{\mathrm{st}}TS^{\mathrm{st}}T = S^{\mathrm{st}}$. This implies $A = \mathrm{id}, S = S^{\mathrm{st}}$.

The proof of the converse statement—that a MF defines a structure of a modular category—is trivial. Indeed, the identity $\tau(\Sigma) = \bigoplus \operatorname{End} V_i$ for Σ being a torus without punctures implies that $\mathcal{C}$ has only finitely many simple objects (since $\tau(\Sigma)$ is finite dimensional). Thus, we only have to check that the matrix $\tilde{s}$, defined in (3.1.1), is non-degenerate. But the identity $S = AS^{\mathrm{st}}$ and the invertibility of S imply that S^{st} is invertible. □

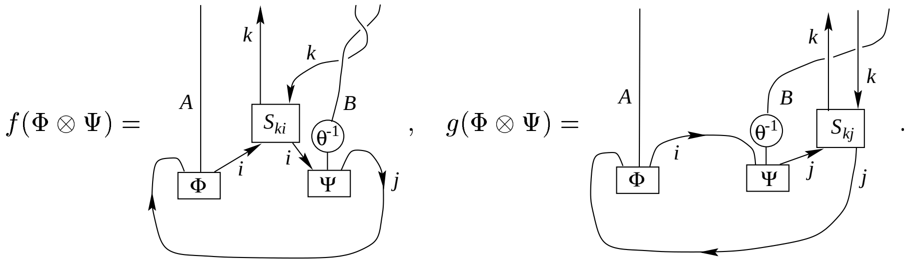
Proof of Lemma 5.5.2. Consider the diagram in Figure 5.15. Let m_1 be the graph in the upper left corner; for convenience, replace the graph m in the lower right corner by $m_2 = F_{c_4}(m)$. Then the vector spaces $\tau(\Sigma, m_1)$ and $\tau(\Sigma, m_2)$ are given by

$$\begin{aligned} \tau(\Sigma, m_1) &= \bigoplus_{i,j} \langle V_j^*, A, V_i\rangle \otimes \langle V_i^*, B, V_j\rangle, \\ \tau(\Sigma, m_2) &= \bigoplus_{k} \langle A, V_k, V_k^*, B\rangle, \end{aligned} \tag{5.5.4}$$

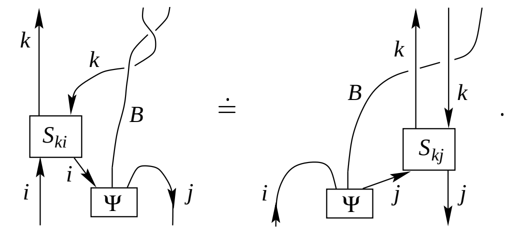
where A, B are the objects assigned to the boundary components α, β respectively (see (5.4.4)).

Then relation MF9 can be written as follows: for every $\Phi \otimes \Psi \in \langle V_j^*, A, V_i\rangle \otimes \langle V_i^*, B, V_j\rangle$, we have $f(\Phi \otimes \Psi) = g(\Phi \otimes \Psi)$, where f is the isomorphism given by the composition of moves forming the left side and the bottom of the commutative diagram, and g—by the moves on the top and the right side. We represent this identity pictorially, using Example 5.4.2, Eq. (5.2.9), and the graphical calculus of Section 2.3.

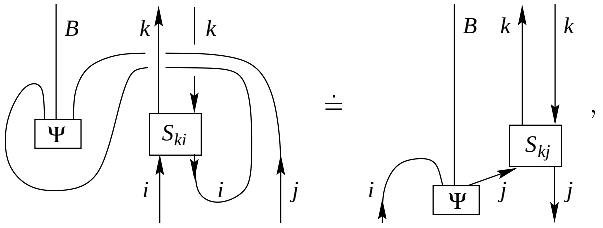
A simple manipulation with figures shows that:



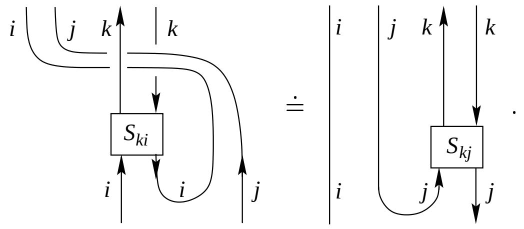
The identity $f(\Phi \otimes \Psi) = g(\Phi \otimes \Psi)$ $\forall \Phi$ is equivalent to:



We manipulate this as follows:



and then cancel Ψ, to get:



From this it is easy to get the statement of the lemma. □

Corollary 5.5.3. *Let $\mathcal{C}$ be an MTC with $p^+ = p^-$. Denote*

$$\tau(g; W_1, \dots, W_n) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, H^{\otimes g} \otimes W_1 \otimes \cdots \otimes W_m)$$

where $H = \bigoplus V_i \otimes V_i^*$. Then we have an action of the pure mapping class group $\Gamma'_{g,n}$ on this space. In particular, for $g = 1, n = 1$ this action coincides with the one defined in Theorem 3.1.17.

Indeed, let $\tau(\Sigma)$ be the modular functor corresponding to $\mathcal{C}$; then it is easy to see, using the gluing axiom, that if Σ is a surface of genus g then $\tau(\Sigma; W_1, \dots, W_n)$ is (not canonically) isomorphic to the space $\tau(g; W_1, \dots, W_n)$ defined above.

REMARK 5.5.4. In fact, Corollary 5.5.3 also holds for modular categories with $p^+/p^- \neq 1$ if we replace the word "action" by "projective action". This will be discussed in Section 5.7.

EXERCISE 5.5.5. Prove the following formula for the dimension of the space $\tau(g)$ for $g \geq 1$ ($n = 0$):

$$\dim \tau(g) = \sum_{i \in I} \left(\frac{1}{s_{0i}^2} \right)^{g-1}. \tag{5.5.5}$$

Hint: Prove that $\dim \tau(g) = \operatorname{tr}(a^{g-1})$, where $a_{ij} = \dim \tau(g = 1; V_i, V_j^*), i, j \in I$. Then prove that $a = \sum_k N_k N_{k^*}$, where N_k is defined as in Proposition 3.1.12, and use the Verlinde formula to diagonalize a.

5.6. Towers of groupoids

Looking at the previous two sections, one is tempted to say that there is some "universal" set of relations which must hold in any weakly ribbon category, and these relations happen to coincide with the relations for the mapping class group. In this section we sketch the appropriate language in which one can formulate this and other related results. Therefore, we do not really prove any new results here, and we allow ourselves to be somewhat informal.

Let us start by considering our main example: the *Teichmüller tower* $\mathcal{T}eich$. By definition, $\mathcal{T}eich$ is a category with objects all extended surfaces, and morphisms isotopy classes of homeomorphisms of extended surfaces (see Definition 5.1.7(i)). This category is a groupoid, i.e., any morphism in $\mathcal{T}eich$ is invertible. It also has some additional structures which played an important role in the previous sections: the disjoint union and gluing of surfaces. The general definition of a tower of groupoids will be modeled on this example, so let us study it in more detail.

Temporarily, let us denote $\mathcal{T}eich$ by $\mathcal{T}$. Below we list its properties.

(a) $\mathcal{T}$ is a groupoid.

(b) The disjoint union of surfaces $\sqcup\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ and the empty surface $\emptyset \in \operatorname{Ob} \mathcal{T}$ provide $\mathcal{T}$ with the structure of a symmetric tensor category.

(c) There is a functor $A\colon \mathcal{T} \to \mathcal{S}ets$: for a surface Σ, $A(\Sigma) = \pi_0(\partial\Sigma)$ is the set of its boundary components. Here $\mathcal{S}ets$ is the groupoid with objects finite sets, and morphisms bijections. Note that $A(\Sigma_1 \sqcup \Sigma_2) = A(\Sigma_1) \sqcup A(\Sigma_2)$ and $A(\emptyset) = \emptyset$ (canonical isomorphisms). In other words, A is a tensor functor.

(d) There is a gluing operation: for every surface $\Sigma \in \operatorname{Ob} \mathcal{T}$ and an unordered pair $\alpha, \beta \in A(\Sigma)$, we have the surface $G_{\alpha,\beta}(\Sigma) = \sqcup_{\alpha,\beta}(\Sigma)$ obtained by identification of the boundary components α, β (cf. Definition 5.1.12(iv)). The gluing satisfies the following properties:

Compatibility with A: $A(G_{\alpha,\beta}(\Sigma)) = A(\Sigma) \setminus \{\alpha, \beta\}$.

Compatibility with $\sqcup$: if $\alpha, \beta \in A(\Sigma_1)$, there is a canonical functorial isomorphism $G_{\alpha,\beta}(\Sigma_1 \sqcup \Sigma_2) = (G_{\alpha,\beta}\Sigma_1) \sqcup \Sigma_2$.

Associativity: if $\alpha, \beta, \gamma, \delta \in A(\Sigma)$ are distinct, then there exists a canonical functorial isomorphism $G_{\alpha,\beta} G_{\gamma,\delta}(\Sigma) = G_{\gamma,\delta} G_{\alpha,\beta}(\Sigma)$.

Functoriality: for each morphism $f\colon \Sigma \to \Sigma'$, we have an isomorphism $G_f : G_{\alpha,\beta}(\Sigma) \to G_{\alpha',\beta'}(\Sigma')$, where $\alpha' = A(f)(\alpha)$, $\beta' \in A(f)(\beta)$ are the corresponding elements in $A(\Sigma')$. These isomorphisms satisfy $G_{f_1 f_2} = G_{f_1} G_{f_2}$ and $G_{\mathrm{id}} = \mathrm{id}$.

DEFINITION 5.6.1. A *tower of groupoids* (or just a *tower*) is the following collection of data:

(i) A groupoid $\mathcal{T}$;

(ii) A "disjoint union" bifunctor $\sqcup\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ and an object $\emptyset \in \operatorname{Ob}\mathcal{T}$ satisfying the axioms of a symmetric tensor category;

(iii) A "boundary functor": a tensor functor $A\colon \mathcal{T} \to \mathcal{S}ets$;

(iv) A "gluing operation": for every $\Sigma \in \operatorname{Ob}\mathcal{T}$ and an unordered pair $\alpha, \beta \in A(\Sigma)$, we have an object $G_{\alpha,\beta}(\Sigma) \in \mathcal{T}$. The gluing should be associative, functorial and compatible with $\sqcup$ and A (see (d) above).

EXAMPLE 5.6.2. $\mathcal{S}ets$ and $\mathcal{T}eich$ are towers of groupoids.

REMARK 5.6.3. Sometimes it is useful to weaken the above definition by considering towers in which the gluing operation $G_{\alpha,\beta}$ is defined not for all but only for some pairs α, β. In this case, the identities $G_{\alpha,\beta}\sqcup = \sqcup(G_{\alpha,\beta} \times \mathrm{Id})$, $G_{\alpha,\beta}G_{\gamma,\delta} = G_{\gamma,\delta}G_{\alpha,\beta}$ in the definition above should be understood in the following way: if one side is defined, then the other one is also defined and they are equal.

An example of such a "partial" tower is given by the the *Teichmüller tower in genus zero*, $\mathcal{T}eich_0$, in which objects are extended surfaces of genus zero and the functor $G_{\alpha,\beta}$ is defined only if α, β belong to different connected components of Σ.

REMARK 5.6.4. One can give a definition of what it means for a tower of groupoids to be presented by generators and relations (but since this is a little boring, we don't do it here). Then the results of Section 5.2 (and [**BK**]) can be reformulated as giving the generators and relations presentation of the Teichmüller tower $\mathcal{T}eich$. One notes that this presentation is much simpler than the presentations for individual mapping class groups $\Gamma(\Sigma)$. The idea of using the Teichmüller tower with the gluing operation for the study of mapping class groups belongs to Grothendieck [**G**]. More results in this direction can be found in [**HLS**].

Before giving more examples of towers, let us reformulate Definition 5.6.1 in a more functorial way. This will be useful later when we define functors between towers.

Let $\mathcal{T}$ be a tower of groupoids. Then $\mathcal{T}$ is a *fibered category* over $\mathcal{S}ets$. For any finite set S, the fiber $\mathcal{T}_S$ over S is the category with objects all pairs (Σ, φ) where $\Sigma \in \operatorname{Ob}\mathcal{T}$ and $\varphi\colon A(\Sigma) \xrightarrow{\sim} S$ is a bijection. A morphism between two objects $(\Sigma_1, \varphi_1), (\Sigma_2, \varphi_2) \in \operatorname{Ob}\mathcal{T}_S$ is a morphism $f \in \operatorname{Mor}_{\mathcal{T}}(\Sigma_1, \Sigma_2)$ such that $\varphi_1 = \varphi_2 \circ A(f)$. Since both $\mathcal{T}$ and $\mathcal{S}ets$ are groupoids, every fiber $\mathcal{T}_S$ is a groupoid.

A bijection of sets $\psi\colon S \xrightarrow{\sim} S'$ gives rise to a functor $\psi_*\colon \mathcal{T}_S \to \mathcal{T}_{S'}$: on objects $\psi_*(\Sigma, \varphi) = (\Sigma, \psi \circ \varphi)$, and on morphisms $\psi_*(f) = f$. It is obvious that

$$(\phi \circ \psi)_* = \phi_* \circ \psi_*, \quad \mathrm{id}_* = \mathrm{id};$$

in particular, all functors ψ_* are isomorphisms of categories.

Conversely, given a collection of groupoids $\{\mathcal{T}_S\}_{S \in \operatorname{Ob}\mathcal{S}ets}$ together with equivariance functors ψ_* as above, one can reconstruct the groupoid $\mathcal{T}$ and the functor $A\colon \mathcal{T} \to \mathcal{S}ets$.

In terms of these data, $\sqcup$ becomes a collection of functors

$$\sqcup^{S,S'}: \mathcal{T}_S \times \mathcal{T}_{S'} \to \mathcal{T}_{S\sqcup S'},$$

while $\emptyset \in \operatorname{Ob}\mathcal{T}_\emptyset$. They satisfy obvious commutativity, associativity and equivariance conditions.

Similarly, the gluing gives a collection of functors

$$G^S_{\alpha,\beta}: \mathcal{T}_S \to \mathcal{T}_{S\setminus\{\alpha,\beta\}}, \qquad S \in \operatorname{Ob}\mathcal{S}ets,\ \alpha,\beta \in S$$

(the pair α, β is unordered). Indeed, for $(\Sigma, \varphi) \in \operatorname{Ob}\mathcal{T}_S$, we define

$$G^S_{\alpha,\beta}(\Sigma,\varphi) = (\Sigma', \varphi|_{A(\Sigma')}) \quad \text{ where } \Sigma' = G_{\varphi^{-1}\alpha,\varphi^{-1}\beta}(\Sigma)$$

(recall that $A(\Sigma') = A(\Sigma)\setminus\{\varphi^{-1}\alpha, \varphi^{-1}\beta\}$). For a morphism $f: (\Sigma_1,\varphi_1) \to (\Sigma_2, \varphi_2)$ in $\mathcal{T}_S$, we define $G^S_{\alpha,\beta}(f) = G_f$ (recall the functoriality of gluing). Now the properties of gluing can be restated as follows.

Compatibility with A: already incorporated in the definition.

Compatibility with $\sqcup$: for any two sets S, S' and $\alpha, \beta \in S$, there exists a canonical isomorphism of functors $G^{S\sqcup S'}_{\alpha,\beta} \circ \sqcup^{S,S'} = \sqcup^{S\setminus\{\alpha,\beta\},S'} \circ (G^S_{\alpha,\beta} \times \mathrm{Id})$.

Associativity: if $\alpha,\beta,\gamma,\delta \in S$ are distinct then there exists a canonical isomorphism of functors $G^{S\setminus\{\gamma,\delta\}}_{\alpha,\beta} \circ G^S_{\gamma,\delta} = G^{S\setminus\{\alpha,\beta\}}_{\gamma,\delta} \circ G^S_{\alpha,\beta}$.

Functoriality: already incorporated in the requirement that $G^S_{\alpha,\beta}$ are functors.

Finally, there is one more property which follows just from the definition of $G^S_{\alpha,\beta}$.

Equivariance: for any bijection of sets $\psi: S \xrightarrow{\sim} S'$, we have $G^{S'}_{\psi\alpha,\psi\beta} \circ \psi_* = (\psi|_{S\setminus\{\alpha,\beta\}})_* \circ G^S_{\alpha,\beta}$.

DEFINITION 5.6.5. A *tower of groupoids* is a collection of groupoids $\{\mathcal{T}_S\}_{S\in\operatorname{Ob}\mathcal{S}ets}$ equipped with the following structure:

(i) Equivariance functors $\psi_*: \mathcal{T}_S \to \mathcal{T}_{S'}$ for any $\psi \in \operatorname{Mor}_{\mathcal{S}ets}(S, S')$, satisfying $(\phi\circ\psi)_* = \phi_* \circ \psi_*$ and $\mathrm{id}_* = \mathrm{id}$.

(ii) An object $\emptyset \in \operatorname{Ob}\mathcal{T}_\emptyset$ and a collection of functors $\sqcup^{S,S'}: \mathcal{T}_S \times \mathcal{T}_{S'} \to \mathcal{T}_{S\sqcup S'}$, satisfying obvious commutativity, associativity and equivariance conditions.

(iii) A collection of functors $G^S_{\alpha,\beta}: \mathcal{T}_S \to \mathcal{T}_{S\setminus\{\alpha,\beta\}}$, satisfying the above associativity, equivariance and compatibility with $\sqcup$.

PROPOSITION 5.6.6. *Definitions* 5.6.1 *and* 5.6.5 *are equivalent.*

PROOF. It was already sketched above. The details are left to the reader as an exercise. $\square$

DEFINITION 5.6.7. A *tower functor* $\mathcal{F}$ between two towers of groupoids $(\mathcal{T}, \sqcup, A, G)$ and $(\mathcal{T}', \sqcup', A', G')$ is a functor $\mathcal{F}: \mathcal{T} \to \mathcal{T}'$ which preserves all the structure. More precisely:

(i) There is an isomorphism of functors $A \simeq A' \circ \mathcal{F}$. Thus $\mathcal{F}$ gives rise to an equivariant collection of functors $\mathcal{F}^S: \mathcal{T}_S \to \mathcal{T}'_S$, $S \in \operatorname{Ob}\mathcal{S}ets$.

(ii) $\mathcal{F}$ is a tensor functor, i.e., the functors $\mathcal{F}\circ\sqcup$ and $\sqcup'\circ(\mathcal{F}\times\mathcal{F}): \mathcal{T}\times\mathcal{T} \to \mathcal{T}'$ are isomorphic.

(iii) For any finite set S, there is an isomorphism of functors $\mathcal{F}^{S\setminus\{\alpha,\beta\}} \circ G^S_{\alpha,\beta} \simeq {G'}^S_{\alpha,\beta} \circ \mathcal{F}^S: \mathcal{T}_S \to \mathcal{T}'_{S\setminus\{\alpha,\beta\}}$. These isomorphisms are equivariant with respect to bijections of S.

EXERCISE 5.6.8. Spell out property (iii) of Definition 5.6.7 in terms of the gluing operations $G_{\alpha,\beta}$ from Definition 5.6.1.

EXAMPLE 5.6.9. $A\colon \mathcal{T} \to \mathcal{S}ets$ is a tower functor for any tower $\mathcal{T}$.

There is an even more economical way to reformulate the definition of a tower. Looking at the equivariance properties of the collections $\{\mathcal{T}_S\}$ and $\{G^S_{\alpha,\beta}\}$, one can notice that they can be combined if we allow more maps between sets. We introduce a category $\mathcal{S}ets_\sharp$ with the same objects as in $\mathcal{S}ets$ (i.e., finite sets), but with more morphisms: all maps between sets that are composed of bijections and the elementary injections $i^S_{\alpha,\beta}\colon S \setminus \{\alpha,\beta\} \hookrightarrow S$. (This definition was inspired by [**BFM**].) Let $\mathcal{S}ets^\sharp$ be the dual category of $\mathcal{S}ets_\sharp$, i.e., the category with the same objects but with all arrows inverted. All morphisms in $\mathcal{S}ets^\sharp$ are composed of bijections and the elementary morphisms

$$\delta^S_{\alpha,\beta}\colon S \to S \setminus \{\alpha,\beta\}, \qquad S \in \operatorname{Ob}\mathcal{S}ets^\sharp,\ \ \alpha,\beta \in S \text{ (unordered)}.$$

Now if we define

$$(\delta^S_{\alpha,\beta})_* = G^S_{\alpha,\beta}\colon \mathcal{T}_S \to \mathcal{T}_{S\setminus\{\alpha,\beta\}},$$

we will have $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ for $\phi,\psi \in \operatorname{Mor}_{\mathcal{S}ets^\sharp}$. Note that $\mathcal{S}ets^\sharp$ is again a symmetric tensor category with respect to $\sqcup$.

PROPOSITION 5.6.10. *A tower of groupoids is the same as a symmetric tensor category $\mathcal{T}$ fibered over $\mathcal{S}ets^\sharp$ such that all fibers $\mathcal{T}_S$ ($S \in \operatorname{Ob}\mathcal{S}ets^\sharp$) are groupoids. In other words, we have parts* (i) *and* (ii) *of Definition* 5.6.5 *with $\mathcal{S}ets$ replaced with $\mathcal{S}ets^\sharp$.*

In this language a *tower functor* $\mathcal{F}$ between two towers is just a collection of functors $\mathcal{F}^S\colon \mathcal{T}_S \to \mathcal{T}'_S$, equivariant with respect to $\operatorname{Mor}_{\mathcal{S}ets^\sharp}$, and such that the corresponding functor $\mathcal{F}\colon \mathcal{T} \to \mathcal{T}'$ is a tensor functor. A *natural transformation* Φ between two tower functors $\mathcal{F},\mathcal{G}\colon \mathcal{T} \to \mathcal{T}'$ is a $\operatorname{Mor}_{\mathcal{S}ets^\sharp}$-equivariant collection of natural transformations Φ^S between the functors $\mathcal{F}^S, \mathcal{G}^S$. Then, as usual, $\mathcal{F}\colon \mathcal{T} \to \mathcal{T}'$ is called an *equivalence of towers* if there exists a tower functor $\mathcal{F}'\colon \mathcal{T}' \to \mathcal{T}$ such that the tower functors $\mathcal{F}\mathcal{F}'$ and $\mathcal{F}'\mathcal{F}$ are isomorphic to Id.

After introducing all this abstract nonsense let us now give some examples and applications.

EXAMPLE 5.6.11. Let $\mathcal{C}$ be an abelian category and $R \in \text{ind}{-}\mathcal{C}^{\boxtimes 2}$ be a symmetric object.[3] We define the tower of groupoids $\mathcal{F}un(\mathcal{C})$ as follows.

Objects: all pairs (S,F) where S is a finite set and F is a functor $\mathcal{C}^{\boxtimes S} \to \mathcal{V}ec_f$.

Morphisms: $\operatorname{Mor}((S_1,F_1),(S_2,F_2))$ consists of all pairs (f,φ) where $\varphi\colon S_1 \xrightarrow{\sim} S_2$ is a bijection, $f\colon F_1 \xrightarrow{\sim} \varphi_*F_2$ is an isomorphism of functors, and φ_*F_2 is the composition $\mathcal{C}^{\boxtimes S_1} \xrightarrow{\varphi_*} \mathcal{C}^{\boxtimes S_2} \xrightarrow{F_2} \mathcal{V}ec_f$.

Boundary functor: $A(S,F) = S$.

Disjoint union: $(S_1 \sqcup S_2, F_1 \otimes F_2\colon \mathcal{C}^{\boxtimes(S_1\sqcup S_2)} \to \mathcal{V}ec_f)$, and similarly for morphisms. The object $\emptyset$ is the obvious one.

Gluing: given by $G_{\alpha,\beta}(S) = S\setminus\{\alpha,\beta\}$ and $G_{\alpha,\beta}(F) = F(\dots,R^{(1)},\dots,R^{(2)},\dots)$, where $R^{(1)},R^{(2)}$ are put in the places corresponding to the indices α,β.

[3] Here and below we use the same notation as in Section 2.4.

DEFINITION 5.6.12. Let $\mathcal{C}$ be an abelian category and $R \in \text{ind}-\mathcal{C}^{\boxtimes 2}$ be a symmetric object. A *representation* of a tower $\mathcal{T}$ in $\mathcal{C}$ is a tower functor $\rho\colon \mathcal{T} \to \mathcal{F}un(\mathcal{C})$.

The following theorem, which follows immediately from the definitions, elucidates the notion of a modular functor.

THEOREM 5.6.13. *A $\mathcal{C}$-extended modular functor is the same as a representation τ of the Teichmüller tower $\mathcal{T}eich$ in $\mathcal{C}$ with the additional normalization condition $\tau(S^2) = \text{id}\colon \mathcal{C}^0 = \mathcal{V}ec_f \to \mathcal{V}ec_f$.*

In a similar way one can rewrite the notion of MS data (see Section 5.3). In order to introduce the corresponding tower of groupoids $\mathcal{MS}$, we will first need the following definition.

DEFINITION 5.6.14. A *marking graph* is a graph m without cycles (a "forest") with the following additional data:

(i) The vertices of m are split into two subsets, "internal" and "external"

$$\text{Vertices}(m) = \text{Int}(m) \sqcup \text{Ext}(m),$$

so that every external vertex is 1-valent, and there are no edges connecting two external vertices.

(ii) For every internal vertex $v \in \text{Int}(m)$, an order on the set of all edges ending at v is given.

REMARK 5.6.15. The marking graphs with 3-valent internal vertices are essentially the same as "Bratelli diagrams" used in physics literature.

Graphs of this type appeared in our discussion of parameterizations of extended surfaces (see Section 5.2). In the figures, we use $*$ for internal vertices and $\bullet$ for external vertices. To show the order, we draw the edges in a clockwise order and mark the first edge by an arrow.

We define a CW complex $\mathcal{M}_0$ in a way parallel to the definition of $\mathcal{M}(\Sigma)$ for genus 0 (see Section 5.2). The vertices of $\mathcal{M}_0$ are all marking graphs. We define the simple moves Z, B, F by Figures 5.5, 5.6 and 5.7, respectively. The relations in $\mathcal{M}_0$ are obtained from MF1–MF7 by forgetting the surfaces.

EXAMPLE 5.6.16. The *Moore–Seiberg tower* $\mathcal{MS}$ is the tower of groupoids defined as follows.

- **Objects:** all marking graphs.
- **Morphisms:** $\text{Mor}(m_1, m_2)$ consists of all paths in the CW complex $\mathcal{M}_0$ connecting m_1 with m_2, modulo homotopy. (In other words, as a groupoid $\mathcal{MS}$ is the fundamental groupoid of $\mathcal{M}_0$.)
- **Boundary functor:** $A(m) = \text{Ext}(m)$.
- **Disjoint union and $\emptyset$:** obvious.
- **Gluing:** if $\alpha, \beta \in \text{Ext}(m)$ are in different connected components, then we define $G_{\alpha,\beta}(m)$ to be the graph obtained by identifying the vertices α and β. The order at the new internal vertex $\alpha = \beta$ is given by $e_\alpha < e_\beta$ where e_α is the edge of m ending at α.

Note that $\mathcal{MS}$ is a "partial" tower in the sense of Remark 5.6.3.

THEOREM 5.6.17. *Let $\mathcal{C}$ be a semisimple abelian category. Then MS data for $\mathcal{C}$ is the same as a non-degenerate representation ρ of the Moore–Seiberg tower $\mathcal{MS}$ in $\mathcal{C}$ with the additional normalization condition $\rho(*) = \text{id}\colon \mathcal{V}ec_f \to \mathcal{V}ec_f$, where $*$ is the marking graph with one vertex and no edges.*

PROOF. Given a collection of MS data, let us construct a representation ρ of the tower $\mathcal{MS}$. For a marking graph m, define the functor $\rho(m)\colon \mathcal{C}^{\boxtimes \mathrm{Ext}(m)} \to \mathcal{V}ec_f$ similarly to (5.4.3). In other words, if W_v are the objects assigned to the external vertices $v \in \mathrm{Ext}(m)$, then we let

$$\rho(m)(\{W_v\}) = \bigotimes_{u \in \mathrm{Int}(m)} \langle X_{e_u^1}, \dots, X_{e_u^{k_u}} \rangle,$$

where $e_u^1, \dots, e_u^{k_u}$ are the edges adjacent to u, in the order defined by u, and $X_e = W_v$ if e connects u with an external vertex v, or $X_e = R$ if e connects two internal vertices.

The definition of the functorial isomorphisms which we assign to the morphisms of graphs is obvious. We also have obvious isomorphisms $\rho(m_1 \sqcup m_2) \simeq \rho(m_1) \otimes \rho(m_2)$ and $\rho(G_{\alpha,\beta}(m)) \simeq G_{\alpha,\beta}(\rho(m))$; in the latter isomorphism both sides coincide with $\rho(m)(\dots, R^{(1)}, \dots, R^{(2)}, \dots)$.

Now, a comparison of the relations MS1–MS7 and the relations MF1–MF7, used in the definition of $\mathcal{M}_0$, shows that the so defined ρ is indeed a representation of $\mathcal{MS}$.

Conversely, given a representation ρ of the tower $\mathcal{MS}$, define the MS data as follows:

$$\langle W_1, \dots, W_n \rangle = \rho(m_n)(W_1, \dots, W_n)$$

where m_n is the "standard" marking graph, with one internal vertex and n external vertices. Again, it is clear how to define the isomorphisms Z, σ, G and check that all the relations are satisfied. □

It is clear by its definition that the tower $\mathcal{MS}$ is just the projection on the level of marking graphs of another tower $\mathcal{PT}eich_0$: the *parametrized Teichmüller tower in genus zero.* On its hand, $\mathcal{PT}eich_0$ is the genus zero part of a tower $\mathcal{PT}eich$ which appeared implicitly in Section 5.2 and which we now proceed to define.

EXAMPLE 5.6.18. The *parameterized Teichmüller tower $\mathcal{PT}eich$* is the tower of groupoids defined as follows.

Objects: all pairs (Σ, M), where Σ is an extended surface and $M = (C, \{\psi_a\})$ is a parameterization of Σ (see Definition 5.2.1).

Morphisms: $\mathrm{Mor}((\Sigma_1, M_1), (\Sigma_2, M_2))$ consists of all pairs (f, φ) where $f\colon \Sigma_1 \xrightarrow{\sim} \Sigma_2$ is a homeomorphism of extended surfaces and φ is a path in $\mathcal{M}(\Sigma_2)$ connecting $f(M_1)$ with M_2. The composition of morphisms is given by $(f, \varphi) \circ (g, \psi) = (f \circ g, \varphi \circ f(\psi))$.

Boundary functor: $A(\Sigma, M) = A(\Sigma) = \pi_0(\partial\Sigma)$ — the set of boundary components of Σ.

Disjoint union and $\emptyset$: the usual ones.

Gluing: $G_{\alpha,\beta}(\Sigma, M) = (\sqcup_{\alpha,\beta}(\Sigma), \sqcup_{\alpha,\beta} M)$, where $\sqcup_{\alpha,\beta}(\Sigma)$ is obtained from Σ by gluing the boundary components α, β, and the parameterization $\sqcup_{\alpha,\beta} M$ is obtained from M by adding $\alpha = \beta$ as a new cut and keeping the homeomorphisms ψ_a unchanged.

Note that by Theorem 5.2.9 the path φ is uniquely defined by f, so we could as well omit φ from the above definition of morphisms. However, it will be useful for us to have the definition in this form.

Now we can reformulate the main results of the previous sections in a much more transparent way.

THEOREM 5.6.19. (i) *The towers of groupoids $\mathcal{T}eich$ and $\mathcal{PT}eich$ are equivalent. Similarly, their genus zero parts $\mathcal{T}eich_0$ and $\mathcal{PT}eich_0$ are equivalent.*
(ii) *The towers $\mathcal{PT}eich_0$ and $\mathcal{MS}$ are equivalent.*

PROOF. (i) To prove the first statement, consider the obvious forgetting functor $\mathcal{PT}eich \to \mathcal{T}eich$. It suffices to check that this functor is bijective on morphisms. By Theorem 5.2.9, for every two parameterizations M, M' of an extended surface Σ there exists a unique path in $\mathcal{M}(\Sigma)$ connecting them. Thus, in a pair $(f, \varphi) \in \mathrm{Mor}_{\mathcal{PT}eich}$, the path φ is uniquely determined by f, which is equivalent to saying that the forgetting functor gives a bijection $\mathrm{Mor}_{\mathcal{PT}eich} \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{T}eich}$. The proof for genus zero is completely parallel.

(ii) To prove the second statement, consider the functor $\mathcal{PT}eich_0 \to \mathcal{MS}$ which assigns to the pair (Σ, M) the marking graph of M. Obviously, every marking graph can be obtained in this way. Thus, it suffices to prove that this functor gives a bijection of the spaces of morphisms. This is immediate from comparing the moves and relations and the following rigidity lemma.

LEMMA 5.6.20. *Let Σ be an extended surface, $M \in M(\Sigma)$ be a parameterization, and m the corresponding marking graph. Let $f\colon \Sigma \xrightarrow{\sim} \Sigma$ be a homeomorphism which preserves the graph m pointwise.*[4] *Then f is homotopic to identity.*

This completes the proof of Theorem 5.6.19. $\square$

A comparison of the theorems above makes the relation between genus zero modular functors and weakly ribbon structures on a semisimple category obvious.

5.7. Central extension of modular functor

In Section 5.5 we have constructed a $\mathcal{C}$-extended modular functor (MF) starting from any modular tensor category $\mathcal{C}$ satisfying $p^+/p^- = 1$. As with TQFT constructed from $\mathcal{C}$, the gluing axiom fails when $p^+/p^- \neq 1$. There are two approaches to deal with the general case.

First, we can content ourselves with a modification of the gluing axiom, which says that it holds only up to a multiplicative factor. This is similar to the notion of a projective representation of a group.

The second approach is to try to construct a kind of a "central extension" of the modular functor. This was done independently by several authors; our exposition follows an unpublished manuscript [**BFM**] by Beilinson, Feigin, and Masur.

We begin with some preliminaries. Let V be a symplectic real vector space of dimension $2g$, $g \in \mathbb{N}$. Let Λ_V be the set of all Lagrangian subspaces of V, i.e., maximal isotropic subspaces of V. This is a compact manifold. Let T_V be the *Poincaré groupoid* of Λ_V; by definition, objects of this groupoid are points of Λ_V and morphisms are homotopy classes of paths connecting two points. It is convenient to define T_V for $V = 0$ as the category with only one object 0 and $\mathrm{Hom}_{T_0}(0,0) = \mathbb{Z}$.

The proof of the following lemma is straightforward and will be omitted.

[4] It is not sufficient to require that $f(m) = m$, as f could interchange components of m.

LEMMA 5.7.1. (i) *For any two symplectic vector spaces* V_1, V_2, *there exists a canonical map* $\Lambda_{V_1} \times \Lambda_{V_2} \to \Lambda_{V_1 \oplus V_2}$.

(ii) *Let* $N \subset V$ *be an isotropic subspace, i.e., such that the restriction of the symplectic form on* N *is* 0. *Then the space* $N^\perp/N$ *is symplectic, and there exists a canonical map* $\Lambda_{N^\perp/N} \to \Lambda_V$ *which assigns to a Lagrangian subspace* $L \subset N^\perp/N$ *the subspace* $\pi^{-1}(L) \subset N^\perp \subset V$, *where* $\pi\colon N^\perp \to N^\perp/N$ *is the natural projection. The induced map of fundamental groupoids* $T_{N^\perp/N} \to T_V$ *is an equivalence.*

COROLLARY 5.7.2. *For any point* $a \in \Lambda_V$, *the fundamental group* $\pi_1(\Lambda_V, a)$ *is isomorphic to* $\mathbb{Z}$.

Corollary 5.7.2 implies that the group $\mathbb{Z}$ acts freely on $\mathrm{Mor}_{T_V}(L_1, L_2)$ for any $L_1, L_2 \in \Lambda_V$. (In other words, $\mathrm{Mor}_{T_V}(L_1, L_2)$ is a $\mathbb{Z}$-*torsor*.) Hence we have a non-canonical identification

$$\mathrm{Mor}_{T_V}(L_1, L_2) \xrightarrow{\sim} \mathbb{Z}. \tag{5.7.1}$$

Let us choose such identifications for all $L_1, L_2 \in \Lambda_V$. If $\varphi\colon L_1 \to L_2$ and $\psi\colon L_2 \to L_3$ are two morphisms in T_V, corresponding to numbers $m, n \in \mathbb{Z}$, then in general $\psi\varphi\colon L_1 \to L_3$ corresponds to some $p \neq m + n$. The difference

$$\mu(L_1, L_2, L_3) := p - m - n \tag{5.7.2}$$

is called the *Maslov index* of the subspaces L_1, L_2, L_3.

Let Σ be an extended surface, as in Section 5.1. We denote by $cl(\Sigma)$ the surface without boundary obtained from Σ by gluing disks to all boundary circles, and let

$$H(\Sigma) := \mathrm{H}_1(cl(\Sigma), \mathbb{R}). \tag{5.7.3}$$

The intersection form makes $H(\Sigma)$ a symplectic space of dimension $2g$ where g is the genus of Σ (i.e., of $cl(\Sigma)$). Introduce the notations

$$\Lambda_\Sigma := \Lambda_{H(\Sigma)}, \quad T_\Sigma := T_{\Lambda_\Sigma}. \tag{5.7.4}$$

When Σ is of genus zero, we have $H(\Sigma) = 0$ and Λ_Σ is a point. In this case, it is convenient to define T_Σ as the category with only one object 0 and $\mathrm{Hom}_{T_\Sigma}(0,0) = \mathbb{Z}$.

The next lemma is left as an exercise.

LEMMA 5.7.3. (i) *There exists a canonical map* $a\colon \Lambda_{\Sigma_1} \times \Lambda_{\Sigma_2} \to \Lambda_{\Sigma_1 \sqcup \Sigma_2}$. (*However, it is not a homeomorphism.*)

(ii) *Let the surface* Σ *be obtained by sewing two surfaces along one boundary component*: $\Sigma = \Sigma_1 \sqcup_{\alpha,\beta} \Sigma_2$. *Then* $H(\Sigma_1 \sqcup \Sigma_2) \simeq H(\Sigma)$. *Therefore, there exists a canonical homeomorphism* $g_{\alpha,\beta}\colon \Lambda_{\Sigma_1 \sqcup \Sigma_2} \xrightarrow{\sim} \Lambda_\Sigma$.

(iii) *Let* Σ *be obtained from* Σ' *by gluing two boundary circles* α_1, α_2 *in the same connected component*: $\Sigma = \sqcup_{\alpha_1,\alpha_2} \Sigma'$. *These two circles give a cycle* $\alpha \in H(\Sigma)$. *Then we claim that* $H(\Sigma') \simeq \alpha^\perp/\mathbb{R}\alpha$. *Therefore, we have a canonical map* $g_{\alpha_1,\alpha_2}\colon \Lambda_{\Sigma'} \to \Lambda_\Sigma$ *which induces an equivalence* $T_{\Sigma'} \xrightarrow{\sim} T_\Sigma$.

EXERCISE 5.7.4. Let Σ be an extended surface, and let C be a cut system on Σ, i.e., a finite set of closed simple non-intersecting curves on Σ such that the connected components Σ_a of $\Sigma \setminus C$ have genus zero (cf. Definition 5.2.1). By Lemma 5.7.3, this defines a map $\prod \Lambda_{\Sigma_a} \to \Lambda_\Sigma$. Since, by definition, each Λ_{Σ_a} is a point, this map gives an element $y_C \in \Lambda_\Sigma$. Show that y_C is the subspace in $\mathrm{H}_1(cl(\Sigma), \mathbb{R})$ spanned by the classes $[c], c \in C$.

Now we can define the "central extension" of the Teichmüller tower which was defined in Section 5.6.

DEFINITION 5.7.5. The central extension $\widetilde{\mathcal{T}eich}$ of the Teichmüller tower $\mathcal{T}eich$ is the tower of groupoids defined as follows.

Objects: all pairs (Σ, y), where Σ is an extended surface and and $y \in \Lambda_\Sigma$.

Morphisms: $\mathrm{Mor}((\Sigma_1, y_1), (\Sigma_2, y_2))$ consists of all pairs (f, ϕ), where $f\colon \Sigma_1 \xrightarrow{\sim} \Sigma_2$ is an orientation preserving homeomorphism and $\phi \in \mathrm{Mor}_{T_{\Sigma_2}}(f_* y_1, y_2)$. Here $f_*\colon \Lambda_{\Sigma_1} \to \Lambda_{\Sigma_2}$ is the map induced from f.

Boundary functor: $A(\Sigma, y) = \pi_0(\partial\Sigma)$ is the set of boundary components of Σ.

Disjoint union: $(\Sigma_1, y_1) \sqcup (\Sigma_2, y_2) = (\Sigma_1 \sqcup \Sigma_2, a(y_1 \oplus y_2))$, where $a\colon \Lambda_{\Sigma_1} \times \Lambda_{\Sigma_2} \to \Lambda_{\Sigma_1 \sqcup \Sigma_2}$ is as in Lemma 5.7.3(i). The object $\emptyset$ is the obvious one.

Gluing: $G_{\alpha,\beta}(\Sigma, y) = (\sqcup_{\alpha,\beta}(\Sigma), g_{\alpha,\beta}(y))$, where $g_{\alpha,\beta}\colon \Lambda_\Sigma \to \Lambda_{\sqcup_{\alpha,\beta}(\Sigma)}$ is as in Lemma 5.7.3(ii), (iii).

This groupoid is a central extension of the usual Teichmüller groupoid in the following sense: we have an obvious functor $\widetilde{\mathcal{T}eich} \to \mathcal{T}eich$ compatible with all the operations, and for each $(\Sigma, y) \in \mathrm{Ob}\,\widetilde{\mathcal{T}eich}$, the kernel of the map $\mathrm{Aut}_{\widetilde{\mathcal{T}eich}}(\Sigma, y) \to \mathrm{Aut}_{\mathcal{T}eich}(\Sigma)$ is $\mathrm{Aut}_{T_\Sigma}(y) = \mathbb{Z}$ (see (5.7.1)). In other words, denoting for an extended surface Σ and $y \in \Lambda_\Sigma$ the extended mapping class group by

$$\hat{\Gamma}(\Sigma, y) := \mathrm{Aut}_{\widetilde{\mathcal{T}eich}}(\Sigma, y), \tag{5.7.5}$$

(up to an isomorphism, this does not depend on the choice of y), we can write the following exact sequence:

$$0 \to \mathbb{Z} \to \hat{\Gamma}(\Sigma, y) \to \Gamma(\Sigma) \to 0. \tag{5.7.6}$$

Note that for Σ of genus zero, Λ_Σ is a point, and we have a canonical isomorphism $\hat{\Gamma}(\Sigma) = \mathbb{Z} \times \Gamma(\Sigma)$, i.e., the above exact sequence splits. For positive genus, this is not so.

EXAMPLE 5.7.6. Let $\Sigma = S_{1,1}$ be the torus with one puncture, and let α, β be the meridian and the parallel of the torus, so that $H(\Sigma) = \mathbb{R}[\alpha] \oplus \mathbb{R}[\beta]$ (see Figure 5.19). Then $\Lambda_\Sigma = \mathbb{RP}^1 = S^1$. Let $s, t \in \Gamma_{1,1}$ be the elements of the mapping class group defined in Example 5.1.11.

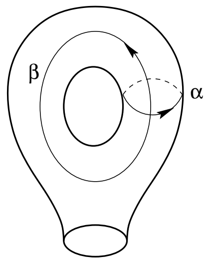


FIGURE 5.19

For $y = [\alpha]$ we will describe the central extension $\hat{\Gamma}(\Sigma, y)$. Note that $t_*([\alpha]) = [\alpha]$, $s_*[\alpha] = [\beta]$. Let us choose a path ϕ in Λ_Σ connecting the points $[\beta]$ and $[\alpha]$.

Now, define elements $\hat{t}, \hat{s}, \hat{c} \in \hat{\Gamma}(\Sigma, y)$ by $\hat{t} = (t, \mathrm{id})$, $\hat{s} = (s, \phi)$, $\hat{c} = (c, \mathrm{id})$, where $c = s^2$ acts on $H(\Sigma)$ by $v \mapsto -v$, and thus, acts trivially on Λ_Σ. Then we claim that the group $\hat{\Gamma}(\Sigma, y)$ is generated by the elements $\hat{t}, \hat{s}, \hat{c}, \gamma$ with the relations

$$\hat{s}^2 = \gamma\hat{c}, \quad (\hat{s}\hat{t})^3 = \hat{s}^2, \quad \gamma, \hat{c} \text{ are central,} \tag{5.7.7}$$

where $\gamma = (\mathrm{id}, \gamma)$ is the generator of the fundamental group $\pi_1(\Lambda_\Sigma, y) = \mathbb{Z}$.

Similarly, if we consider a torus without punctures, then the mapping class group $\Gamma(S_{1,0}, y)$ is generated by the same elements with the additional relation $\hat{c}^2 = 1$. The proof of both of these statements is left to the reader as an exercise.

Remark 5.7.7. One sees that for $\Sigma = S_{1,1}$, the exact sequence (5.7.6) trivially splits. For $\Sigma = S_{1,0}$, we have $\Gamma(\Sigma) = \mathrm{SL}_2(\mathbb{Z})$, and one can check that the above exact sequence does not split, but it "splits over $\mathbb{Q}$": if we denote by $\hat{\Gamma}(\Sigma, y)_{\mathbb{Q}} = \hat{\Gamma}(\Sigma, y) \times_{\mathbb{Z}} \mathbb{Q}$ the group obtained by adding to $\hat{\Gamma}(\Sigma)$ fractional powers of γ, then the exact sequence

$$0 \to \mathbb{Q} \to \hat{\Gamma}(\Sigma, y)_{\mathbb{Q}} \to \Gamma(\Sigma) \to 0$$

does split. However, it can be shown that for $g > 1$ the exact sequence (5.7.6) for $\Gamma_{g,0}$ does not split even over $\mathbb{Q}$.

Now we can formulate the notion of a modular functor with a central charge. Recall that we have defined the notion of a representation of a tower of groupoids in an abelian category $\mathcal{C}$ (see Definition 5.6.12), and the modular functor can be defined as a representation of the Teichmüller tower (see Theorem 5.6.13).

Definition 5.7.8. Let $\mathcal{C}$ be an abelian category. A *$\mathcal{C}$-extended modular functor with (multiplicative) central charge* $K \in k^\times$ is a representation of the tower $\widetilde{\mathcal{T}eich}$, with the additional normalization condition $\tau(S^2) = k$, and such that for every extended surface Σ and $y \in \Lambda_\Sigma$ the generator γ of $\mathrm{Aut}_{\mathcal{T}_\Sigma}(y) = \mathbb{Z} \subset \mathrm{Aut}_{\widetilde{\mathcal{T}eich}}(\Sigma, y)$ acts as multiplication by K.

For those readers who do not like the language of towers of groupoids, this definition can be spelled out explicitly as follows.

Definition 5.7.9. A *modular functor with (multiplicative) central charge* $K \in k^\times$ is the following collection of data:

(i) Let Σ be a compact oriented surface with boundary, with a point and an object of $\mathcal{C}$ attached to any boundary circle, and let $y \in \Lambda_\Sigma$. To any such (Σ, y) the modular functor assigns a finite dimensional vector space $\tau(\Sigma, y)$.

(ii) To any morphism $\tilde{f}\colon (\Sigma, y) \to (\Sigma', y')$ the modular functor assigns an isomorphism of the corresponding vector spaces $\tilde{f}_*\colon \tau(\Sigma, y) \xrightarrow{\sim} \tau(\Sigma', y')$.

(iii) Functorial isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(\Sigma_1 \sqcup \Sigma_2, y_1 \oplus y_2) \xrightarrow{\sim} \tau(\Sigma_1, y_1) \otimes \tau(\Sigma_2, y_2)$.

(iv) A symmetric object $R \in \mathrm{ind}\text{-}\mathcal{C}^{\boxtimes 2}$ (see Section 2.4).

(v) **Gluing isomorphism:** Let Σ' be the surface obtained from Σ by cutting Σ along a circle. Then we require that there is an isomorphism

$$\tau(\Sigma', y; R^{(1)}, R^{(2)}) \to \tau(\Sigma, g(y)) \tag{5.7.8}$$

where g is as in Lemma 5.7.3(ii), (iii).

These data have to satisfy the same axioms as in Definition 5.1.12 and the following additional relation. Note that for every (Σ, y) the group $\pi_1(\Lambda_\Sigma, y)$ is canonically isomorphic to $\mathbb{Z}$. (The orientation of Σ gives a choice for the sign of the generator γ.) Then we require that $\gamma_*\colon \tau(\Sigma, y) \to \tau(\Sigma, y)$ be a multiplication by K.

THEOREM 5.7.10. *Any modular tensor category gives rise to a modular functor with central charge $K = p^+/p^-$. Conversely, if τ is a $\mathcal{C}$-extended modular functor with central charge K, then it defines on $\mathcal{C}$ a structure of a weakly ribbon category. If this category is rigid, then $\mathcal{C}$ is a modular category with $p^+/p^- = K$.*

PROOF. The proof is similar to the proof in the case of zero central charge ($p^+ = p^-$). It is based on an analogue of Theorem 5.2.9, giving the set of moves and relations among the parameterizations. However, now we have to extend the notion of parameterization as follows.

Let Σ be an extended surface and $y \in \Lambda_\Sigma$. An *extended parameterization* $\hat{M}$ is a pair (M, φ), where M is a parameterization of Σ (see Definition 5.2.1), and $\varphi \in \mathrm{Mor}_{T_\Sigma}(y, y_M)$, where $y_M \in \Lambda_\Sigma$ is the Lagrangian subspace defined by the cut system C of M (see Example 5.7.4).

Since the moves B, F, Z do not change y_M, we can lift each of them to a move between extended parameterizations by letting $\hat{B} = (B, \mathrm{id})$, etc. We also have a new move $\gamma\colon (M, \varphi) \rightsquigarrow (M, \gamma \circ \varphi)$, where γ is the generator of $\mathrm{Aut}_{T_\Sigma}(y_M, y_M) = \mathbb{Z}$. Finally, the move S can be lifted to a move $\hat{S}$ as in Example 5.7.6. Then each of relations MF1–MF7 makes sense as a relation among the moves $\hat{Z}, \dots, \hat{F}$. As for relations MF8, MF9, they can be uniquely lifted to relations among the moves between the extended parameterizations by replacing $Z, \dots S$ by $\hat{Z}, \dots, \hat{S}$ and inserting an appropriate power of γ to make it into a closed loop in $\hat{M}(\Sigma)$. We will denote the corresponding axioms by MF$\hat{8}$, MF$\hat{9}$. Let us also add an axiom MF$\hat{10}$ requiring that γ be central. Then it is easy to deduce from Theorem 5.2.9 that the corresponding 2-complex $\hat{\mathcal{M}}(\Sigma)$ is connected and simply-connected.

Now to show that every MTC defines a modular functor, we can follow the same approach as before, i.e., first define $\tau(\Sigma, y, \hat{M})$, and then assign to every move $\hat{E}\colon \hat{M} \rightsquigarrow \hat{M}'$ an isomorphism $\tau(\Sigma, y, \hat{M}) \to \tau(\Sigma, y, \hat{M}')$ so that all the relations MF1–MF$\hat{10}$ are satisfied.

Let us define $\tau(\Sigma, y, \hat{M}) = \tau(\Sigma, M)$ (thus, it does not depend on the choice of y and φ) and assign to the moves $\hat{Z}, \hat{B}, \hat{F}$ the same isomorphisms as before (i.e., Z, σ, G). Assign to γ the isomorphism given by multiplication by p^+/p^-. Finally, assign to $\hat{S}$ the operator $S/\sqrt{p^+/p^-}$, where S is defined in Theorem 3.1.17. Explicit calculation shows that for so defined $\hat{S}$, relations MF$\hat{8}$, MF$\hat{9}$ are satisfied. For MF$\hat{8}$, this calculation essentially coincides with the one done in Example 5.7.6.

The proof in the opposite direction is absolutely parallel to the one for the genus zero case; thus, we omit it. □

5.8. From 2D MF to 3D TQFT

Starting from a modular tensor category $\mathcal{C}$ with $p^+/p^- = 1$, we have constructed a $\mathcal{C}$-extended 3-dimensional Topological Quantum Field Theory (Section 4.4) and a $\mathcal{C}$-extended 2-dimensional modular functor (Section 5.1). We have also showed that conversely, if $\mathcal{C}$ is a semisimple abelian category then any $\mathcal{C}$-extended 2-dimensional modular functor gives rise to a structure of a modular category on $\mathcal{C}$ (provided that the rigidity condition is satisfied).

Schematically, we have:

$$
\begin{tikzcd}
 & \mathcal{C}\text{-extended 3D TQFT} \\
\text{MTC } \mathcal{C} \arrow[ur] \arrow[dr, Leftrightarrow] & \\
 & \mathcal{C}\text{-extended 2D MF}
\end{tikzcd}
.
$$

This indicates that there must be also a direct construction relating ($\mathcal{C}$-extended) 3D TQFT with ($\mathcal{C}$-extended) 2D MF.

3D TQFT $\to$ 2D MF. This implication has already been discussed before: in fact, the axioms of 2D MF (except the gluing axiom) are part of the axioms of 3D TQFT, cf. Remark 5.1.2. To prove that the gluing axiom also follows from the axioms of 3D TQFT, we again use the version of extended surface from Definition 5.1.10.

Let Σ'_V be the surface obtained from a surface Σ by cutting a circle from it and labeling the two new boundary components with objects V and V^*, as in Definition 5.1.12 (see Figure 5.20).

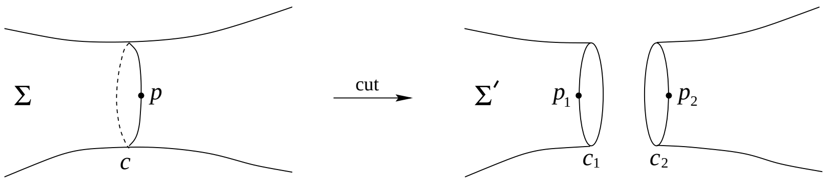


FIGURE 5.20

In accordance with the proof of Proposition 5.1.8, instead of Σ'_V we consider the surface $\Sigma'' = \Sigma''_V$ obtained from Σ'_V by replacing the boundary circles with marked points with tangent vectors at them. We can shrink Σ'', so that it is "inside" Σ, as in Figure 5.21 below.

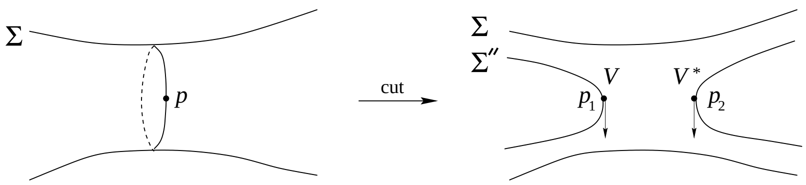


FIGURE 5.21

Then we "fill in the space between Σ and Σ''", i.e., we consider a 3-manifold M with boundary $\partial M = \Sigma \sqcup \overline{\Sigma''}$ (see Figure 5.22). This M is a $\mathcal{C}$-marked 3-manifold, hence it gives a vector

$$\tau(M) \in \tau(\partial M) \simeq \mathrm{Hom}_k(\tau(\Sigma''), \tau(\Sigma)).$$

Considered as a map $\tau(\Sigma'_V) \to \tau(\Sigma)$, this gives the required gluing map (5.1.1). One can easily check that this definition is correct and satisfies all the properties of Definition 5.1.12.

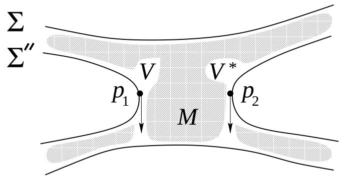


FIGURE 5.22

2D MF → 3D TQFT. This implication is much more difficult and, to the best of our knowledge, no complete construction of it is known. There are two approaches: the first one, due to L. Crane [**C**] (see also [**Ko**]), is based on the Heegaard splitting; the second one, due to M. Kontsevich and to I. Frenkel (unpublished), is based on Morse theory.

Following Crane [**C**], we will construct (non-extended) 3D TQFT starting from a $\mathcal{C}$-extended 2D MF. We do not know how to extend this construction to a $\mathcal{C}$-extended 3D TQFT.

We will use the following well-known theorem in topology (for references, see [**Cr**]).

THEOREM 5.8.1 (Reidemeister–Singer). *Let M be a connected closed oriented 3-manifold. Then*:

(i) *M can be presented as a result of gluing of two solid handlebodies*:

$$M = M_\varphi = H_1 \sqcup_\varphi H_2,$$

where $\varphi\colon \partial H_1 \xrightarrow{\sim} \overline{\partial H_2}$. Such a presentation is called a Heegaard splitting.

(ii) *Two such M_φ and $M_{\varphi'}$ are homeomorphic iff $\varphi\colon \partial H_1 \xrightarrow{\sim} \overline{\partial H_2}$ can be obtained from $\varphi'\colon \partial H_1' \xrightarrow{\sim} \overline{\partial H_2'}$ by a sequence of the following moves*:

(a) *$H_1 = H_1'$, $H_2 = H_2'$, φ' is isotopic to φ.*

(b) *$H_1 = H_1'$, $H_2 = H_2'$, $\varphi' = y \circ \varphi \circ x$, where $x \in N_{H_1}$, $y \in N_{H_2}$ and*

$$N_H := \{\textit{homeomorphisms of } \partial H \textit{ which extend to } H\}.$$

(c) Stabilization. *Let $H_1' = H_1 \# T$, $H_2' = H_2 \# T$, where T is a solid torus and $\#$ denotes a connected sum of topological spaces (see Figure 5.23 below). Let $\varphi' = \varphi \# s$, where $s\colon \partial T \xrightarrow{\sim} \partial T$ is the homeomorphism of the 2-torus which has a matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the standard basis $\{\alpha, \beta\}$ of $\mathrm{H}_1(\partial T, \mathbb{R})$. Then $M_{\varphi'} \simeq M_\varphi \# S^3 \simeq M_\varphi$.*

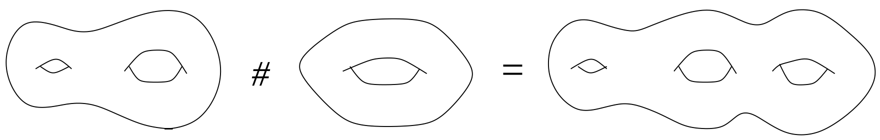


FIGURE 5.23. Connected sum of 3-manifolds.

Now suppose that we have a $\mathcal{C}$-extended modular functor. Let H be a solid handlebody whose boundary ∂H is a surface of genus g. We will construct a vector $v_0(H) \in \tau(\partial H)$ as follows.

Choose some non-intersecting "cuts", i.e., disks embedded in H, which cut H into contractible pieces. This also gives a system of cuts on ∂H and thus, a decomposition of ∂H into spheres with holes: $\partial H = \bigcup \Sigma_a$. Consider all possible labelings $i : \{\text{cuts}\} \to I$ of the cutting circles by simple objects of $\mathcal{C}$ (see Figure 5.24).

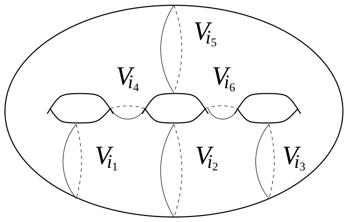


FIGURE 5.24

Then, by the gluing axiom,

$$\tau(\partial H) \simeq \bigoplus_i \bigotimes_a \tau(\Sigma_a; \{V_{i_c}^{\varepsilon}\}_{c \subset \partial \Sigma_a}).$$

Here Σ_a are the components of ∂H, the notation $c \subset \partial \Sigma_a$ means that the cut c is one of the boundary components of Σ_a, and V^{ε} is either V or V^* chosen so that every V_{i_c} appears in the tensor product once as V_{i_c} and once as $V_{i_c}^*$.

Let us choose all $i_c = 0$, i.e., all $V_{i_c} = \mathbf{1}$. Then $\tau(\Sigma_a; \mathbf{1}, \ldots, \mathbf{1}) = k$. Therefore, this gives a vector

$$v_0(H) = \bigotimes_a (1 \in \tau(\Sigma_a; \mathbf{1}, \ldots, \mathbf{1})) \in \tau(\partial H).$$

(compare with Remark 4.5.4).

THEOREM 5.8.2 (Crane [**C**]). *The vector $v_0(H)$ does not depend on the choice of the cuts. Moreover, $v_0(H)$ is N_H-invariant.*

PROOF. Obviously, any two systems of cuts of H into a union of solid balls can be related to one another by a sequence of the following moves:

(a) the action of N_H, and (b) the F-move.

It is easy to see that $v_0(H)$ does not change under the move (b). As for (a), one needs a description of the generators of N_H. Such a description is known [**Su**]. Then one checks that $v_0(H)$ is invariant under these generators—this is not difficult—we refer to [**C**], [**Ko**] for the details.

The fact that $v_0(H)$ is N_H-invariant follows from (a). $\square$

Now we will use Theorems 5.8.1 and 5.8.2 to construct invariants of closed 3-manifolds.

Let $M = M_\varphi = H_1 \sqcup_\varphi H_2$ be as in 5.8.1. The map $\varphi \colon \partial H_1 \xrightarrow{\sim} \overline{\partial H_2}$ gives an isomorphism of vector spaces $\varphi_* \colon \tau(\partial H_1) \xrightarrow{\sim} \tau(\overline{\partial H_2}) = \tau(\partial H_2)^*$. We define

$$\tau(M) := D^{g-1} \left(\varphi_*(v_0(H_1)), v_0(H_2)\right), \tag{5.8.1}$$

where $D = s_{00}^{-1}$ is defined by (3.1.15).

The prefactor D^{g-1} is chosen in order that $\tau(M)$ be invariant under the stabilization move 5.8.1(c). Indeed, let $H' = H \# T$. Then $\partial H' = \partial H \# \partial T$, where ∂T is the 2-torus. By the construction of $v_0(H')$ it is clear that

$$v_0(H') = v_0(H) \otimes v_0(T).$$

Then

$$\begin{aligned}\tau(M') &= D^g\left((\varphi\# s)_*(v_0(H_1)\otimes v_0(T)), v_0(H_2')\right)\\ &= \tau(M)D\left(s_* v_0(T), v_0(T)\right) = \tau(M)Ds_{00} = \tau(M).\end{aligned}$$

Therefore, we have constructed an invariant τ of closed 3-manifolds. To construct 3D TQFT, we have to define $\tau(M)$ for any 3-manifold M with boundary. To do so, we need a variant of Heegaard splitting for 3-manifolds with boundary. There is such a theorem, due to Motto [**Mo**]. His result is similar to what we had before, only one has to consider not only handlebodies but also "hollow handlebodies". A hollow handlebody is a handlebody with some parts of its interior cut out. Hence, it has both "inner" and "outer" boundary. We glue two such hollow handlebodies by identifying their outer boundaries, the remaining inner boundaries give the boundary of the resulting 3-manifold.

Then we can repeat the above construction of $\tau(M)$ for manifolds M with boundary. This gives the implication

$$\mathcal{C}\text{-extended 2D MF} \to \text{(non-extended) 3D TQFT}.$$

In order to go one step further, i.e., to construct a $\mathcal{C}$-extended 3D TQFT, one needs an analog of Heegaard splitting and Reidemeister–Singer theorem for manifolds with boundary and marked points. To the best of our knowledge, such a result is not available at the moment. Hopefully, this is only a temporary difficulty. Finally, let us note that if we start with a non-extended 2D MF, without gluing axiom, the construction of 3D TQFT would fail.

CHAPTER 6

Moduli Spaces and Complex Modular Functor

In this chapter, we will rewrite the definition of modular functor in algebro-geometric terms, i.e., in terms of flat connections with regular singularities on the moduli spaces, instead of the topological surfaces and mapping class groups in Chapter 5. In fact, this is how the modular functor originally appeared in conformal field theory, see, e.g, [**MS1**], [**S**]. The exposition in this chapter is based on the unpublished manuscript [**BFM**]; similar ideas were also introduced in Deligne's letter to Drinfeld.

The complex version of modular functor is best formulated using the language of connections with regular singularities on the Deligne–Mumford compactification of the moduli space of complex curves. For readers' convenience, we give a short introduction to the theory of moduli spaces and connections with regular singularities.

In this chapter, "complex curve" means "complex projective curve" (thus, it is compact); unless stated otherwise, the curves are assumed to be connected and non-singular. We remind that by Riemann's theorem, every (non-singular) compact Riemann surface is projective. However, unless otherwise specified, we will consider all manifolds with analytic topology. We assume that the reader is familiar with some basic notions of algebraic geometry, such as coherent $\mathcal{O}$-modules, vector bundles, etc.; all the necessary prerequisites can be found, for example, in [**GH**].

6.1. Moduli spaces and complex Teichmüller tower

In this section, we give a definition of the Teichmüller tower of groupoids in terms of moduli spaces of complex curves. Let us first recall the relation between the moduli space and the mapping class group.

Let Σ be a compact oriented topological surface without boundary. A *complex structure* on Σ is an isomorphism class of pairs (C,φ) where C is a smooth compact complex curve and $\varphi\colon C \xrightarrow{\sim} \Sigma$ is a homeomorphism preserving orientation. Equivalently, a complex structure on a smooth surface can be defined as a polarization of the complexified tangent space, i.e., a one-dimensional complex vector sub-bundle $T^{\mathbb{C}}\Sigma \subset (T^{\mathbb{R}}\Sigma)\otimes_{\mathbb{R}}\mathbb{C}$ such that $(T^{\mathbb{R}}\Sigma)\otimes_{\mathbb{R}}\mathbb{C} = T^{\mathbb{C}} \oplus \overline{T^{\mathbb{C}}}$.

We identify two complex structures on Σ if they can be obtained one from another by an isotopy of Σ; in other words, we let $(C,\varphi)\simeq(C',\varphi')$ if there exists a commutative square

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \Sigma \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow{\varphi'} & \Sigma \end{array}$$

where f is an isomorphism of complex varieties, and g is an automorphism of Σ which is homotopic to identity. The set of all complex structures on Σ up to isotopy is called the *Teichmüller space* and will be denoted by $T(\Sigma)$. For a connected surface of genus g, we will also use the notation T_g.

Denote by $\mathcal{M}_g$ the set of isomorphism classes of complex curves of genus g. It is well known that this set has a natural structure of an analytic variety. We will call $\mathcal{M}_g$ the *moduli space* of curves of genus g (to be more precise, it is a *coarse* moduli space in the terminology of Mumford—see Theorem 6.1.8). The following result immediately follows from the definitions.

PROPOSITION 6.1.1. *The moduli space $\mathcal{M}_g$ is isomorphic to T_g/Γ_g, where Γ_g is the mapping class group of a surface of genus g.*

The next result is classical, see, e.g., [**Ab**].

THEOREM 6.1.2 (Teichmüller). *The set T_g of all complex structures (up to isotopy) on a connected surface Σ of genus g has a natural structure of a complex analytic manifold such that the action of Γ_g is holomorphic. In particular, this gives a structure of an analytic variety on $\mathcal{M}_g$.*

As a real *analytic manifold, T_g is isomorphic to $\mathbb{R}^{6g-g}$ for $g > 1$.*

Note that $T_g \not\simeq \mathbb{C}^{3g-3}$ as a complex analytic manifold.

EXAMPLE 6.1.3. For Σ of genus 1, i.e., a torus

$$T_1 \simeq \mathcal{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \simeq \mathbb{R}^2.$$

Then $\mathcal{M}_1 = \mathcal{H}/\mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{C}$.

The above results can be generalized to surfaces with marked points.

DEFINITION 6.1.4. A *pointed curve* is a complex curve C with an ordered set of marked points $y_1, \dots, y_n \in C$ and with a non-zero tangent vector v_i given at each point.

Note that choosing a non-zero tangent vector is equivalent to choosing a non-zero cotangent vector: it can be defined by $\langle v_i, v_i^* \rangle = 1, v_i \in T_{y_i}C, v_i^* \in T^*_{y_i}C$.

One defines isomorphism of pointed curves in an obvious way. Let us denote

$$\begin{aligned} \mathcal{M}_{g,n} = &\text{the set of isomorphism classes of pointed curves} \\ &\text{of genus } g \text{ with } n \text{ marked points.} \end{aligned} \tag{6.1.1}$$

As before, we will call $\mathcal{M}_{g,n}$ the *moduli space of pointed curves*.

REMARK 6.1.5. This moduli space is different from the moduli space considered in [**Kn**]. The latter space, which we will denote $\mathcal{M}_g^n$, is defined as the set of isomorphism classes of curves of genus g with n marked points, but without tangent vectors. However, they are closely related: for example, if g, n are such that there are no non-trivial automorphisms of a n-pointed genus g curve, then $\mathcal{M}_{g,n}$ is a $(\mathbb{C}^\times)^n$ bundle over $\mathcal{M}_g^n$, so all the results of [**Kn**] can be easily reformulated for $\mathcal{M}_{g,n}$. One can also define more general moduli spaces $\mathcal{M}_{g,r}^n$ in an obvious way; they will not be used in our work.

Let us define the *Teichmüller space* $T_{g,n}$ to be the set of all complex structures on a surface Σ of genus g with n marked points and tangent vectors up to an isotopy which fixes the marked points and vectors. This space has a natural structure of an analytic manifold. Then the previous results can be generalized as follows:

THEOREM 6.1.6. (i) *The Teichmüller space $T_{g,n}$ is contractible.*

(ii) *Let $\Gamma'_{g,n} \subset \Gamma_{g,n}$ be the group of automorphisms of an extended topological surface of genus g with n boundary components which act trivially on the set of boundary components. Then this group acts holomorphically on $T_{g,n}$, and the stabilizer of every point is finite.*

(iii) *As a complex variety, $\mathcal{M}_{g,n} \simeq T_{g,n}/\Gamma'_{g,n}$. In particular, $\mathcal{M}_{g,n}$ is connected.*

REMARK 6.1.7. In fact, it is shown in [**DM**], [**Kn**], that $\mathcal{M}_{g,n}$ is an irreducible quasiprojective algebraic variety over $\mathbb{C}$—this is a difficult theorem.

If the action of $\Gamma'_{g,n}$ on the Teichmüller space $T_{g,n}$ were free, then $\pi_1(\mathcal{M}_{g,n})$ would be equal to $\Gamma'_{g,n}$. Unfortunately, the action of $\Gamma'_{g,n}$ is not free: the stabilizer of a point coincides with the group of automorphisms of the corresponding complex curve. Therefore, in general $\pi_1(\mathcal{M}_{g,n}) \neq \Gamma'_{g,n}$, as can be seen already for $g = 1$: in this case, $\pi_1(\mathcal{M}_{1,0}) = \{1\}$, while $\Gamma_{0,1} \simeq \mathrm{SL}_2(\mathbb{Z})$.

Now, let us discuss in what sense $\mathcal{M}_{g,n}$ is the moduli space of curves. Let us recall (see, e.g., [**Ha**]) that a family of curves C over a smooth variety U by definition is a variety C_U with a proper flat morphism $\pi\colon C_U \to U$, such that $\pi^{-1}(t) = C_t$ is a compact complex curve (unless specified otherwise, we will assume that the fibers are connected). Note that $\pi^{-1}(t)$ can be singular even if C_U is smooth, as shown by the example of the surface $xy = tu$ in $\mathbb{P}^4$. Similarly, a family of pointed curves is a family C_U together with n non-intersecting sections $p_i\colon U \to C_U$ and a non-vanishing vertical vector field v_i on $p_i(U)$ (vertical means that $\pi_*(v_i) = 0$).

THEOREM 6.1.8. *$\mathcal{M}_{g,n}$ is the* coarse moduli space *of curves in the sense of* [**MFK**]: *for every family of pointed curves C_U over U, the induced map $U \to \mathcal{M}_{g,n}$, $t \mapsto [C_t]$, is analytic.* (*Here $[C]$ denotes the isomorphism class of a curve C.*)

Unfortunately, it is not true that the construction above gives a bijection

$$\{\text{families of curves over } U\} \xrightarrow{\sim} \{\text{maps } U \to \mathcal{M}_{g,n}\};$$

in other words, $\mathcal{M}_{g,n}$ is not the fine moduli space. The reason for the failure is that $\mathcal{M}_{g,n}$ carries no information about the automorphisms of a curve.

EXERCISE 6.1.9. Let C be a pointed curve, and σ—a non-trivial automorphism of C. Construct a family of curves C_t over $\mathbb{C}^\times$ such that $C_t \simeq C$ for any t, but this family is not isomorphic to the direct product $C \times \mathbb{C}^\times$.

It turns out that this was the only problem: if we assume that the curves have no non-trivial automorphisms, then $\mathcal{M}_{g,n}$ is the fine moduli space.

THEOREM 6.1.10. *Assume that $g > 0, n > 0$ or $G = 0, n > 1$. Then:*

1. *For every complex curve C of genus g with n marked point, the group of automorphisms is trivial.*
2. *The action of the group $\Gamma'_{g,n}$ on the corresponding Teichmüller space is free, so $\mathcal{M}_{g,n} = T_{g,n}/\Gamma'_{g,n}$ is smooth.*
3. *$\mathcal{M}_{g,n}$ is the fine moduli space: for every variety S, the functors*

$$S \mapsto \textit{families of curves of genus } g \textit{ with } n \textit{ marked points over } S$$

and

$$S \mapsto \mathrm{Mor}(S, \mathcal{M}_{g,n})$$

are canonically isomorphic. In other words, every family of curves on S can be obtained as a pull-back of a universal family over $\mathcal{M}_{g,n}$ for a unique map $\psi\colon S \to \mathcal{M}_{g,n}$.

It turns out that one can also define a suitable "fine moduli space" $M_{g,n}$ under less restrictive assumptions that (g,n) is stable, i.e.

$$(g,n) \neq (0,0),(0,1),(1,0). \tag{6.1.2}$$

In this case, the group of automorphisms of every curve $C \in \mathcal{M}_{g,n}$ is finite. It turns out that under this assumption, it is possible to account for these automorphisms and define a "fine" moduli space, if we allow the moduli space to be not a variety, but a *stack*, as defined in [**DM**], [**Ar**]. Intuitively, this means that every point of $\mathcal{M}_{g,n}$ has some additional structure, which encodes the group of automorphisms of the corresponding curve. Unfortunately, an accurate exposition of the theory of algebraic stacks goes far beyond the scope of this book; we can only refer the reader to the Appendix to [**Vi**] for an introduction to this language. Another approach, which applies if g,n are such that the generic curve $C \in \mathcal{M}_{g,n}$ has no automorphisms, is to consider $\mathcal{M}_{g,n}$ as an *orbifold*, or V*-variety* (see [**Sat**]). Finally, the third possibility, used in [**TUY**], is to consider "local universal families of curves", which can be viewed as local charts of the algebraic stack. For our purposes, we can use any of these approaches: all of them will yield the same results, and each has its own advantages and disadvantages. We chose to use the language of algebraic stacks.

We will denote by $M_{g,n}$ the *moduli stack* of pointed curves of genus g with n marked points; as we said, we will not explain what it is, referring the reader to [**DM**] instead. Nevertheless, we can say what are points, vector bundles, etc., on $M_{g,n}$. Namely: for a complex manifold S, a morphism $S \to M_{g,n}$ is by definition the same as a family of pointed complex curves of genus g with n marked points over S. In particular, this implies that the set of (closed) points of $M_{g,n}$ is $\mathcal{M}_{g,n}$. Similarly, a vector bundle E on $M_{g,n}$ is the same as a collection of vector bundles ϕ^*E for every morphism $\phi : S \to M_{g,n}$, such that this collection is functorial in S. Local systems, flat connections, etc., can be defined in a similar way. We can also define a divisor $D \subset M_{g,n}$ as a compatible collection of divisors $\phi^*D \subset S$ for every étale (i.e., finite unramified covering) $\phi : S \to M_{g,n}$. Finally, we define the fundamental group of $M_{g,n}$ by

$$\pi_1(M_{g,n}, C) = \{(C_t)_{0\le t\le 1}, \varphi_0 : C_0 \xrightarrow{\sim} C, \varphi_1 : C_1 \xrightarrow{\sim} C\}/\text{homotopy}.$$

Here C_t is a C^∞ family of complex curves, i.e., a C^∞ real manifold Σ with a map $\pi\colon \Sigma \to [0,1]$ such that $d\pi \neq 0$, and for every $t \in \mathbb{R}$, $C_t = \pi^{-1}(t)$ is a smooth compact oriented surface, and with a family of complex structures μ_t in C_t such that μ_t is a C^∞ function of t (this should be modified in an obvious way for pointed curves). Later we will show that in fact, $\pi_1(M_{g,n}) = \Gamma'_{g,n}$.

Of course, we are just hiding the real problem: why so defined $M_{g,n}$ is a reasonable geometric object, i.e., why the standard results about, say, sheaves on varieties apply to $M_{g,n}$? This is indeed a difficult question, and the best we can do here is to refer to [**DM**]. Their results show that as far as we are concerned, $M_{g,n}$ can be treated in the same way as a non-singular variety: all the standard results from algebraic geometry we will be using apply to $M_{g,n}$.

As was mentioned above, for $g > 0, n > 0$ or $g = 0, n > 1$, $\mathcal{M}_{g,n}$ itself is the fine moduli space, so in this case we have $\mathcal{M}_{g,n} = M_{g,n}$. In general, this is not true.

From now on, we will use $M_{g,n}$ as the moduli space, and all geometric constructions will be understood in the stack sense. A reader who is not too experienced in this language can just think of $M_{g,n}$ as a smooth manifold.

Sometimes, it is convenient to define a slightly different space. Let A be a finite set. Denote $\mathcal{M}_{g,A} = \{(C, f)\}$, where C is a genus g complex curve with n **unordered** marked points and non-zero tangent vectors, and f is a bijection ?! $A(C) \xrightarrow{\sim} A$, where $n = |A|$, and $A(C)$ is the set of marked points of C. In other words, $\mathcal{M}_{g,A}$ is the moduli space of curves of genus g with n marked points labeled by elements of A, and with non-zero tangent vectors at these points. Obviously, for $A = \{1, \dots, n\}$, this coincides with the definition of $\mathcal{M}_{g,n}$. One defines the stack $M_{g,A}$ in a similar way.

We will also consider the moduli space $\mathcal{M}_{*,n}$ of not necessarily connected n-pointed curves, and the space $\mathcal{M} = \sqcup_{n\geq 0}\mathcal{M}_{*,n}$. One easily sees that $\mathcal{M}_{*,n}$ is an unramified finite cover over the space

$$\sqcup(\mathcal{M}_{g_1,n_1} \times \dots \times \mathcal{M}_{g_k,n_k}),$$

where the disjoint union is taken over all finite sequences $(g_1, n_1), \dots, (g_k, n_k)$ such that $\sum n_i = n$, up to permutation. Thus, we can easily define the stacks $M_{*,n}, M_{*,A}$. The last stack will be frequently used later, so we state its definition explicitly:

$$\begin{aligned} M_{*,A} = &\text{moduli stack of stable smooth possibly disconnected curves}\\ &\text{with unordered marked points and non-zero tangent vectors}\\ &\text{and a bijection (marked points)} \xrightarrow{\sim} A \end{aligned} \tag{6.1.3}$$

Recall that we have defined the notion of a tower of groupoids, which is just a groupoid Γ with a functor $A\colon \Gamma \to \mathcal{S}ets$ and with the functors of disjoint union and gluing (see Definition 5.6.1). In particular, we have defined the Teichmüller tower $\mathcal{T}eich$, in which the objects are (topological) surfaces with boundary, morphisms are homeomorphisms of surfaces, and gluing is the gluing of two boundary components (see Definition 5.1.7, Section 5.6). The formula $\pi_1(M_{g,n}) = \Gamma'_{g,n}$, which is an obvious corollary of Theorem 6.1.6 if the action of $\Gamma'_{g,n}$ is free, suggests that the same tower can be defined in terms of the moduli spaces $M_{g,n}$.

Recall that for a topological space M, its *Poincaré groupoid* (also called the *fundamental groupoid*) is defined as the groupoid with objects: points of M, and morphisms: homotopy classes of paths in M connecting two points.

DEFINITION 6.1.11. The *complex Teichmüller tower of groupoids* $\mathcal{T}eich^{\mathbb{C}}$ is the fundamental groupoid of the stack M, i.e.,

$$\operatorname{Ob}\mathcal{T}eich^{\mathbb{C}} = \text{pointed complex curves}$$

and

$$\operatorname{Mor}(C', C'') = \{(C_t)_{0\leq t\leq 1}, \varphi_0 : C_0 \xrightarrow{\sim} C', \varphi_1 : C_1 \xrightarrow{\sim} C''\}/\text{homotopy},$$

where, as before, C_t is a C^∞ family of pointed curves. We also define the functor $A\colon \mathcal{T}eich^{\mathbb{C}} \to \mathcal{S}ets$ by $A(C, y_i, v_i) = \{y_i\}$, and the disjoint union and empty set in an obvious way.

In a similar way, for a finite set A we define the groupoid $\mathcal{T}eich^{\mathbb{C}}_A$ as the fundamental groupoid of the stack $M_{*,A}$.

Note that in particular, for every curve C we have a canonical map $\operatorname{Aut} C \to \operatorname{Mor}(C, C)$, which assigns to $\sigma \in \operatorname{Aut} C$ the data $C_t = C \times [0,1], \varphi_0 = \mathrm{id}, \varphi_1 = \sigma$, which explains why we introduced φ_0, φ_1 in the definition.

To complete the definition, we also have to define the gluing functor. This will be done in the next section.

Now we can compare this complex Teichmüller groupoid with the groupoid $\mathcal{T}eich$ defined in the previous chapter in terms of topological surfaces with boundary. Note, however, that since we have imposed the stability condition (6.1.2), it only makes sense to compare $\mathcal{T}eich^{\mathbb{C}}$ with the subgroupoid $\mathcal{T}eich^{stab} \subset \mathcal{T}eich$, formed by topological surfaces all connected components of which satisfy the stability condition (6.1.2).

THEOREM 6.1.12. *The towers of groupoids $\mathcal{T}eich^{stab}$ and $\mathcal{T}eich^{\mathbb{C}}$ are equivalent. In particular, $\pi_1(M_{g,n}) = \Gamma'_{g,n}$.*

PROOF. The proof essentially repeats the proof of the fact that for a simply-connected T, one has $\pi_1(T/\Gamma) = \Gamma$.

First we construct a functor $\mathcal{T}eich^{stab} \to \mathcal{T}eich^{\mathbb{C}}$ as follows. Let Σ be an object of $\mathcal{T}eich^{stab}$, i.e., an extended surface. We will use Definition 5.1.10 of extended surface; thus, Σ is a topological surface with marked points and non-zero tangent vectors at these points. Fix a complex structure μ on the surface Σ. Let C_μ be the complex curve obtained from Σ with the complex structure μ. It is a pointed curve, with the same marked points and tangent vectors as Σ (recall that a complex structure defines an $\mathbb{R}$-linear isomorphism of the real tangent space $T^{\mathbb{R}}_p\Sigma$ and the complex tangent space $T^{\mathbb{C}}_pC$; for example, for $\Sigma = \mathbb{R}^2$ and the standard complex structure, we get $\partial_x \mapsto \partial_z, \partial_y \mapsto \mathrm{i}\partial_z$). This construction depends on the choice of μ. By Theorem 6.1.6, the set $T(\Sigma)$ of all complex structures on Σ is contractible. Therefore, every two complex structures μ_0, μ_1 can be connected by a unique path μ_t in $T(\Sigma)$. This gives a canonical family of curves C_{μ_t} connecting C_{μ_0} with C_{μ_1}, or a canonical morphism $C_{\mu_0} \to C_{\mu_1}$ in $\mathcal{T}eich^{\mathbb{C}}$. Thus, we have assigned to a topological surface C a collection of objects $C_\mu \in \mathcal{T}eich^{\mathbb{C}}$, canonically isomorphic to each other. As was discussed before (see Definition 1.1.11, Lemma 1.1.12), such a collection can be viewed as an object of $\mathcal{T}eich^{\mathbb{C}}$.

This defines the functor $\mathcal{T}eich^{stab} \to \mathcal{T}eich^{\mathbb{C}}$ on the objects of $\mathcal{T}eich^{stab}$. To define it on morphisms, we note that any homeomorphism of extended surfaces $f\colon \Sigma \xrightarrow{\sim} \Sigma'$ gives an identification of the Teichmüller spaces $f_*\colon T(\Sigma) \xrightarrow{\sim} T(\Sigma')$. Thus, for any $\mu \in T(\Sigma), \mu' \in T(\Sigma')$ there is a unique path connecting $f_*\mu$ with μ' in $T(\Sigma')$, and thus, a unique path connecting C_μ with $C_{\mu'}$ in the moduli stack M.

The inverse functor $\mathcal{T}eich^{\mathbb{C}} \to \mathcal{T}eich^{stab}$ is constructed as follows. On objects, it is just the forgetful functor, which assigns to a complex curve C the underlying topological surface. To define it on morphisms, let $C_t, t \in [0,1]$ be a family of curves. As before, let us forget the complex structure and view it as a family of extended surfaces. Then each of the surfaces Σ_t is homeomorphic to Σ_0, and the homeomorphism is unique if we additionally require that it depends continuously on t (this follows from the discreteness of the mapping class group). This gives a family of homeomorphisms $\varphi_t\colon \Sigma_0 \xrightarrow{\sim} \Sigma_t$. In particular, this defines $\varphi_1\colon \Sigma_0 \to \Sigma_1$.

It is easy to check that the above two functors are inverse to each other and are compatible with gluing (see the next section). $\square$

REMARK 6.1.13. Sometimes we will use an alternative definition of pointed curve. Recall that extended surface can be defined in any of the following three ways: 1) as a surface with boundary and a point on each boundary component; 2) as a surface with boundary and a parametrization of every boundary component; 3) as a surface without boundary but with marked points and non-zero tangent vectors. All these definitions give rise to equivalent groupoids (see Proposition 5.1.8).

Similarly, in the complex situation we can use the following definition of pointed curve: a pointed curve is a complex curve with marked points y_i and a local parameter z_i near each of these points. The corresponding moduli space (which is infinite-dimensional) will be denoted $\mathcal{M}_{g,n}^{(\infty)}$; similarly, one defines $\mathcal{M}^{(\infty)}$, and the groupoid $\mathcal{T}eich^{\mathbb{C}(\infty)}$. One has an obvious forgetting functor $\mathcal{T}eich^{\mathbb{C}(\infty)} \to \mathcal{T}eich^{\mathbb{C}}\colon (C, y_i, z_i) \mapsto (C, y_i, v_i)$, where the vector v_i is defined by $\langle v_i, dz_i\rangle = 1$. Since the set $\{f = z + \sum_{n>1} a_n z^n \mid f$ converges in a neighborhood of $0\}$ is contractible, this functor is an equivalence. Therefore, we can use either definition of the Teichmüller groupoid.

Finally, there exists yet one more definition: a pointed curve is a topological surface Σ with a boundary, with a complex structure μ and with parametrizations $\pi_i\colon (\partial\Sigma)_i \to S^1$ of the boundary components which are analytic with respect to the complex structure μ. We leave it to the reader to check that this definition is equivalent to the two previous ones.

6.2. Compactification of the moduli space and gluing

In this section, we will define the gluing functor for the complex Teichmüller groupoid. A straightforward approach would be to cut from a curve small disks around the marked points, and glue the boundary circles together (this was first suggested by Vafa, see [**V1**]). However, there is a much better way of defining the gluing, which uses the so-called Deligne–Mumford compactification of the moduli space.

Following [**DM**], let us call a possibly singular complex curve C *stable* if its only singularities are ordinary double points, and its group of automorphisms is finite. (We recall that $p \in C$ is called an *ordinary double point* if locally C is isomorphic to the coordinate cross $xy = 0$ in $\mathbb{C}^2$ with $p = (0,0)$. Such a singularity is also called a *node*, or a *quadratic singularity*.) For such a curve C, one can define its genus g as the genus of the smooth curve obtained by deforming away all double points; in local coordinates, this can be described by replacing the equation $xy = 0$ by $xy = \varepsilon$ (for readers familiar with algebraic geometry, we note that so defined genus coincides with the "arithmetic genus" of C). Thus, we can consider the set $\overline{\mathcal{M}}_g$ of all isomorphism classes of connected stable curves of genus g. Similarly, one can define stable pointed curves (the marked points must be non-singular, i.e., they can not coincide with the double points), and define the space $\overline{\mathcal{M}}_{g,n}$ of isomorphism classes of such curves. This moduli space (or, rather, the analogous space $\overline{\mathcal{M}}_g^n$—see Remark 6.1.5) was considered by Deligne and Mumford [**DM**] and Knudsen [**Kn**]. Note that for non-singular curves, the stability condition automatically follows from the condition (6.1.2) which we imposed in the previous section.

As before, we can also define the corresponding stack $\overline{M}_{g,n}$ by $\mathrm{Mor}(S, \overline{M}_{g,n}) = \{$families of (possibly singular) stable pointed curves over $S\}$.

THEOREM 6.2.1 ([**DM**], [**Kn**]). (i) *$\overline{M}_{g,n}$ is a stack in the sense of* [**DM**].
(ii) *$\overline{M}_{g,n}$ is connected.*

(iii) *The complement* $D = \overline{M}_{g,n} \setminus M_{g,n}$ *is a divisor with normal crossings in* $\overline{M}_{g,n}$ *in the stack sense.* (*Often* D *is also called the* boundary, *or* points at infinity, *of the moduli stack* $M_{g,n}$.)

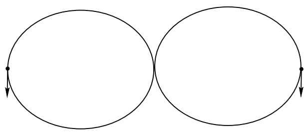
Let us recall that a *divisor with normal crossings* $D \subset M$, where M is an analytic manifold, is a union of a finite number of smooth codimension 1 subvarieties D_i, such that if p lies in the intersection of k components, then one can introduce local coordinates $z_1, \dots, z_n$ ($n = \dim M$) near p such that D is given by the equation $z_1 \cdots z_k = 0$. This definition should be properly modified for stacks.

As far as we are concerned, part (i) of the theorem could read as follows: "most standard results about smooth manifolds apply to $\overline{M}_{g,n}$".

REMARK 6.2.2. As in Remark 6.1.5, we note that in [**DM**], [**Kn**], a different space is considered—they consider the space $\overline{\mathcal{M}}_g^n$, which is a compactification of the moduli space $\mathcal{M}_g^n$. It is shown in [**Kn**] that $\overline{\mathcal{M}}_g^n$ is a projective variety; in particular, it is compact. Our space $\overline{\mathcal{M}}_{g,n}$ is not compact for obvious reasons: the tangent vectors lie in a punctured affine space, which is not compact. However, other results from [**DM**], [**Kn**],—most importantly, the fact that $D = \overline{M}_{g,n} \setminus M_{g,n}$ is a divisor with normal crossings—can be easily generalized to our situation. One could further extend $\overline{\mathcal{M}}_{g,n}$, replacing $T_p^\times(C)$ by a projective space. However, this is not necessary for our purposes.

The stack $\overline{M}_{g,n}$ is called the *Deligne–Mumford compactification* of the moduli space (warning: it is not compact unless $n = 0$—see the remark above).

EXAMPLE 6.2.3. Let $g = 0, n = 2$. Then one easily sees that $M_{0,2} = \mathcal{M}_{0,2} = \mathbb{C}^\times$, and $\overline{\mathcal{M}}_{0,2} = \overline{M}_{0,2} = \mathbb{C}$. The "infinite point" is the singular curve shown below.



Note that every divisor with normal crossings is naturally stratified: its codimension k (in D) stratum D^k consists of points which lie in the intersection of $k+1$ components. In particular, the set of non-singular points of D is given by

$$D^0 = \sqcup_i D_i^0, \tag{6.2.1}$$

where $D_i^0 = D_i \setminus (\cup_{j \neq i} D_j)$, D_i being the components of D.

For stacks, this definition should be suitably modified. It can be shown that for $\overline{M}_{g,n}$ the stratification of D is given by

$$D^k = \{\text{curves with exactly } k+1 \text{ double points}\}. \tag{6.2.2}$$

In particular, the open stratum $D^0 \subset D$ consists of the curves with exactly one double point. Every such curve is obtained by identifying two distinct points of a stable non-singular curve $C^\vee$ (*normalization* of C). In other words, if we denote by $M^2_{*,A}$ the set of stable possibly not connected curves with $|A| + 2$ marked points, out of which all but two are labeled by elements of A and have non-zero tangent vector, and the remaining two are not labeled or ordered and have no tangent vector assigned, then we have a natural isomorphism $S\colon M^2_{*,A} \xrightarrow{\sim} D^0$. This shows that the components of D are in bijection with the components of $M^2_{*,A}$, which are easy to describe. For example, for $g = 0$, we get the following result: the irreducible components of $D \subset \overline{M}_{0,A}$ are given by

$$\textit{Irr } D \leftrightarrow \{(A', A'') \mid A', A'' \subset A, A = A' \sqcup A'', |A'|, |A''| \geq 1\} \tag{6.2.3}$$

(here (A', A'') is an unordered pair). The corresponding component of D is defined by

$$D^0_{A',A''} = S(M^1_{0,A'} \times M^1_{0,A''});$$

in other words, these are curves which can be obtained by identifying a point on $C^{(1)} \in M_{0,A'}$ with a point on $C^{(2)} \in M_{0,A''}$, as in Example 6.2.3.

Let us show how such a curve can be obtained as a limit of a family of non-singular curves.

EXAMPLE 6.2.4. Let $z_1, \ldots, z_n \in \mathbb{C}$, $v_1, \ldots, v_n \in \mathbb{C}^\times$ be such that $z_i \neq z_j$. Denote

$$\begin{aligned} (\mathbb{P}^1; z_1, \ldots, z_n; v_1, \ldots, v_n) = {} & \text{the projective line } \mathbb{P}^1 \\ & \text{with the standard coordinate } z \in \mathbb{C} \cup \infty, \\ & \text{with marked points } z = z_i \text{ and tangent vectors } v_i \partial_z. \end{aligned} \tag{6.2.4}$$

Obviously, any curve $C \in M_{0,n}$ can be written in such form.

Now, choose $z'_1 \ldots, z'_k, a, z''_1, \ldots, z''_m$ such that $z'_i \neq z'_j, z'_i \neq a, z''_i \neq z''_j$. Choose $q \in \mathbb{C}^\times$ small enough and define the curve C_q by

$$C_t = (\mathbb{P}^1; z'_1, \ldots, z'_k, a + qz''_1, \ldots, a + qz''_m; v'_1, \ldots, v'_k, qv''_1, \ldots, qv''_m).$$

Then we claim that the limit $C_0 = \lim_{q \to 0} C_q$ exists in $\overline{M}_{0,k+m}$, and is given by the singular curve obtained by identifying the point $a \in C^{(1)}$ with $\infty \in C^{(2)}$, where

$$\begin{aligned} C^{(1)} = & (\mathbb{P}^1; z'_1, \ldots, z'_k; v'_1, \ldots, v'_k), \\ C^{(2)} = & (\mathbb{P}^1; z''_1, \ldots, z''_m; v''_1, \ldots, v''_m). \end{aligned}$$

Speaking informally, one can say that as $q \to 0$, the curve C_q looks to a bare eye as $C^{(1)}$ and m points $a + qz''_i$ all collapsed at a; looking at the point a with a microscope, one can separate these m points and see that their relative position is described by $C^{(2)}$.

SKETCH OF PROOF. For simplicity, we will disregard the tangent vectors and will take $a = 0$. First of all, recall that the topology in $\overline{M}_{g,n}$ is defined so that every map $S \to M$ is continuous. In particular, if one can construct an analytic family $C_q, q \in U$, of curves over a disk U in the q-plane, then $\lim_{q \to 0} C_q = C_0$.

Let us construct such a family with C_0 defined above. Let Σ be the surface in $\mathbb{P}^2 \times U$ given by the equation

$$uv = qw^2, \quad (u : v : w) \in \mathbb{P}^2, q \in U \tag{6.2.5}$$

and the marked points given by $\{((z'_i)^2 : q : z'_i), (q(z''_i)^2 : 1 : z''_i)\}_{i=1}^n$. Obviously, this is a smooth family of pointed curves over U, with an obvious projection to U; define $\pi^{-1}(q) = \tilde{C}_q$. We claim that for $q \neq 0$, the curve $\tilde{C}_q \simeq C_q$. Explicitly, the isomorphism $C_q \to \tilde{C}_q$ is given by $(z : s) \mapsto (u : v : w) = (z^2 : qs^2 : zs)$. Similarly, for $q = 0$, $\tilde{C}_0$ can be identified with C_0 by $\psi_1 : C^{(1)} \to \tilde{C}_0, \psi_2 : C^{(2)} \to \tilde{C}_0$ given by $\psi_1 \colon (z : s) \mapsto (z : 0 : s), \psi_2 \colon (w : s) \mapsto (0 : s : w)$. This completes the proof. We leave it to the reader to check that the above construction in fact also gives correct tangent vectors. □

Let us denote by $N(D)$ the normal bundle to D in $\overline{M}_{g,n}$: for $C \in D, N_C(D) = T_C\overline{M}_{g,n}/T_C D$; and let $N^\times(D)$ be the complement to the zero section: $N_C^\times(D) = N_C(D) \setminus \{\text{zero section}\}$.

LEMMA 6.2.5. *If C is a stable singular curve with only one double point a, then $N_C D$ is one-dimensional, and can be canonically identified with $T_a^{(1)}C \otimes T_a^{(2)}C$, where $T_a^{(1)}C$, $T_a^{(2)}C$ are the tangent spaces to the two components of C at a.*

SKETCH OF PROOF. By definition, the space $N_C D$ is the set of equivalence classes of one-parameter families of curves C_U, defined over a disk U in the complex plane, such that $C_0 = C$ and C_q is non-singular for $q \neq 0$. A typical example of such a family is given by (6.2.5).

Let $a \in C_0 \subset C_U$ be the double point. Then it can be shown that one can always introduce local coordinates x_1, x_2 on Σ near a such that $x_1 x_2 = q$; when restricted to C_0, these coordinates become the local coordinates on the two components of C_0. Now, define the map $N_C D \to T_a^{(1)}C \otimes T_a^{(2)}C$ by $\partial_q \mapsto \partial_{x_1} \otimes \partial_{x_2}$. We leave it to the reader to check that this map does not depend on the choice of local coordinates t_1, t_2. □

?! Informally, the family C_q corresponding to the vector $v \in N_C^\times D$ can be presented as "thickening" of the double point, as shown in the figure below.

THERE WILL BE A FIGURE HERE

FIGURE 6.1. Family of smooth curves converging to a singular curve.

More generally, if $C \in \overline{M}_{g,n}$ is a curve with k double points $a_1, \dots, a_k$ (equivalently, C lies in the intersection of k components of D: $C \in D_1 \cap \dots \cap D_k$), then $N_C D = T_C\overline{M}_{g,n}/\cap T_C D_i$ is k-dimensional, and we have a canonical isomorphism

$$N_C D \simeq \bigoplus_{i=1}^{k} T_{a_i}^{(1)}C \otimes T_{a_i}^{(2)}C. \tag{6.2.6}$$

Using this lemma, we can now define the gluing functor for the complex Teichmüller groupoid. This is done in two steps.

First, let A be a finite set, $\alpha, \beta \in A$—an unordered pair. Then we define the "clutching" map

$$\begin{aligned} S_{\alpha\beta}\colon M_{*,A} &\to N(D^0), \quad D^0 \subset \overline{M}_{*,A\setminus\{\alpha,\beta\}}, \\ C^\vee &\mapsto (C, v), \end{aligned} \tag{6.2.7}$$

where $C \in D^0$ is the singular curve obtained by identifying the marked points α, β of $C^\vee$, and $v = v_\alpha \otimes v_\beta \in T_\alpha C^\vee \otimes T_\beta C^\vee \simeq N_C(D)$. The map S is a $\mathbb{C}^\times$-bundle over $N(D^0)$. We will also denote by $S_{\alpha,\beta}$ the corresponding functor between fundamental groupoids:

$$S_{\alpha,\beta}\colon \mathcal{T}eich_A^{\mathbb{C}} \to \pi_1(N^\times D),$$

where $\pi_1(X)$ denotes the fundamental groupoid of X.

The second step is to pass from $N^\times D \subset \overline{M}$ to M. Choose some tubular neighborhood N_ε of D in $N(D)$, and a C^∞ embedding

$$i\colon N_\varepsilon \to \overline{M} \tag{6.2.8}$$

such that i is identity on the normal bundles (note that the normal bundle to D in N_ε is canonically identified with $N(D)$). Such a map exists; moreover, it can be shown that the set of all such maps is contractible. Restricting this map to $N_\varepsilon^\times(D) = N_\varepsilon \setminus D$, we get a well-defined functor between the fundamental groupoids:

$$i\colon \pi_1(N_\varepsilon^\times(D)) \to \pi_1(M) = \mathcal{T}eich^{\mathbb{C}}.$$

Since the embedding $N_\varepsilon^\times(D) \to N^\times D$ is a homotopy equivalence, $\pi_1(N_\varepsilon^\times(D)) \simeq \pi_1(N^\times D)$. Thus, we can view i as a functor

$$i\colon \pi_1(N^\times D) \to \mathcal{T}eich^{\mathbb{C}}. \tag{6.2.9}$$

Now, let us define the gluing functor for the complex Teichmüller groupoid as the composition

$$F_{\alpha,\beta}\colon \mathcal{T}eich_A^{\mathbb{C}} \xrightarrow{S_{\alpha,\beta}} \pi_1(N^\times D) \xrightarrow{i} \mathcal{T}eich^{\mathbb{C}}_{A\setminus\{\alpha,\beta\}}. \tag{6.2.10}$$

Note that it is defined only for those curves $C \in M_{*,A}$ for which $S_{\alpha,\beta}(C)$ is stable.

EXAMPLE 6.2.6. Let us describe the gluing map for genus zero. Let $A' = \{\infty', 1', \dots, k', a\}$, $A'' = \{\infty'', 1'', \dots, m''\}$. Then the gluing map

$$F_{a,\infty''}\colon M_{0,A'} \times M_{0,A''} \to M_{0,B}$$

where $B = (A' \sqcup A'') \setminus \{a, \infty''\} = \{\infty', 1', \dots, k', 1'', \dots, m''\}$, can be described explicitly as follows. Choose for any $C^{(1)} \in M_{0,A'}$ a presentation in the form

$$C^{(1)} = (\mathbb{P}^1; \infty, z_1', \dots, z_k', a; v_\infty, v_1', \dots, v_k', t)$$

as in (6.2.4), where the tangent vector at ∞ is given by $v_\infty = -\partial_{1/z}$. (More formally, choose a section of the projection $X_{k+1} \to M_{0,A'}$, where $X_{k+1} = (\mathbb{C}^{k+1} \setminus \text{diagonals}) \times (\mathbb{C}^\times)^{k+1}$.) Do the same for $M_{0,A''}$. Then a simple generalization of the arguments of Example 6.2.4 shows that the gluing functor $F_{a,\infty''}$ is given by

$$\begin{aligned} C^{(1)} \sqcup C^{(2)} \mapsto (\mathbb{P}^1; &\infty, z_1', \dots, z_k', a + tz_1'', a + tz_m'';\\ &v_\infty', v_1', \dots, v_k', tv_1'', \dots, tv_m''). \end{aligned} \tag{6.2.11}$$

This map is well defined only for small enough t, and depends on the choice of presentation of $C^{(2)}$ in the form (6.2.4); however, the induced functor of fundamental groupoids is well defined up to a unique isomorphism.

The gluing operation satisfies the associativity property formulated in Definition 5.6.1, i.e., for distinct $\alpha, \beta, \gamma, \delta \in A$, the functors $F_{\alpha,\beta}F_{\gamma,\delta}, F_{\gamma,\delta}F_{\alpha,\beta}\colon \mathcal{T}eich_A^{\mathbb{C}} \to \mathcal{T}eich_{A'}^{\mathbb{C}}$, where $A' = A \setminus \{\alpha, \beta, \gamma, \delta\}$, are canonically isomorphic. The proof of this fact can be obtained from noting that each of them is isomorphic to the composition

$$\pi_1(M_{*,A}) \to \pi_1(N^\times(D^1)) \to \pi_1(M_{*,A'}) = \mathcal{T}eich_{A'}^{\mathbb{C}},$$

where D^1 is the strata of the boundary $D = \overline{M}_{*,A'} \setminus M_{*,A'}$ consisitng of curves with two double points, and the first arrow is given by identifying the points $\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta$ of C, thus producing a curve with two double points, and taking the normal vector to be $(v_\alpha \otimes v_\beta) \otimes (v_\gamma \otimes v_\delta)$ (see (6.2.6)). The second map is defined as in (6.2.9). The details are left to the reader.

It is easy to check that the gluing operation is also compatible with the disjoint union and empty set. Thus, the groupoid $\mathcal{T}eich^{\mathbb{C}}$ is a tower of groupoids in the sense of Definition 5.6.1. It is also easy to verify that the equivalence $\mathcal{T}eich^{stab} \to \mathcal{T}eich^{\mathbb{C}}$, constructed in Theorem 6.1.12, identifies this gluing operation with the gluing in $\mathcal{T}eich$. Thus, $\mathcal{T}eich^{stab} \xrightarrow{\sim} \mathcal{T}eich^{\mathbb{C}}$ as towers of groupoids.

The construction of gluing above requires that all the curves we use (including the singular ones) be stable; otherwise, the moduli spaces of curves are not stacks in the sense of [**DM**], which makes life much more difficult. In particular, we can not define the gluing $M_{0,1} \times M_{g,n} \to M_{g,n-1}$ because $M_{0,1}$ is not a DM-stack. Note, however, that in the topological approach the groupoid $\mathcal{T}eich_{0,1}$ is trivial (i.e., equivalent to the group with one element), and the operation of gluing $\mathcal{T}eich \times \mathcal{T}eich_{0,1}$ coincides with the operation of erasing a marked point (or patching a hole, depending on what definition of an extended surface was used). This operation is also well-defined as a functor $\mathcal{T}eich^{\mathbb{C}}_A \to \mathcal{T}eich^{\mathbb{C}}_{A\setminus\alpha}$ in the complex Teichmüller groupoid.

6.3. Connections with regular singularities

In this section, we briefly give the main definitions and results regarding flat connections with regular singularities. This will be used in the next section to define the modular functor in terms of connections on the moduli spaces of curves. Most of these results are due to Deligne and can be found in [**De1**] or in the review [**Ma**]. We assume that the reader is familiar with basic geometric notions such as vector bundles, sheaves, and $\mathcal{O}$-modules (as usual, we denote by $\mathcal{O}$ the structure sheaf, i.e., the sheaf of germs of analytic functions on M). As before, the word "manifold" stands for complex analytic manifold, and vector bundles are holomorphic vector bundles, etc. The notation $s \in \mathcal{F}$ means that s is a local section of the sheaf $\mathcal{F}$.

Let M be a manifold. By definition, a *local system* on M is a representation of the Poincaré groupoid of M. It is well-known that this is the same as a locally constant sheaf of vector spaces on M.

A convenient way of constructing local systems on a manifold is given by vector bundles with flat connections. Recall that a *connection* in a vector bundle E over M is a morphism of sheaves

$$\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1, \tag{6.3.1}$$

such that

$$\nabla(sf) = (\nabla s)f + s \otimes df, \qquad s \in \mathcal{E}, f \in \mathcal{O}_M,$$

where $\mathcal{E}$ is the sheaf of sections of E, and Ω^n is the sheaf of differential forms of degree n,

We can extend ∇ to a map from $\mathcal{E} \otimes \Omega^n$ to $\mathcal{E} \otimes \Omega^{n+1}$, $n = 0, 1, \dots$. The connection ∇ is called *flat* if the resulting $\mathcal{E} \otimes \Omega^\bullet$ is a complex, i.e., if $\nabla^2 = 0$.

For any vector field X on M, (6.3.1) gives a linear morphism

$$\nabla_X : \mathcal{E} \to \mathcal{E} \tag{6.3.2}$$

such that

$$\nabla_X(sf) = (\nabla_X s)f + s\,X(f).$$

Then ∇ is flat iff $X \mapsto \nabla_X$ is a Lie algebra homomorphism, i.e., $[\nabla_X, \nabla_Y] = \nabla_{X,Y}$. In local coordinates x_i, $X_i = \partial/\partial x_i$, $\nabla_i = \nabla_{X_i}$, this means $[\nabla_i, \nabla_j] = 0$. In other words, a flat connection is the same as an action of the sheaf Θ_M of vector fields on sections of E, compatible with action of $\mathcal{O}$.

We can say that a flat connection is a consistent system of partial differential equations. Any flat connection gives rise to a locally constant sheaf—the sheaf of solutions to this system of differential equations; its sections are $\{s \in \mathcal{E} \mid \nabla_X s = 0 \text{ for all } X\}$; usually they are referred to as "flat sections".

If M is a C^∞ real manifold, then it is known that the converse is also true: any local system can be obtained from a vector bundle with a flat connection. More formally, one can say that in this case the categories of local systems and of vector bundles with flat connections are equivalent. The same holds if M is an analytic complex manifold, and we consider holomorphic vector bundles with holomorphic flat connections.

Recalling the definition (6.2.10) of gluing for the complex Teichmüller groupoid, we see that in order to describe it in terms of flat connections we need somehow to extend our flat connections on $M_{g,n}$ to the boundary $D = \overline{M} \setminus M$. In the simplest example when M is one-dimensional and D is a point, it is well known that though one can not define a value of a flat section at D, one has a well-defined notion of asymptotics provided that our system of differential equations has regular singularities (see, e.g., [**CL**]). Thus, it is natural to expect that in order to define the gluing axiom, one has to introduce local systems with regular singularities.

Definition 6.3.1. Let $\overline{M}$ be a complex analytic manifold, $D \subset \overline{M}$ be a divisor with normal crossings, and $M = \overline{M} \setminus D$. Let E be a holomorphic vector bundle on $\overline{M}$ with a holomorphic flat connection defined over M. This flat connection is said to have *logarithmic singularities* at D (*log* D connection for short) if in a neighborhood of every $p \in D$ the bundle E admits a trivialization such that the connection has the form

$$\begin{aligned} \nabla_i &= \frac{\partial}{\partial z_i} + \frac{A_i(z)}{z_i}, \qquad && 1 \le i \le k, \\ \nabla_i &= \frac{\partial}{\partial z_i} + A_i(z), && k+1 \le i \le n, \end{aligned} \tag{6.3.3}$$

where z_i are local coordinates near p chosen so that the divisor D is given by the equation $z_1 \cdots z_k = 0$, and $A_i(z)$ are regular matrix-valued functions in a neighborhood of p.

The following lemma describes (locally) the local system of flat sections of a *log* D connection.

Lemma 6.3.2. *In the notation of Definition* 6.3.1*, let* U *be a small ball around* p*, and* $U^0 = U \setminus (U \cap D)$*. Then*:

(i) $\pi_1(U^0) = \mathbb{Z}^k$*, with the generators* $\gamma_i \colon z_i \mapsto e^{i\varphi} z_i$*,* $0 \le \varphi \le 2\pi$*,* $1 \le i \le k$*.*

(ii) *Let* $i \le k$*. Then the conjugacy class of the matrix* $A_i(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n)$ *does not depend on* $z_1, \dots, z_n$*. In particular, the eigenvalues* $\lambda_i^a, a = 1, \dots, \dim E$ *of* $A_i|_{z_i=0}$ *do not depend on* z_j*.*

(iii) *Let us assume that the connection* ∇ *satisfies the following non-integrality condition*:

$$\lambda_i^a - \lambda_{i'}^{a'} \notin (\mathbb{Z} \setminus \{0\}) \quad \text{for any } i, i' = 1, \dots, k, \quad a, a' = 1, \dots, \dim E. \tag{6.3.4}$$

(*Note that multiple eigenvalues are allowed.*) *Then the corresponding representation of* $\pi_1(U^0)$ *is given by* $\gamma_i \mapsto e^{-2\pi i A_i(z_i=0)}$*.*

The proof of this lemma is not too difficult and essentially follows from the one-dimensional case. We refer the reader to [**Ma**], [**De1**] for details.

Note that part (iii) of the lemma may fail if we do not impose the non-integrality condition.

EXAMPLE 6.3.3. Let $\overline{M} = \mathbb{C}, D = \{0\}$. Let E be the trivial 2-dimensional vector bundle over $\mathbb{C}$ with the connection given by $\nabla = d - \frac{A(z)}{z}dz$, with

$$A(z) = \begin{pmatrix} 0 & 0 \\ z & 1 \end{pmatrix}.$$

Show that this connection has nontrivial monodromy even though $e^{2\pi \mathrm{i} A(0)} = 1$.

One defines morphisms between bundles with connections in an obvious way. However, it is convenient also to introduce a more general notion of a morphism as follows. Let E, F be two holomorphic vector bundles over $\overline{M}$ with flat connections defined over M. Let $\mathcal{E}[D]$ be the sheaf of **meromorphic** sections of E which are holomorphic outside of D. Assume that the connections preserve $\mathcal{E}[D], \mathcal{F}[D]$ (this holds automatically for $log\, D$ connections). We define *meromorphic morphisms* between E and F to be the morphisms of sheaves $\mathcal{E}[D] \to \mathcal{F}[D]$ which commute with the connection (note that this is more general than the usual definition of a morphism between two vector bundles). We will say that (E, ∇_E) is *meromorphically equivalent* to (F, ∇_F) if there exists an invertible meromorphic morphism $E \to F$.

EXERCISE 6.3.4. Let $\overline{M} = \mathbb{C}, D = \{0\}$, and let $\nabla^s = d + s\frac{dz}{z}$ be the connection in the trivial one-dimensional vector bundle. Show that ∇^s is meromorphically equivalent to ∇^t iff $s - t \in \mathbb{Z}$.

Let E be a vector bundle on $\overline{M}$ with a flat connection ∇ defined on M.

DEFINITION 6.3.5. The connection ∇ has *regular singularities at D* if (E, ∇) is meromorphically equivalent to a bundle with a $log\, D$ connection (see Definition 6.3.1).

EXERCISE 6.3.6. Show that if $\dim M = 1$, then ∇ has regular singularities iff ∇ is a $log\, D$ connection. (For $\dim M > 1$, this is not true.)

For brevity, we will refer to the pair (E, ∇) in the definition as a "connection on $\overline{M}$ with regular singularities at D". The category of such connections with respect to meromorphic morphisms will be denoted by $\mathcal{RS}(\overline{M}, M)$ (or just $\mathcal{RS}(M)$ when there is no ambiguity). Note that meromorphic morphisms do not change monodromy, and thus we have a well-defined functor $\mathcal{RS}(\overline{M}, M) \to \{\text{local systems on } M\}$.

REMARK 6.3.7. If we considered algebraic theory rather than analytic one, then any vector bundle on M admits a unique meromorphic continuation to $\overline{M}$, so the category of sheaves on $\overline{M}$ up to meromorphic equivalence is the same as the category of sheaves on M. Moreover, in this case it was proved by Deligne that the notion of connection with regular singularities on M can be defined purely in terms of M, without using $\overline{M}$ at all. In analytic situation, it is not true.

We quote here without proofs several important results of Deligne about connections with regular singularities. Proofs and details can be found in [**De1**] or in [**Ma**].

THEOREM 6.3.8. *Let $(E, \nabla) \in \mathcal{RS}(\overline{M}, M)$. For every $z \in \mathbb{C}/\mathbb{Z}$, choose a representative $\tau(z) \in \mathbb{C}$ (τ needs not to be continuous). Then there is a unique vector bundle $\tilde{E}$ with a flat $\log D$ connection $\tilde{\nabla}$ such that $(\tilde{E}, \tilde{\nabla})$ is meromorphically equivalent to (E, ∇), and all eigenvalues $\tilde{\lambda}_i^a$ (see Lemma 6.3.2) lie in the image of τ.*

COROLLARY 6.3.9. *Every flat connection with regular singularities is meromorphically equivalent to a $\log D$ connection which satisfies the non-integrality property (6.3.4).*

THEOREM 6.3.10. *In the notation of Definition 6.3.1, let D^0 be the smooth part of the divisor D (cf. (6.2.1)). Let E be a vector bundle on $\overline{M}$, and ∇ be a flat connection with regular singularities at D^0. Then ∇ has regular singularities at D.*

In other words, it suffices to check the regularity condition only for the open part of D. (Note: the proof of this theorem in [**De1**] contains a mistake, which Deligne later corrected.)

THEOREM 6.3.11. *In the notation of Definition 6.3.1, any holomorphic vector bundle on M with a flat connection can be extended to a vector bundle on $\overline{M}$ with a connection which has regular singularities at D. This extension is unique up to a meromorphic isomorphism.*

COROLLARY 6.3.12 (The Riemann–Hilbert correspondence). *The natural functor*

$$\mathcal{RS}(\overline{M}, M) \to \text{local systems on } M$$

is an equivalence.

In practical applications, it is convenient to use the following criterion of regularity, which is easy to prove.

LEMMA 6.3.13. *Let E, ∇ be as in Definition 6.3.5. Then ∇ has regular singularities iff for every holomorphic map $u \colon U \to \overline{M}$, where U is a disk, and $u^{-1}(D) = \{0\}$, the induced connection $u^*\nabla$ on U has regular singularities at 0.*

In fact, due to Theorem 6.3.10, it suffices to check this condition for $u(0) \in D^0$.

It will be convenient to rewrite the notion of flat connection with regular singularities in terms of $\mathcal{D}$-modules. As was noted before, a connection ∇ in a vector bundle E is flat if it defines for every open $U \subset M$ an action of the Lie algebra $\Theta(U)$ of vector fields on U on the space of sections $\mathcal{E}(U)$, which is compatible with multiplication by functions. Such an action is the same as an action on $\mathcal{E}$ of the sheaf of associative algebras $\mathcal{D}$ of differential operators on M. Thus, if E is a vector bundle with a flat connection, then the sheaf $\mathcal{E}$ of its sections is naturally a module over the sheaf $\mathcal{D}$ of differential operators, or a *$\mathcal{D}$-module* for short. Conversely, it is easy to see that if $\mathcal{E}$ is a $\mathcal{D}$-module which is locally free of finite rank as $\mathcal{O}$-module, then $\mathcal{E}$ is a the sheaf of sections of some vector bundle with a flat connection. It is known (but not easy) that it suffices to require that $\mathcal{E}$ be *coherent*, i.e., locally finitely generated over $\mathcal{O}$—for $\mathcal{D}$-modules, this automatically implies that $\mathcal{E}$ is locally free of finite rank. we refer the reader to [**Ber**], [**Bor**], [**Bjo**] for the proof of this and other facts about $\mathcal{D}$-modules.

In a similar way, it is easy to show that a connection with logarithmic singularities at D is the same as a sheaf of modules over the sheaf

$$\mathcal{D}^0_{\overline{M}} = \{\partial \in \mathcal{D}_{\overline{M}} \mid \partial I \subset I\} \tag{6.3.5}$$

where I is the sheaf of functions on $\overline{M}$ vanishing at D. The sheaf $\mathcal{D}^0_{\overline{M}}$ is generated as a sheaf of algebras by $\mathcal{O}$ and vector fields tangent to D. For example, for $\overline{M} = \mathbb{C}, D = \{0\}$, the sheaf $\mathcal{D}^0_{\overline{M}}$ is generated by $\mathcal{O}_{\mathbb{C}}$ and the vector field $q\partial_q$.

Now, let us show how connections with regular singularities allow us to pass to the boundary of the moduli space. Before doing so, we need to introduce one more notion.

DEFINITION 6.3.14. Let X be a $\mathbb{C}^\times$ bundle over M. A monodromic flat connection on X is a pair (E, ∇), where E is a $\mathbb{C}^\times$-equivariant vector bundle on X, and ∇ a connection which commutes with the action of $\mathbb{C}^\times$.

In local coordinates, such a connection can always be written as $\nabla = d + \sum A_i(x)dx_i + A(x)\frac{du}{u}$, where x_i are coordinates in M, and u is coordinate along the fibers of X.

LEMMA 6.3.15. (i) *Let $D \subset \overline{M}$ be a smooth divisor. Then there exists a natural specialization functor*

$$Sp_D \colon \mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(ND, N^\times D)$$

such that $Sp_D(\nabla)$ is monodromic along the fibers of the projection $ND \to D$ and has the same monodromy around D as ∇.

(ii) *Let D be a divisor with normal crossings: $D = \cup D_i$. Fix one of the components D_i and let $D_i^0 = D_i \setminus (\cup_{j\neq i} D_j)$. Then we have a natural specialization functor*

$$Sp_{D_i} \colon \mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(ND_i, N^\times D_i^0)$$

with the same properties as above.

PROOF. (i) The easiest way to define this functor is to use the terminology of $\mathcal{D}$-modules. By Corollary 6.3.9, we can assume that ∇ has logarithmic singularities and satisfies the non-integrality property (6.3.4). First of all, note that one can describe the structure sheaf of $\mathcal{O}_{ND}$ in terms of the restriction of the structure sheaf $\mathcal{O}_M$ to D. Namely, the latter sheaf is naturally filtered by the powers of the ideal I: $\mathcal{O}_M = I^0 \supset I \supset I^2 \supset \dots$. We claim that $\mathcal{O}_{ND} = \widehat{\bigoplus}_{n\geq 0} I^n/I^{n+1}$ is the completion of the associated graded algebra (we need completion to get all analytic functions, not just polynomial). Similarly, the sheaf of differential operators on ND which preserve I is nothing but the (completion of) associated graded sheaf for $\overline{M}$: $\mathcal{D}^0_{ND} = \widehat{\bigoplus}_{n\geq 0} I^n \mathcal{D}^0_{\overline{M}}/I^{n+1}\mathcal{D}^0_{\overline{M}}$

As was mentioned above, a flat connection with first order poles at D is the same as a $\mathcal{O}_M$-coherent $\mathcal{D}^0_{\overline{M}}$-module $\mathcal{E}$. Such a module is also naturally filtered, and action of $\mathcal{D}^0_{\overline{M}}$ preserves this filtration. Now define

$$Sp_D(\mathcal{E}) = \widehat{\bigoplus_{n\geq 0}} I^n\mathcal{E}/(I^{n+1}\mathcal{E}).$$

This is naturally a $\mathcal{D}^0_{ND}$ module, and thus a sheaf of sections of a vector bundle on ND with a flat connection which has first order poles at D.

Here is a more explicit construction. Choose coordinates $z_1, \dots, z_n$ in a neighborhood of the point $p \in D$ such that D is given by the equation $z_1 = 0$. This also gives coordinates $t, z_2, \dots, z_n$ in ND, where $t(a, v) = \langle v, dz_1 \rangle, a \in D, v \in T_a M$.

Choose a trivialization of E near p; then ∇ is given by (6.3.3), with $k = 1$. Define the connection $Sp\nabla = Sp_D(\nabla)$ in $N^\times D$ by

$$(Sp\nabla)_t = \partial_t + A_1(0, z_2, \dots)/t,$$
$$(Sp\nabla)_{z_i} = \partial_{z_i} + A_i(0, z_2, \dots), \quad i = 2, \dots, n.$$

One easily sees that this connection is flat, invariant with respect to the action of $\mathbb{C}^\times$ by dilations on ND, and does not depend on the choice of coordinates up to a unique isomorphism. By Lemma 6.3.2, $Sp\nabla$ has the same monodromy as ∇. This definition uses a choice of a local coordinate system and an extension of the vector bundle to D. However, it can be shown that this construction coincides with the previous one (described by passing from filtered modules to associated graded ones) and thus $Sp\nabla$ does not depend on these choices.

In terms of the corresponding local systems, the specialization functor is defined as $Sp_D = i^*$, where i is is an identification of ND with a neighborhood of D in $\overline{M}$, as in (6.2.8).

Equivalently, the same flat connection can be defined by specifying its flat sections. Let f be a flat section of the original local system on M, i.e., $f(z)$ is a solution of the system $\nabla_i f = 0$. Let us restrict this solution to the curve $z(t) = p+tv, t \in \mathbb{R}_{>0}, (p, v) \in N^\times(D)$. By the classical theory of ODE's with regular singularities (see, e.g., [**CL**]), there exists a vector $g(p, v)$ such that $f(p + tv) = F(t)g(p, v)$, where $F(t)$ is the fundamental (matrix) solution; usually, g is called asymptotics of f along this curve. Then $g(p, v)$ is a flat section of the connection $Sp_D(\nabla)$.

To prove (ii), we need to check that $Sp_{D_i}(\nabla)$ has regular singularities at $D_i \cap D_j$, which can be done explicitly. □

REMARKS 6.3.16. (i) The specialization functor can be easily described in terms of the functor of nearby cycles for $\mathcal{D}$-modules (see [**KasS**].

(ii) Note that the specialization functor is defined even if the eigenvalues of $A_i(z_i = 0)$ differ by a non-zero integer. However, in this case this functor is not so easy to describe: one first needs to replace the flat connection by a meromorphically equivalent one which satisfies the non-integrality condition. Such a connection exists by Corollary 6.3.9, but explicitly constructing it can be difficult.

Finally, it is natural to consider the following question. Suppose we have a divisor with normal crossings, and D_1, D_2 are two of the components. Is it true that Sp_{D_1} and Sp_{D_2} commute?

In order for this question to make sense, one must first define the composition $Sp_{D_1} Sp_{D_2}$. For simplicity, let us assume that $\dim \overline{M} = 2$, $D_1 \cap D_2 = \{p\}$. Let $N_1 = N(D_1)$; it contains the divisor $D_{12} = N_p(D_1)$. Let N_{12} be the normal bundle to D_{12} inside N_1.

LEMMA 6.3.17. (i) *There exists a canonical homeomorphism $N_{12} \simeq T_p$, where for brevity we denoted $T_p = T_p\overline{M}$. Thus, one can define the specialization functor $Sp_{12} : \mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(T_p, T_p^\times)$, where $T_p^\times = T_p \setminus (T_pD_1 \cup T_pD_2)$, as the composition*

$$\mathcal{RS}(\overline{M}, M) \to \mathcal{RS}(N_1, N_1^\times) \to \mathcal{RS}(N_{12}, N_{12}^\times) \simeq \mathcal{RS}(T_p, T_p^\times).$$

(ii) *The functors Sp_{12}, Sp_{21}, defined as in part* (i), *are canonically isomorphic.*

The proof of this lemma is not difficult and is left as an exercise.

6.4. Complex analytic modular functor

In this section, we give a definition of modular functor in terms of flat connections on the moduli spaces.

Let $\mathcal{C}$ be a semisimple abelian category over $\mathbb{C}$, and $R \in \text{ind}-\mathcal{C}^{\boxtimes 2}$ be a symmetric object, as in Section 2.4. Recalling the definition of a $\mathcal{C}$-extended modular functor (Definition 5.1.12) and the results of the previous section, we can rewrite the definition of modular functor as follows.

DEFINITION 6.4.1. A *complex $\mathcal{C}$-extended modular functor* is the following collection of data:

(i) For every finite set A and $W \in \mathcal{C}^{\boxtimes A}$, a finite-dimensional vector bundle over $\overline{M}_{*,A}$ with a flat connection with regular singularities. This bundle is called the *bundle of conformal blocks*; its fiber at a point $C \in \overline{M}_{*,A}$ will be denoted by $\langle W\rangle_C$.

(ii) Isomorphisms $\langle X\rangle_{C'} \otimes \langle Y\rangle_{C''} \simeq \langle X \boxtimes Y\rangle_{C' \sqcup C''}$.

(iii) **Gluing isomorphisms.** Let A be a finite set, $\alpha, \beta \in A$—an unordered pair, $A' = A \setminus \{\alpha, \beta\}, W \in \mathcal{C}^{\boxtimes A'}$. For every such collection, we require an isomorphism of vector bundles with connections on $M_{*,A}$:

$$G_{\alpha,\beta}\colon \langle W \boxtimes R\rangle \xrightarrow{\sim} S^*_{\alpha,\beta} Sp_D \langle W\rangle, \tag{6.4.1}$$

where $S_{\alpha,\beta}\colon M_{*,A} \to N(D^0), D^0 \subset M_{*,A'}$, is the "clutching" (6.2.7), and R is placed at positions with indices α, β. (Since $S_{\alpha,\beta}$ is a $\mathbb{C}^\times$-bundle, the definition of $S^*_{\alpha,\beta}$ causes no problems.)

(iv) **Vacuum propagation.** We have a distinguished element $\mathbf{1} \in \operatorname{Ob}\mathcal{C}$, and for every $\alpha \in A$, we require an isomorphism of vector bundles with connections on $M_{*,A}$:

$$G_\alpha\colon \langle W \boxtimes \mathbf{1}\rangle \xrightarrow{\sim} S^*_\alpha \langle W\rangle, \tag{6.4.2}$$

where $S_\alpha\colon M_{*,A} \to M_{*,A\setminus\alpha}$ is the operator of erasing the point α.

These data have to satisfy the following properties:

- **Functoriality:** $W \mapsto \langle W\rangle_C$ is functorial in W, and the isomorphisms (ii)-(iv) are functorial isomorphisms.
- **Equivariance:** $W \mapsto \langle W\rangle_C$ is equivariant with respect to the action of the symmetric group S_A.
- **Compatibility:** the isomorphisms (ii)-(iv) are compatible with each other and with the commutativity, associativity, and unit morphisms in $\mathcal{V}ec_f$ (cf. Definition 4.2.1).
- **Normalization:** $\langle \mathbf{1}, \mathbf{1}\rangle_{\mathbb{P}^1} = \mathbb{C}$.

We can now formulate the main result of this section.

THEOREM 6.4.2. *The notions of a (topological) $\mathcal{C}$-extended modular functor and a complex $\mathcal{C}$-extended modular functor are equivalent.*

PROOF. By Lemma 5.6.13, a modular functor is the same as a functor $\mathcal{T}eich \to \mathcal{F}un(\mathcal{C})$. We leave it to the reader to check that the same definition can be rewritten in terms of the groupoid $\mathcal{T}eich^{stab}$ (that is, without using surfaces of type $(g,n) = (0,0),(0,1),(1,0)$) if, in addition to the gluing of such surfaces, we also add an operation of erasing a marked point. This operation makes up for the operation of gluing a sphere with one hole, i.e., a disk, in $\mathcal{T}eich$.

Using the equivalence $\mathcal{T}eich^{stab} \simeq \mathcal{T}eich^{\mathbb{C}}$, we see that modular functor is the same as a functor $\mathcal{T}eich^{\mathbb{C}} \to \mathcal{F}un(\mathcal{C})$. Thus, for every fixed $W \in \mathcal{C}^{\boxtimes A}$, we get a local system on $M_{*,A}$. By the results of the previous section, every local system can be presented by a unique up to meromorphic equivalence holomorphic vector bundle on $\overline{M}_{*,A}$ with a flat connection on $M_{*,A}$ which has regular singularities at $D = \overline{M}_{*,A} \setminus M_{*,A}$. Then all the properties, except for the gluing axiom, are obvious reformulations of the axioms of modular functor.

To show that the definition of gluing given above coincides with the one given for topological modular functor, recall that the gluing functor $F_{\alpha,\beta}\colon \mathcal{T}eich^{\mathbb{C}}_A \to \mathcal{T}eich^{\mathbb{C}}_{A'}$ was defined using the composition

$$M_{*,A} \xrightarrow{S_{\alpha,\beta}} N^{\times}(D) \to M_{*,A'},$$

see (6.2.10). In the language of local systems, the gluing isomorphism should identify the vector spaces $\langle W \boxtimes R\rangle_C \simeq \langle W\rangle_{C'}$ in such a way that it agrees with morphisms in $\mathcal{T}eich_A$. This is equivalent to saying that it must be an isomorphism of local system on $M_{*,A}$, $\langle W \boxtimes R\rangle \simeq F^*_{\alpha,\beta}\langle W\rangle$. Replacing local systems by connections with regular singularities, we note that the identification $i\colon N^{\times}(D) \to M_{*,A'}$ was defined so that i^* is exactly the specialization functor. This leads to the definition of the gluing isomorphism given above.

We leave ti to the reader to check the equivalence of normalization axioms in topological and complex-analytic settings. □

REMARK 6.4.3. Let us restrict the gluing functor to

$$S_{\alpha,\beta}\colon M_{g_1,A} \times M_{g_2,B} \to N^{\times}(D^0), \quad D \subset M_{g_1+g_2,C},$$

where $\alpha \in A, \beta \in B$, and $C = (A \sqcup B) \setminus \{\alpha, \beta\}$. Then the gluing axiom says that the bundle of conformal blocks $\langle W_A \boxtimes W_B\rangle$ on $M_{g_1+g_2,C}$ factors into tensor product of bundles $\langle W_A, R^{(1)}\rangle$ on $M_{g_1,A}$ and $\langle R^{(2)}, W_B\rangle$ on $M_{g_2,B}$ as we approach the corresponding component of the boundary in $\overline{M}_{g_1+g_2,C}$.[1] This is known as the *factorization property* of the bundle of conformal blocks and was first introduced in [**FS**].

COROLLARY 6.4.4. *Every modular tensor category $\mathcal{C}$ over $\mathbb{C}$ with $p^+/p^- = 1$ gives rise to a complex $\mathcal{C}$-extended modular functor such that*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) = \langle W_1, \dots, W_n\rangle_C$$

where C is the standard n-punctured sphere

$$C = (\mathbb{P}^1; z_1, \dots, z_n; v_1, \dots, v_n) \tag{6.4.3}$$

with $z_1 < \cdots < z_n \in \mathbb{R}, v_i > 0$. The Dehn twist θ_{W_i} corresponds to the monodromy around the loop $v_i \mapsto e^{i\varphi}v_i, 0 \le \varphi \le 2\pi$, and the braiding $\sigma_{W_i,W_{i+1}}$ corresponds to the holonomy around the path b_i shown in Figure 6.2

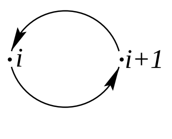


FIGURE 6.2. Braiding for the complex modular functor.

[1] We are using the same notation as in Definition 5.1.12.

Conversely, if $\mathcal{C}$ is a semisimple abelian category over $\mathbb{C}$, then every complex $\mathcal{C}$-extended modular functor gives rise to a structure of weakly ribbon category on $\mathcal{C}$ such that the above properties hold; if this category is rigid, then it is also modular with $p^+/p^- = 1$.

This corollary is nothing but the reformulation of Theorem 5.5.1 in the language of complex modular functor.

REMARK 6.4.5. Equivalently, one can describe the relation between conformal blocks and the spaces of homomorphisms in $\mathcal{C}$ as follows:

$$\mathrm{Hom}_{\mathcal{C}}(W_\infty^*, W_1 \otimes \cdots \otimes W_n) = \langle W_\infty, W_1, \dots, W_n \rangle_C$$

where $C = (\mathbb{P}^1; \infty, z_1, \dots, z_n; v_\infty, v_1, \dots, v_n)$, with $z_1 < \cdots < z_n, v_1 > 0, \dots, v_n > 0$ and the tangent vector at ∞ given by $v_\infty = -\partial_{1/z}$. Indeed, this curve can be reduced to the $(n+1)$-punctured standard sphere (6.4.3) by the change of variables $z \mapsto -1/z$.

6.5. Example: Drinfeld's category

In this section, we study one example of modular functor in genus zero, associated with a simple Lie algebra $\mathfrak{g}$. This modular functor is defined in terms of the Knizhnik–Zamolodchikov equations; the corresponding tensor category is the Drinfeld's category $\mathcal{D}$ defined in Chapter 1.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and let $\mathcal{C} = \mathcal{R}ep_f\mathfrak{g}$ be the category of finite dimensional $\mathfrak{g}$-modules. Let $R = \bigoplus_{\lambda \in P_+} V_\lambda \boxtimes V_\lambda^*$, where V_λ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ, and $*$ is the usual duality for $\mathcal{R}ep_f\mathfrak{g}$. Fix $\varkappa \in \mathbb{C} \setminus \mathbb{Q}$. We will construct a (complex) modular functor in genus 0 for $\mathcal{C}$.

First, note that the moduli space $M_{0,n}$ of pointed curves of genus zero is given by $M_{0,n} = X_n/\mathrm{PSL}_2(\mathbb{C})$, where

$$X_n = \{z_1, \dots, z_n \in \mathbb{P}^1, v_i \in T_{z_i}^\times \mathbb{P}^1 \mid z_i \neq z_j\}. \tag{6.5.1}$$

This, in particular, implies that $M_{0,n}$ is smooth for $n > 0$; for $n = 0$, $M_{0,0} = \{pt\}$ and this case will not be considered. We will start by constructing the bundle of conformal blocks on the open part

$$X_n^0 = \{(\mathbf{z}, \mathbf{v}) \in X_n \mid z_i \neq \infty\} = \{z_1, \dots, z_n \in \mathbb{C} \mid z_i \neq z_j\} \times (\mathbb{C}^\times)^n. \tag{6.5.2}$$

For $(\mathbf{z}, \mathbf{v}) \in X_n^0$, the corresponding curve is $\mathbb{P}^1$ with marked points $z_1, \dots, z_n$ and tangent vectors v_i (since $T_z\mathbb{C} = \mathbb{C}$).

Let $W_1, \dots, W_n$ be the representations of $\mathfrak{g}$ assigned to these points. Define the bundle of conformal blocks to be the trivial vector bundle over X_n^0 with fiber $(W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}}$ and with the *Knizhnik–Zamolodchikov connection* (cf. (KZ_n)):

$$\begin{aligned} \nabla_{z_j} &= \frac{\partial}{\partial z_j} - \frac{1}{\varkappa} \sum_{\substack{1 \le k \le n \\ k \neq j}} \frac{\Omega_{jk}}{z_j - z_k}, \qquad 1 \le j \le n, \\ \nabla_{v_j} &= \frac{\partial}{\partial v_j} - \frac{1}{2\varkappa} \frac{D_j}{v_j}. \end{aligned} \tag{6.5.3}$$

Here D is the Casimir element of $\mathfrak{g}$ defined by (1.4.4); all other notation is as in (KZ_n). This connection, which is originally defined in the bundle with fiber

$W_1 \otimes \cdots \otimes W_n$, commutes with the action of $\mathfrak{g}$ and therefore can be restricted to the sub-bundle of invariants $(W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}} \subset W_1 \otimes \cdots \otimes W_n$.

LEMMA 6.5.1. (i) *The KZ connection* (6.5.3) *on X_n^0 is flat and S_n-equivariant.*
(ii) *This connection can be uniquely extended to a* $\mathrm{PSL}_2(\mathbb{C})$*-invariant connection on X_n.*

PROOF. The first statement easily follows from the flatness of the usual KZ connection—see Lemma 1.4.7. The second can be checked explicitly; it suffices to check that under a change of variables $w = (az+b)/(cz+d)$, the KZ equations in terms of w have the same form as in terms of z. Details can be found, for example, in [**EFK**]. Note that the second statement fails for the KZ connection in the form of Chapter 1: to ensure projective invariance, one needs to add the equation in v_i and restrict to $\mathfrak{g}$-invariants. $\square$

Thus, the connection (6.5.3) defines a flat connection on the moduli space $M_{0,n}$, which will also be called the KZ connection.

REMARK 6.5.2. It is more convenient to describe the same connection in a slightly different way. Namely, the map

$$(z_1, \dots, z_n; v_1, \dots, v_n) \mapsto (\mathbb{P}^1; \infty, z_1, \dots, z_n; v_\infty, v_1, \dots, v_n),$$

where, as before, $v_\infty = -\partial_{1/z}$, gives an identification $M_{0,n+1} = X_n^0/\mathbb{C}$, where $\mathbb{C}$ acts on X_n^0 by $z_i \mapsto z_i + a$. Let us fix $W_\infty, W_1, \dots, W_n$ and consider the connection (6.5.3) in $W_1 \otimes \cdots \otimes W_n$. This connection induces a connection in $(W_\infty \otimes W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}}$, which is obviously translation invariant and thus defines a connection on $M_{0,n+1}$. One easily checks that this connection coincides with the KZ connection defined above.

THEOREM 6.5.3. *The KZ connection, considered as a connection on the trivial vector bundle with fiber $(W_1 \otimes \cdots \otimes W_n)^{\mathfrak{g}}$ over the compactification $\overline{M}_{0,n}$, has first order poles and satisfies the gluing axiom, with $R = \bigoplus V_\lambda \boxtimes V_\lambda^*$ and the gluing isomorphism given by*

(6.5.4)
$$\bigoplus_\lambda (W_1 \otimes \ldots \otimes W_k \otimes V_\lambda)^{\mathfrak{g}} \otimes (V_\lambda^* \otimes W_{k+1} \otimes \ldots \otimes W_n)^{\mathfrak{g}} \xrightarrow{\sim} (W_1 \otimes \ldots \otimes W_n)^{\mathfrak{g}}$$
$$(w_1 \otimes \ldots \otimes w_k \otimes v) \otimes (v^* \otimes w_{k+1} \otimes \ldots \otimes w_n) \mapsto (v, v^*) w_1 \otimes \ldots \otimes w_n.$$

PROOF. By Theorem 6.3.10, it suffices to check the regularity condition for the open strata of $D = \overline{M}_{0,n} \setminus M_{0,n}$, i.e., for the curves with one double point. Thus, it suffices to check the regularity and the gluing axiom for the gluing functor $F_{a,\infty''}\colon M_{0,k+2} \times M_{0,m+1} \to M_{0,k+m+1}$ considered in Example 6.2.6. Using the notation from that example, we represent $M_{0,k+2} = X_{k+1}^0/\mathbb{C}$, $M_{0,m+1} = X_m^0/\mathbb{C}$ as in Remark 6.5.2. Under this identification, the gluing functor is given by

$$\begin{aligned} &F_{a,\infty''}\colon X_{k+1}^0 \times X_m^0 \to X_{k+m}^0, \\ &(z', a; v', t) \times (z''; v'') \mapsto (z_1', \dots, z_k', a + t z_1'', \dots, a + t z_m''; \\ &\qquad\qquad v_1', \dots, v_k', t v_1'', t v_1'', \dots, t v_m''). \end{aligned} \tag{6.5.5}$$

By definition, the KZ connection on $M_{0,A'}$, $M_{0,A''}$ is given by (6.5.3) in the variables $z_1', \dots, z_k'$, $a = z_{k+1}'$ and $z_1'', \dots, z_m''$, respectively, while on $M_{0,B}$ it is given by (6.5.3) in the variables $z_1, \dots, z_{k+m}$.

Then to prove the theorem it suffices to check that

$$\begin{aligned}Sp_{t=0}F^*_{a,\infty''}\langle W'_\infty, W'_1, \dots, W'_k, W''_1, \dots, W''_m\rangle \\ = \bigoplus_i \langle W'_\infty, W'_1, \dots, W'_k, V_i\rangle \otimes \langle V_i^*, W''_1, \dots, W''_m\rangle\end{aligned}$$

as vector bundles with connections. To obtain the left hand side, we need to substitute in (6.5.3)

$$\begin{aligned} z_i &= z'_i, & v_i &= v'_i, & i &\le k;\\ z_{k+i} &= a + t z''_i, & v_{k+i} &= t v'_i, & i &\le m, \end{aligned}$$

and then specialize to $t = 0$.

Explicit calculation shows that this substitution gives the following connection in terms of the variables z', z'', a, v', v'', t:

$$\begin{aligned}
\nabla_{z'_i} &= \frac{\partial}{\partial z'_i} - \frac{1}{\varkappa}\left(\sum_{j\neq i}\frac{\Omega_{ij}}{z'_i - z'_j} + \sum_q \frac{\Omega_{ik+q}}{z'_i - (a + t z''_q)}\right),\\
\nabla_{v'_i} &= \frac{\partial}{\partial v'_i} - \frac{1}{2\varkappa}\frac{D_i}{v'_i},\\
\nabla_{z''_p} &= \frac{\partial}{\partial z''_p} - \frac{t}{\varkappa}\left(\sum_j \frac{\Omega_{jk+p}}{(a + t z''_p) - z'_j} + \sum_{q\neq p}\frac{\Omega_{k+pk+q}}{t(z''_p - z''_q)}\right),\\
\nabla_{v''_p} &= \frac{\partial}{\partial v''_p} - \frac{1}{2\varkappa}\frac{D_{k+p}}{v''_p},\\
\nabla_a &= \frac{\partial}{\partial a} - \frac{1}{\varkappa}\sum_j\sum_q \frac{\Omega_{jk+q}}{(a + t z''_q) - z'_j},\\
\nabla_t &= \frac{\partial}{\partial t} - \frac{1}{\varkappa}\sum_p\left(\sum_i \frac{z''_p\Omega_{ik+p}}{(a + t z''_p) - z'_i} + \sum_{q\neq p}\frac{z''_p\Omega_{k+pk+q}}{t(z''_p - z''_q)} + \frac{D_{k+p}}{2t}\right),
\end{aligned}$$

where the indices $i, j = 1, \dots, k$ and $p, q = 1, \dots, m$.

It is obvious that this is a connection with regular singularities at $t = 0$. Let us specialize it to $t = 0$ as described in Lemma 6.3.15; we will check the non-integrality condition later. This gives:

$$\nabla_{z'_i} = \frac{\partial}{\partial z'_i} - \frac{1}{\varkappa}\left(\sum_{j\neq i}\frac{\Omega_{ij}}{z'_i - z'_j} + \frac{\Omega_{i,(2)}}{z'_i - a}\right),$$
$$\nabla_{v'_i} = \frac{\partial}{\partial v'_i} - \frac{1}{2\varkappa}\frac{D_i}{v'_i},$$
$$\nabla_{z''_p} = \frac{\partial}{\partial z''_p} - \frac{1}{\varkappa}\left(\sum_{q\neq p}\frac{\Omega_{k+pk+q}}{z''_p - z''_q}\right),$$
$$\nabla_{v''_p} = \frac{\partial}{\partial v''_p} - \frac{1}{2\varkappa}\frac{D_{k+p}}{v''_p},$$
$$\nabla_a = \frac{\partial}{\partial a} - \frac{1}{\varkappa}\sum_j \frac{\Omega_{j,(2)}}{a - z'_j},$$
$$\nabla_t = \frac{\partial}{\partial t} - \frac{1}{2\varkappa t}\sum_{p,q}\Omega_{k+p,k+q},$$

where we denoted $\Omega_{i,(2)} = \sum_p \Omega_{i,k+p} = \Omega_{W'_i, W''_1\otimes\cdots\otimes W''_m}$.

Let us identify

$$\begin{aligned}(W'_\infty \otimes W'_1 \otimes \cdots \otimes W'_k \otimes W''_1 \otimes \cdots \otimes W''_m)^{\mathfrak{g}} \\ = \bigoplus_\lambda (W'_\infty \otimes W'_1 \otimes \cdots \otimes W'_k \otimes V_\lambda)^{\mathfrak{g}} \otimes (V^*_\lambda \otimes W''_1 \otimes \cdots \otimes W''_m)^{\mathfrak{g}}.\end{aligned}$$

Under this identification, one has:

$$\Omega_{j,(2)} = \Omega_{W'_j, V_\lambda},$$
$$\sum_{p,q}\Omega_{k+p,k+q} = D_{V^*_\lambda} = D_{V_\lambda}.$$

Thus, we see that the specialization exactly coincides with the product of KZ connections on $X^0_{k+1} \times X^0_m$.

Finally, in order to justify that our calculation of the specialization functor is valid, we have to check that our connection satisfies the non-integrality property (6.3.4). This follows from the fact that in our case, the operator $A(t = 0)$ is given by $\frac{1}{2\varkappa}D_{V_i}$. Thus, its eigenvalues are of the form $\langle\lambda, \lambda + 2\rho\rangle/2\varkappa$. Since $\langle\lambda, \lambda + 2\rho\rangle \in \mathbb{Q}$, and $\varkappa \notin \mathbb{Q}$, these eigenvalues can not differ by a non-zero integer (this is the only place where we use the condition $\varkappa \notin \mathbb{Q}$!).

This completes the proof of the gluing axiom. □

Therefore, we see that the KZ connection given above does define a genus zero complex modular functor. Thus, it defines a structure of a weakly rigid tensor category on $\mathcal{R}ep_f\mathfrak{g}$.

PROPOSITION 6.5.4. *The weakly rigid tensor structure on $\mathcal{R}ep_f(\mathfrak{g})$ defined as above coincides with the Drinfeld category structure defined in Section* 1.4.

The proof of this proposition immediately follows from the definition of Drinfeld's category. This, in particular, shows that this category is rigid.

The reader might notice that the proof of the gluing axiom given above is essentially the same proof we used in Chapter 1 to prove the associativity axiom for the Drinfeld's category—only now we have the language of connections with regular singularities and specialization functors in our disposition, which allows us to make all the arguments absolutely rigorous.

6.6. Twisted $\mathcal{D}$-modules

So far, we have only discussed modular functors in which the bundle of confor-mqal blocks carries a natural flat connection ("modular functors with zero central charge"). Comparing this with the discussion of Section 5.7, we see that these modular functors correspond to categories with $p^+/p^- = 1$. However, most interesting examples — for example, the category of representations of quantum groups at roots of unity — do not satisfy this property. As was discussed in Section 5.7, the way to incorporate modular categories with $p^+/p^- \neq 1$ is to define modular functor with central charge. In topological language, it was defined as a projective representation of the tower of mapping class groups, or, more precisely, as a representation of a suitable central extension of this tower with the central element acting by the fixed constant K (multiplicative central charge).

In order to give an analogous description of the modular functor with central charge, we need to introduce the appropriate formalizm — namely, the notion of twisted $\mathcal{D}$-modules. This is done in this section; in the next section, we will use this formalizm to define modular functor with central charge.

As before, the simplest way to describe such modular functors is to replace the requirement that the bundle carry a flat connection by a *projectively flat* connection, i.e., a connection such that $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is an operator of multiplication by a function (depending on X, Y). Equivalently, we can say that the sheaf of sections carries a projective action of the algebra of vector fields, or that the sheaf of sections is a projective $\mathcal{D}$-module.

However, we want to describe the failure of the connection to be flat more precisely, by describing the corresponding central extension of the Lie algebra of vector fields. It can be done as follows.

First, let us say that an $\mathcal{O}_S$-module is a sheaf of Lie algebras if we have a Lie algebra structure on local sections; the Lie bracket does not have to be $\mathcal{O}_S$-linear. For example, the sheaf Θ_S of vector fields on S is a sheaf of Lie algebras. A map of Lie algebra sheaves is just a sheaf morphism which preserves both the Lie bracket and $\mathcal{O}_S$-module structure.

DEFINITION 6.6.1. A central extension of Θ_S is a sheaf of Lie algebras $\mathcal{A}$ on S along with given maps of sheaves of Lie algebras giving a short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\psi} \mathcal{A} \xrightarrow{\varepsilon} \Theta_S \to 0 \tag{6.6.1}$$

(here $\mathcal{O}_S$ is considered as sheaf of Lie algebras with zero bracket) such that:

1. $\psi(1)$ is central in $\mathcal{A}$.
2. For $a, b \in \mathcal{A}, f \in O_S$, we have $[a, fb] = f[a,b] + (\varepsilon(a)f)b$.

A module $\mathcal{F}$ over $\mathcal{A}$ is a quasicoherent $\mathcal{O}$-module with the action of $\mathcal{A}$ (as a Lie algebra) on $\mathcal{F}$ which agrees with the $\mathcal{O}$-module structure: $\psi(f)s = fs,\ f \in \mathcal{O}, s \in \mathcal{F}$.

REMARK 6.6.2. This is a special case of a more general notion of *Atiyah algebra*, in which $\mathcal{O}_S$ is replaced by an arbitrary sheaf of associative algebras over $\mathcal{O}$, see [**BS**] for details.

One can easily see that locally in S, we can choose a lifting, i.e. a morphism of $\mathcal{O}_S$-modules $a\colon \Theta \to \mathcal{A}$; then the bracket can be written as $[a(X), a(Y)] = a([X,Y]) + c(X,Y)$, where $c(X,Y) \in \mathcal{O}$ is a 2-cocycle on Θ_S.

EXAMPLES 6.6.3.

1. Let $\mathcal{A}_{\mathcal{O}} = \mathcal{O}_S \oplus \Theta_S$ (direct sum as $\mathcal{O}_S$-modules), with the bracket given by $[X+f, Y+g] = [X,Y] + X(g) - Y(f)$, where $X, Y \in \Theta, f, g \in \mathcal{O}$, and $[X,Y]$ is the usual bracket of vector fields. This plays the role of a trivial central extension.
2. Let L be a line bundle on S, $\mathcal{L}$— sheaf of sections of L. Define $\mathcal{A}_{\mathcal{L}}$ as the algebra of first order differential operators in L. If we choose a local trivialization of L, then sections of $\mathcal{A}$ have the form $\partial = X + f, X \in \Theta_S, f \in \mathcal{O}_S$. In other words, choice of trivialization $\mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}|_U$ defines an isomorphism $\mathcal{A}_{\mathcal{L}}|_U \xrightarrow{\sim} \mathcal{A}_{\mathcal{O}}|_U$.
3. Let $\mathcal{A}$ be a central extension of Θ, $k \in \mathbb{C}^\times$. Then we can define the central extension $\mathcal{A}^k$; as a sheaf of Lie algebras, it coincides with $\mathcal{A}$, but the embedding $\mathcal{O} \to \mathcal{A}^k$ given by ψ/k, where ψ is the embedding $\mathcal{O} \to \mathcal{A}$. Equivalently, if we locally choose a lifting $\Theta \to \mathcal{A}$ so that the extension $\mathcal{A}$ is given by a 2-cocycle $c(X,Y)$, then $\mathcal{A}^k$ is given by the 2-cocycle $kc(X,Y)$, which also shows that it is well-defined for $k=0$. One can easily check that for integer k, one has $\mathcal{A}_{\mathcal{L}^k} = (\mathcal{A}_{\mathcal{L}})^k$. Using this, we will define for any $k \in \mathbb{C}$ the "sheaf of first order differential operators in $\mathcal{L}^k$" by
$$\mathcal{A}_{\mathcal{L}^k} = (\mathcal{A}_{\mathcal{L}})^k.$$

Now one can easily see that every projectively flat connection ∇ in a vector bundle E defines a central extension $\mathcal{A}$ of Θ such that ∇ defines a true action of $\mathcal{A}$ by first order differential operators in E. In other words, failure of a projectively flat connection to be flat can be described by a central extension $\mathcal{A}$ of the Lie algebra of vector fields.

EXERCISE 6.6.4. Let L be a line bundle on S. Show that $\mathcal{A}_{\mathcal{L}}$ is isomorphic to the Lie algebra of vector fields on the total space of L which commute with the action of $\mathbb{C}^\times$ on L by dilations; locally, such vector fields have the form $\partial = X + f(s)u\partial_u, X \in \Theta_S, f \in \mathcal{O}_S$, where u is coordinate along the fibers of L. Using this, show that an action of $\mathcal{A}_{\mathcal{L}^k}$ on a vector bundle E on S is the same as a monodromic flat connection on the pullback π^*E of E to $L^\times = L \setminus \{\text{zero section}\}$ such that the monodromy of this connection around the zero section is equal to $e^{-2\pi i k}$.

As with the usual flat connections, we can also use the language of $\mathcal{D}$-modules. The appropriate generalization of the notion of $\mathcal{D}$-module is the notion of twisted $\mathcal{D}$-module.

DEFINITION 6.6.5. A *twisted sheaf of differential operators* on S is a sheaf of associative algebras $\mathcal{U}$ on S and an embedding $\mathcal{O}_S \hookrightarrow \mathcal{U}$ such that locally, the pair $(\mathcal{U}, \mathcal{O} \hookrightarrow \mathcal{U})$ is isomorphic to $(\mathcal{D}, \mathcal{O} \hookrightarrow \mathcal{D})$. A *twisted $\mathcal{D}$-module* is a sheaf of modules over a twisted sheaf of differential operators.

It turns out that the notions of twisted sheaves of differential operators and of central extensions of Θ are equivalent. Namely, for a twisted sheaf $\mathcal{U}$ of d.o., we can define the subsheaf $\mathcal{U}_1$ of differential operators of first order. A reader can easily check that $\mathcal{U}_1$ is closed under the Lie bracket $[a,b] = ab - ba$ and the action of $\mathcal{U}_1$

on $\mathcal{O}$ by $\partial(f) = [\partial, f]$ defines an isomorphism $\mathcal{U}_1/\mathcal{O} \xrightarrow{\sim} \Theta$, and thus $\mathcal{U}_1$ is a central extension of Θ. Conversely, if $\mathcal{A}$ is a central extension of Θ, then define

$$\mathcal{U}(\mathcal{A}) = U(\mathcal{A})/(f - \psi(f)), \qquad f \in \mathcal{O},$$

where $U(\mathcal{A})$ is the universal enveloping algebra of $\mathcal{A}$ (as an $\mathcal{O}$-module, it is isomorphic to $\mathcal{O} \oplus \mathcal{A} \oplus S^2(\mathcal{A}) \oplus \dots$).

LEMMA 6.6.6. *The functors $\mathcal{U} \mapsto \mathcal{U}_1$, $\mathcal{A} \mapsto \mathcal{U}(\mathcal{A})$ are inverse to each other and thus give an equivalence of categories of twisted algebras of differential operators and central extensions of Θ. In particular, twisted $\mathcal{D}$-modules are the same as modules over central extensions of Θ.*

We refer the reader to [**BS**] for the proof (easy) and more details.

EXAMPLE 6.6.7. For a line bundle L, let $\mathcal{D}_L$ be the sheaf of differential operators in L. This is a twisted sheaf of d.o., which corresponds to the central extension $\mathcal{A}_{\mathcal{L}}$.

In a similar way, we can define the "sheaf $\mathcal{D}_{\mathcal{L}^k}$ of differential operators in L^k" as the twisted sheaf of d.o. corresponding to the central extension $\mathcal{A}_{\mathcal{L}^k}$ (see Example 6.6.3):

$$\mathcal{D}_{\mathcal{L}^k} = \mathcal{U}(\mathcal{A}_{\mathcal{L}^k}). \tag{6.6.2}$$

EXERCISE 6.6.8. Let us choose local trivializations of L: $\varphi_\alpha\colon \mathcal{L}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}|_{U_\alpha}$ (where $\{U_\alpha\}$ is an open cover of S), and let $f_{\alpha\beta} = \varphi_\beta\varphi_\alpha^{-1} \in \mathcal{O}(U_\alpha \cap U_\beta)$ be the corresponding transition functions. Show that this defines isomorphisms $\varphi_\alpha\colon \mathcal{D}_{L^k}|_{U_\alpha} \xrightarrow{\sim} \mathcal{D}|_{U_\alpha}$, and the transition functions $\varphi_\beta\varphi_\alpha^{-1}\colon \mathcal{D}(U_\alpha \cap U_\beta) \to \mathcal{D}(U_\alpha \cap U_\beta)$, when restricted to vector fields, are given by

$$\varphi_\beta\varphi_\alpha^{-1}\colon v \mapsto v + k\frac{v(f_{\alpha\beta})}{f_{\alpha\beta}}.$$

Note that the right-hand side is not a vector field but a first order differential operator.

EXERCISE 6.6.9. Show that if L admits a flat connection, then $\mathcal{D}_L \simeq \mathcal{D}$, $\mathcal{A}_{\mathcal{L}} \simeq A_{\mathcal{O}}$.

As for usual $\mathcal{D}$-modules, we can define the category $RS_{L^c}(\overline{M}, M)$ (where L is a vector bundle over $\overline{M}$) as the category of vector bundles over $\overline{M}$ (up to meromorphic equivalence) with an action of $\mathcal{D}_{L^c}$ on the sheaf of sections such that E admits a trivialization such that the action of vector fields has first order poles. Indeed, the regularity condition is local, and locally a twisted sheaf of differential operators is isomorphic to the usual sheaf $\mathcal{D}$ of differential operators.

Similarly, we can define the specialization functor

$$Sp_D\colon RS_{L^c}(\overline{M}, M) \to RS_{L^c}(ND, N^\times D).$$

Note that restriction of L to the fiber $N_dD \simeq \mathbb{C}^\times$ is necessarily trivial, so that there is no need to change the definition of monodromic connection.

6.7. Modular functor with central charge

In order to apply the technique of twisted D-modules to modular functors, we must choose a line bundle L on each of the moduli spaces $M_{g,n}$ in a consistent way. We will use the so-called *determinant line bundle* introduced by Grothendieck (see [**KM**] for details). As before, in order to define a line bundle on $M_{g,n}$ we need to define a line bundle on S for every family of curves C_S over S.

Before doing so, let us introduce some notation. Let L be a finite-dimensional vector space; we define one dimensional vector space $\det L$ as the highest exterior power of L:

$$\det L = \Lambda^{\dim L} L.$$

More generally, for a finite complex of finite-dimensional vector spaces $L^\bullet = \ldots \to L_{i-1} \to L_i \to L_{i+1} \to \ldots$, we denote $\det L^\bullet = \bigotimes (\det L_i)^{(-1)^i}$, where, for a one-dimensional vector space X, we let $X^1 = X$, $X^{-1} = X^*$.

EXERCISE 6.7.1. Show that there is a canonical isomorphism

$$\det L^\bullet = \bigotimes (\det H^i(L^\bullet))^{(-1)^i}.$$

This definition can be generalized to vector bundles over a smooth base S: if E is a vector bundle of dimension n over S, then we define line bundle $\det L$ by $\det L = \Lambda^n E$. Again, this can be trivially generalized to complexes of vector bundles, and we have the following proposition.

LEMMA 6.7.2. *Let $E^\bullet, F^\bullet$ be finite complexes of vector bundles over S, and let $f\colon E^\bullet \to F^\bullet$ be a morphism of complexes of vector bundles which is a quasi-isomorphism, i.e., it induces isomorphism of the cohomology sheaves $\mathcal{H}^i f\colon \mathcal{H}^i(\mathcal{E}^\bullet) \xrightarrow{\sim} \mathcal{H}^i(\mathcal{F}^\bullet)$. Then f defines an isomorphism of the line bundles* $\det E^\bullet \simeq \det F^\bullet$.

This lemma allows one to define $\det \mathcal{E}$ for arbitrary coherent $\mathcal{O}$-module $\mathcal{E}$, generalizing the case when $\mathcal{E}$ is the sheaf of sections of a vector bundle E. Indeed, every coherent $\mathcal{O}$-module admits a resolution by vector bundles: we can find a complex of vector bundles $F^\bullet$ such that $\mathcal{H}^0(\mathcal{F}^\bullet) = \mathcal{E}, \mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for $i \neq 0$. By definition, let $\det \mathcal{E} = \det F^\bullet$. Lemma 6.7.2 shows that this is independent of the choice of resolution. The same argument shows that we can define $\det \mathcal{E}^\bullet$ for a complex of coherent $\mathcal{O}$-modules $\mathcal{E}^\bullet$, and that it only depends on the quasi-isomorphism class of $\mathcal{E}^\bullet$; in other words, det is well defined on the derived category of coherent $\mathcal{O}$-modules (see [**KM**] for details).

REMARK 6.7.3. In [**KM**], the determinant line bundle is defined as a pair, consisting of a line bundle a and "parity", i.e. an element of $\mathbb{Z}/2\mathbb{Z}$. Parity is important for tracking correct signs in isomorphisms like $\det(F \oplus G) \simeq \det F \otimes \det G$. However, for us these signs are not important, and therefore we omit parity.

DEFINITION 6.7.4. Let C_S be a family of pointed curves over S. We define the corresponding *determinant line bundle* Q_S by

$$Q = (\det R\pi_* \mathcal{O}_{C_S})^{-1} = \bigotimes (\det R^i \pi_* \mathcal{O}_{C_S})^{(-1)^{i+1}}, \tag{6.7.1}$$

where π is the projection $C_S \to S$.

Note that this definition does not use the marked points. Also, this definition is valid even if the family C_S is singular: in this case, $R^i \pi_* \mathcal{O}_{C_S}$ needs not be a vector bundle, but is always a coherent $\mathcal{O}_S$-module, and thus $\det R^i\pi_*\mathcal{O}_{C_S}$ is a line bundle.

For readers who prefer to avoid using the notion of higher direct images, it suffices to say that the fiber of this line bundle at point $s \in S$ is

$$Q_s = \bigotimes(\det H^i(C_s, \mathcal{O}_{C_s}))^{(-1)^{i+1}}.$$

For a smooth family, this definition can be simplified. Let us assume that C_S is a non-singular family with connected fiber. Then $H^0(C_s, \mathcal{O}) = \mathbb{C}$, and $H^i(C_s, \mathcal{O}) = 0$ for $i > 1$ (since $\mathcal{O}$ is coherent). Thus, in this case $Q = \det R^1\pi_*(\mathcal{O}_{C_S})$, so its fiber at point s is given by

$$Q_s = \det(H^1(C_s, \mathcal{O}_{C_s})). \tag{6.7.2}$$

These spaces are well known (see, e.g., [**GH**]): for a compact complex curve of genus g, $H^1(C, \mathcal{O}_C)$ is a vector space of complex dimension g. In particular, for $g = 0, H^1(C, \mathcal{O}_C) = 0$ and thus, the determinant line bundle is trivial.

One easily sees that definition of the determinant bundle Q_S is functorial in S, and thus, we have a well-defined line bundle Q_M over the moduli stack $M_{g,n}$. The same definition also works for singular curves, and thus Q_M is well defined over the completion $\overline{M}_{g,n}$.

Finally, let us discuss the behavior of the determinant bundle with respect to gluing. Let A be a finite set, $\alpha, \beta \in A$—an unordered pair, $A' = A \setminus \{\alpha, \beta\}$. Such a pair defines a "clutching" map $S_{\alpha,\beta}\colon M_{*,A} \to N(D^0)$, where D^0 is the corresponding component of the boundary in $\overline{M}_{*,A'}$ (see (6.2.7)).

PROPOSITION 6.7.5. *Let $S_{\alpha,\beta}$ be as above, Q–the determinant line bundle on $M_{*,A'}$. Let us also denote by Q the corresponding line bundle on $N(D^0)$. Then $S^*_{\alpha,\beta}(Q)$ is canonically isomorphic to the determinant line bundle over $M_{*,A}$.*

We omit the proof of this proposition, referring the reader to [**BFM**].

Now, we are ready to formulate the definition of the modular functor with central charge, which is parallel to Definition 6.4.1, but with replacement of vector bundles with flat connection by vector bundles with the action of the central extension $\mathcal{A}_{Q^c}$ of the Lie algebra of vector fields, or, equivalently, with the action of $\mathcal{D}_{Q^c}$. As before, let $\mathcal{C}$ be an abelian category over the field $\mathbb{C}$, and R – a symmetric object in ind$-\mathcal{C}^{\boxtimes 2}$.

DEFINITION 6.7.6. A *complex $\mathcal{C}$-extended modular functor with* (*additive*) *central charge* $a \in \mathbb{C}$ is the following collection of data:

(i) For every finite set A and $W \in \mathcal{C}^{\boxtimes A}$, a finite-dimensional vector bundle over $\overline{M}_{*,A}$ with an action of $\mathcal{D}_{Q^a}$. This bundle is called the *bundle of conformal blocks*; its fiber at a point $C \in \overline{M}_{*,A}$ will be denoted by $\langle W\rangle_C$.

(ii) Isomorphisms $\langle X\rangle_{C'} \otimes \langle Y\rangle_{C''} \simeq \langle X \boxtimes Y\rangle_{C' \sqcup C''}$.

(iii) **Gluing isomorphisms.** Let A be a finite set, $\alpha, \beta \in A$—an unordered pair, $A' = A \setminus \{\alpha, \beta\}, W \in \mathcal{C}^{\boxtimes A'}$. For every such collection, we require an isomorphism of $\mathcal{D}_{Q^a}$-modules on $M_{*,A}$:

$$G_{\alpha,\beta}\colon \langle W \boxtimes R\rangle \xrightarrow{\sim} S^*_{\alpha,\beta} Sp_D \langle W\rangle, \tag{6.7.3}$$

where $S_{\alpha,\beta}\colon M_{*,A} \to N(D^0), D^0 \subset M_{*,A'}$, is the "clutching" (6.2.7), and R is placed at positions with indices α, β. (Since $S_{\alpha,\beta}$ is a $\mathbb{C}^\times$-bundle, the definition of $S^*_{\alpha,\beta}$ causes no problems.)

(iv) **Vacuum propagation.** We have a distinguished element $\mathbf{1} \in \operatorname{Ob}\mathcal{C}$, and for every $\alpha \in A$, we require an isomorphism of $\mathcal{D}_{Q^a}$-modules on $M_{*,A}$:

$$G_\alpha : \langle W \boxtimes \mathbf{1} \rangle \xrightarrow{\sim} S_\alpha^* \langle W \rangle, \tag{6.7.4}$$

where $S_\alpha : M_{*,A} \to M_{*,A\setminus\alpha}$ is the operator of erasing the point α.

These data have to satisfy the following properties:

Functoriality: $W \mapsto \langle W \rangle_C$ is functorial in W, and the isomorphisms (ii)-(iv) are functorial isomorphisms.

Compatibility: the isomorphisms (ii)-(iv) are compatible with each other and with the commutativity, associativity, and unit morphisms in $\mathcal{V}ec_f$ (cf. Definition 4.2.1).

Normalization: $\langle \mathbf{1}, \mathbf{1} \rangle_{\mathbb{P}^1} = \mathbb{C}$.

Note that the requirement that $\langle W \rangle$ be a $\mathcal{D}_{Q^a}$ is equivalent to saying that the pull-back of $\langle W \rangle$ to the total space of the line bundle Q has a monodromic flat connection with the monodromy $e^{-2\pi a}$ around the zero section (see Exercise 6.6.4).

Let us now relate it to the topological formulation of the modular functor with central charge. To do it, let us recall the definition of the central charge for the modular functor in the topological setup (see Section 5.7).

It was defined using the central extension of the usual tower of mapping class groups by the groupoid $T_\Sigma = T_{H^1(\Sigma,\mathbb{R})}$, where Σ is a closed oriented topological surface of genus g, and for a symplectic real vector space V, T_V is the Poincare (fundamental) groupoid $T_V = \pi_1(\Lambda_V)$ of the set Λ_V of all Lagrangian subspaces in V. Recall also that for every $L \in T_V, \operatorname{Hom}_{T_V}(L, L) \simeq \mathbb{Z}$.

It is well known (see, e.g., [**GH**]) that for a connected compact complex curve the natural map of sheaves $\mathbb{R} \to \mathcal{O}$ induces an isomorphism of the cohomology spaces $H^1(C, \mathbb{R}) \simeq H^1(C, \mathcal{O})$, where both sides are considered as vector spaces over $\mathbb{R}$. In other words, a choice of a complex structure on a topological surface Σ defines a complex structure on the $2g$-dimensional real vector space $V = H^1(\Sigma, \mathbb{R})$.

THEOREM 6.7.7. *Let C be a complex curve. Then one has a canonical equivalence of groupoids*

$$T_\Sigma = \pi_1(Q_C^{\otimes 2} \setminus \{0\})$$

where, as before, $Q_C = \det H^1(C, \mathcal{O}_C)$.

PROOF. Let us note that the complex structure on $V = H^1(C, \mathbb{R})$ defined by the identification $V \simeq H^1(C, \mathcal{O})$, agrees with the symplectic structure in V as follows:

$$\langle \cdot, i_C \cdot \rangle \text{ is symmetric positive definite} \tag{6.7.5}$$

where $\langle \cdot, \cdot \rangle$ is the symplectic form, and $i_C \in \operatorname{End}_{\mathbb{R}}(V)$ is the operator of multiplication by $\mathrm{i} = \sqrt{-1}$ in the complex structure defined by C.

Let V be an arbitrary symplectic real vector space. Denote by H_V the set of all complex structures on V satisfying the condition above. It is usually called *Siegel upper half plane of* V. We quote without proof the following standard result, which can be found, for example, in [**GH**].

THEOREM 6.7.8. *H_V is a contractible space.*

EXERCISE 6.7.9. Show that for $V = \mathbb{R}^2$ with the standard symplectic form, H_V can be identified with the upper half-plane of $\mathbb{C}$.

Let us consider the line bundle λ over H_V, whose fiber at point $h \in H_V$ is $\lambda_h = \det_{\mathbb{C}} V$; here V is considered as g-dimensional vector space over $\mathbb{C}$ with the complex structure given by h. Consider the total space of $\lambda^{\otimes 2} \setminus \{\text{zero section}\}$.

PROPOSITION 6.7.10. *For every $h \in H_V$, one has canonical equivalence of groupoids*

$$\pi_1(\lambda^{\otimes 2} \setminus \{\textit{zero section}\}) \simeq \pi_1(\lambda_h^{\otimes 2} \setminus \{0\}) \simeq T_V. \tag{6.7.6}$$

Obviously, this proposition immediately implies the statement of the theorem.

Let us first construct equivalence $\pi_1(\lambda^{\otimes 2} \setminus \{\text{zero section}\}) \simeq \pi_1(\lambda_h^{\otimes 2} \setminus \{0\})$. This equivalence follows from the fact that H_V is contractible, and therefore, embedding $\lambda_h^{\otimes 2} \setminus \{0\} \hookrightarrow \lambda^{\otimes 2} \setminus \{\text{zero section}\}$ is a homotopy equivalence.

Next, let us construct equivalence $\pi_1(\lambda_h^{\otimes 2} \setminus \{0\}) \simeq T_V$. To do so, let us rewrite the definition of T_V as follows. Let L be the tautological vector bundle of dimension g over Λ_V: it is a sub-bundle of the trivial bundle $V \times \Lambda_V$ such that its fiber at point $L \in T(V)$ is the subspace $L \subset V$. Consider the one-dimensional vector bundle $\lambda_{\mathbb{R}} = \det_{\mathbb{R}} L = \Lambda_{\mathbb{R}}^g L$ (to avoid confusion, we use subscript $\mathbb{R}$ for determinant of vector spaces over $\mathbb{R}$). Then $\lambda_{\mathbb{R}}^{\times} / \pm 1$ is a bundle with fiber $\mathbb{R}_+$ over Λ_V; thus, $\lambda_{\mathbb{R}}^{\times} / \pm 1 \to \Lambda_V$ is a homotopy equivalence, and $T_V = \pi_1(\lambda_{\mathbb{R}}^{\times} / \pm 1)$.

Now let $h \in H_V$ be a complex structure on V, $\lambda_h = \det V = \Lambda^g V$, where V is considered as a vector space over $\mathbb{C}$ using the complex structure h. Thus, we have a well-defined map $\lambda_{\mathbb{R}} = \Lambda_{\mathbb{R}}^g L \to \lambda_h = \Lambda^g V$; the left-hand side is one dimensional real vector space, the right hand side is one-dimensional complex vector space. This gives rise to a map $\lambda_{\mathbb{R}}^{\times} / \pm 1 \to \lambda^{\otimes 2} \setminus \{0\}$.

This completes the proof of the proposition. On the other hand, the proposition immediately implies the statement of the theorem. □

EXAMPLE 6.7.11. Let $g = 1$, so Σ is a torus (or, equivalently, C is an elliptic curve). Then $V = \mathbb{R}^2$ with the canonical symplectic form, so Λ_V is the set of all real one-dimensional subspaces in $\mathbb{R}^2$, thus $\Lambda_V \simeq S^1$. On the other hand, in this case $\lambda_h = \det V = V$ (as a complex vector space). Thus, in this case the canonical map $\pi_1(\Lambda_V) \to \pi_1(\lambda_h^{\otimes 2} \setminus \{0\})$ is obvious.

As an immediate corollary of Theorem 6.7.7, we see get the following result. As before, let $\mathcal{C}$ be an abelian category over $\mathcal{C}$.

THEOREM 6.7.12. *The notions of $\mathcal{C}$-extended topological MF with multiplicative central charge K and of $\mathcal{C}$-extended complex MF with additive central charge a are equivalent if $K = e^{\pi \mathrm{i} a}$.*

COROLLARY 6.7.13. *Any MTC category over $\mathbb{C}$ gives rise to a $\mathcal{C}$-extended complex MF with additive central charge a such that*

$$e^{\pi i a} = p^+/p^-. \tag{6.7.7}$$

Conversely, every $\mathcal{C}$-extended complex MF with additive central charge a defines a weakly ribbon structure on $\mathcal{C}$; if this category is rigid, then it is modular, and p^+/p^- is given by the formula above.

Note that for modular functors coming from rational conformal field theory, the additive central charge is given by

$$a = c/2,$$

where c is the Virasoro central charge of the theory (we will illustrate it in the next chapter, where the Wess-Zumino-Witten model is considered). Combining this with the corollary above, we see that in this case one has

$$K = \frac{p^+}{p^-} = e^{\pi \mathrm{i} c/2}$$

(cf. Remark 3.1.20).

CHAPTER 7

Wess–Zumino–Witten Model

In this chapter, we give a construction of what is probably the best known example of a modular functor. This modular functor is based on the category of integrable representations of an affine Lie algebra and appears naturally in the Wess–Zumino–Witten model of conformal field theory; abusing the language, we will call it the *WZW modular functor*. The literature devoted to this measures in hundreds of papers; the most prominent among them are [**KZ**], [**MS1**], [**TUY**], [**BFM**]. For more detailed exposition of conformal field theory in general and WZW model in particular, we refer the reader to [**FMS**] and references therein.

The main goal of this chapter is to prove the following result. Fix a simple complex Lie algebra $\mathfrak{g}$, and let $\mathcal{O}_k^{\text{int}}$ be the category of integrable modules of level $k \in \mathbb{Z}_+$ over the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$.

THEOREM 7.0.1. *The category $\mathcal{O}_k^{\text{int}}$ has a structure of a modular tensor category.*

Of course, in this form the theorem is not very precise since we have not defined the tensor product (which is usually called the *fusion product*, and denoted $\dot{\otimes}$, to distinguish it from the usual tensor product of vector spaces). We will give a precise definition later (see Corollary 7.9.11).

Another important result, which, unfortunately, we will not prove, is the following. Recall that in Section 3.3 we defined a structure of a modular tensor category on a certain subquotient $\mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$ of the category of representations of the quantum group $U_q\mathfrak{g}$, $q = e^{\pi \mathrm{i}/m\varkappa}$.

THEOREM 7.0.2 ([**F**]). *The category $\mathcal{O}_k^{\text{int}}$ is equivalent to the category $\mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$ as a modular tensor category for $\varkappa = k + h^\vee$, where $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$.*

Because of the importance of these two theorems, we will comment here on their history. They have appeared in somewhat vague form in physics literature in the 1980s. The accurate construction of the tensor structure on $\mathcal{O}_k^{\text{int}}$ first appeared in [**MS1**]; however, Moore and Seiberg did not give a complete proof.

To the best of our knowledge, there are three known proofs of Theorem 7.0.1. The first one, which we present in this chapter, is based on the use of the notion of modular functor. The corresponding modular functor (which, as we mentioned above, naturally appears in the Wess–Zumino–Witten model of conformal field theory) is defined in terms of the spaces of coinvariants. The crucial step in proving that these spaces satisfy the axioms of a modular functor is checking the gluing axiom, which was done by Tsuchiya, Ueno, and Yamada [**TUY**]. Another proof of the gluing axiom can be obtained by suitably modifying the proof for the minimal models given in [**BFM**].

The second proof of Theorem 7.0.1 was given by Finkelberg [**F**], who based his approach on the series of papers of Kazhdan and Lusztig [**KL**]. They proved that for negative integer level k, the category $\mathcal{O}_k$ is a ribbon category, which is equivalent to the category $\mathcal{C}(\mathfrak{g}, \varkappa)$ of representations of the quantum group $U_q\mathfrak{g}$. Therefore, this category contains a subquotient category which is equivalent to the MTC $\mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$. Combining this result with a certain duality between the categories $\mathcal{O}_k$ and $\mathcal{O}_{-2h^\vee-k}$, Finkelberg showed that this subquotient is dual to the category $\mathcal{O}_k^{\text{int}}$, thus establishing simultaneously Theorems 7.0.1 and 7.0.2.

Finally, the third proof of Theorem 7.0.1, based on the theory of vertex operator algebras, was recently given by Huang and Lepowsky [**HL**].

Unfortunately, none of these proofs is easy. Finkelberg's proof is based on a 250 pages long series of papers [**KL**], which is very tersely written; few people (if any at all) have expertise and patience to follow all the details of this proof. Similarly, the proof of Huang and Lepowsky is heavily based on a number of their previous papers on vertex operator algebras, which can sometimes get rather technical. The modular functor approach seems to be the easiest of all three, but it still requires all the formalism of modular functors and their relation with tensor categories (which took the previous 140 pages of this book) and some non-trivial algebraic geometry used in [**TUY**], also not an easy reading.

The proof given in this chapter is based on the modular functor approach; however, our proof of the gluing axiom follows the ideas of [**BFM**] rather than [**TUY**]. This proof was never published before; however, for the most part it closely follows the arguments in [**BFM**], so all the credit belongs to Beilinson, Feigin, and Mazur. Modifying their arguments for WZW model was rather straightforward; according to private communications from Beilinson and Feigin, they intended to include the proof for WZW model in the final version of the manuscript. Unfortunately, it is not clear when (and if) such a final version appears, so we include this proof here.

7.1. Preliminaries on affine Lie algebras

The aim of this subsection is just to fix the notation, we refer to the book of Kac [**K1**] for a comprehensive treatment.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $\langle \cdot, \cdot \rangle$ be an invariant bilinear form on $\mathfrak{g}$ normalized so that $\langle \alpha, \alpha \rangle = 2$ for long roots of $\mathfrak{g}$. We will use the same notations (and notions) as in Section 1.3.

Let $\mathfrak{g}((t)) \equiv \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$ be the *loop algebra* of $\mathfrak{g}$. Then the *affine Lie algebra* of $\mathfrak{g}$ is

$$\widehat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}K \tag{7.1.1}$$

with commutation relations

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg + \langle a, b \rangle \operatorname{Res}_0(df\, g)K, \qquad [K, \widehat{\mathfrak{g}}] = 0.$$

For brevity, we often use the notation $x[n] = x \otimes t^n, x \in \mathfrak{g}$.

We let $\widehat{\mathfrak{g}}^+ = t\mathfrak{g}[[t]]$, $\widehat{\mathfrak{g}}^- = t^{-1}\mathfrak{g}[t^{-1}]$. We have a decomposition of $\widehat{\mathfrak{g}}$ into subalgebras

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^+ \oplus \mathfrak{g} \oplus \mathbb{C}K \oplus \widehat{\mathfrak{g}}^-.$$

We will be interested in $\widehat{\mathfrak{g}}$-modules of level $k \in \mathbb{C}$, i.e., modules V such that $K|_V = k\, \mathrm{id}_V$; this is equivalent to considering modules over $U(\widehat{\mathfrak{g}})_k = U\widehat{\mathfrak{g}}/U\widehat{\mathfrak{g}}(K-k)$.

We will denote by $\mathcal{O}_k$ the category of $\widehat{\mathfrak{g}}$-modules of level k which have weight decomposition with finite-dimensional weight subspaces, such that the action of $\widehat{\mathfrak{g}}^+$ is locally nilpotent and the action of $\mathfrak{g}$ is integrable.

Of special interest for us are two classes of modules from $\mathcal{O}_k$: Weyl modules and integrable modules. Weyl module $V^k_\lambda, \lambda \in P_+$, is defined by

$$V^k_\lambda = \operatorname{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}\oplus\widehat{\mathfrak{g}}^+\oplus\mathbb{C}K} V_\lambda, \tag{7.1.2}$$

where V_λ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ, which we consider as a module over $\mathfrak{g}\oplus\widehat{\mathfrak{g}}^+\oplus\mathbb{C}K$ by letting $\widehat{\mathfrak{g}}^+$ act as 0 and K act as $k\,\mathrm{id}$. The Weyl module is free over $\widehat{\mathfrak{g}}^-$.

If $k \notin \mathbb{Q}$, then Weyl modules are irreducible and the category $\mathcal{O}_k$ is semisimple. We will be mostly interested in the other extreme case $k \in \mathbb{Z}_+$. In this case, we can also consider integrable highest-weight modules. We will denote by $\mathcal{O}^{\mathrm{int}}_k \subset \mathcal{O}_k$ the subcategory of integrable modules, i.e., such modules that for every root α, $n \in \mathbb{Z}$, the action of $e_\alpha[n]$ is locally nilpotent. It is known that $\mathcal{O}^{\mathrm{int}}_k$ is semisimple with simple objects $L^k_\lambda, \lambda \in P^k_+$, where P^k_+ is the *positive Weyl alcove*

$$P^k_+ = \{\lambda \in P_+ \mid (\lambda, \theta^\vee) \le k\}, \tag{7.1.3}$$

see [**K1**]. (Note that P^k_+ is the same set which we denoted by C in Section 3.3.) The modules L^k_λ are irreducible and can be described as the quotient $L^k_\lambda = V^k_\lambda/Z_\lambda$, where Z_λ is the unique maximal proper submodule of V^k_λ. It is known that Z_λ is generated by one vector: $Z_\lambda = U\widehat{\mathfrak{g}}(e_\theta[-1])^{a+1}v_{\lambda,k}$, where $a = k - (\lambda, \theta^\vee)$.

It is useful to note that both V^k_λ and L^k_λ have a natural $\mathbb{Z}_-$-grading (sometimes called the *homogeneous grading*), defined by $\deg v_{\lambda,k} = 0, \deg a[n] = n, a \in \mathfrak{g}, n \in \mathbb{Z}$. It is easy to see that homogeneous components of V^k_λ (and, in fact, any module in the category $\mathcal{O}_k$) are finite-dimensional.

Finally, we will define the duality in the category $\mathcal{O}_k$ by $DV = (V^*)^\natural$, where V^* is the restricted dual to V, i.e. the direct sum of the dual spaces to homogeneous components of V, and $\natural$ is defined as follows: for a $\widehat{\mathfrak{g}}$ module W, the module $W^\natural$ coincides with W as a vector space, and the action of $\widehat{\mathfrak{g}}$ is twisted by the automorphism

$$\natural\colon x[n] \mapsto (-1)^n x[-n], \qquad K \mapsto -K. \tag{7.1.4}$$

It is easy to see that D is an anti-automorphism of the category $\mathcal{O}_k$ which preserves $\mathcal{O}^{\mathrm{int}}_k$. In particular, for an integrable module L^k_λ, DL^k_λ is also an irreducible integrable module, whose top homogeneous component is V^*_λ. It is (non-canonically) isomorphic to $L^k_{-w_0(\lambda)}$.

7.2. Reminders from algebraic geometry

In this section we briefly list some facts from algebraic geometry which will be used below. All of them are quite standard, so a reader who has even basic knowledge of algebraic geometry over $\mathbb{C}$ can safely skip this section.

All varieties considered in this section are considered with analytic topology; as before, we use the words "manifold" and "non-singular variety" as synonyms. For a variety S, we denote by $\mathcal{O}_S$ the structure sheaf of S (i.e., the sheaf of analytic functions on S). We assume that the reader is familiar with the notion of a $\mathcal{O}$-module and a coherent $\mathcal{O}$-module. As usual, for a point $s \in S$ we define by $\mathcal{O}_{S,s}$ the local ring at s, i.e. the ring of germs of analytic functions at s, and by m_s the maximal ideal of this ring, which consists of functions vanishing at s. We also denote by $\widehat{\mathcal{O}}_{S,s}$ the completion of the local ring with respect to topology given by the powers of the maximal ideal. In particular, if $\dim S = 1$, $s \in S$ is a regular point, and t is a *local parameter at* s, i.e., an analytic function in a neighborhood of s such that $t(s) = 0, (dt)_s \neq 0$, then $\widehat{\mathcal{O}}_{S,s} \simeq \mathbb{C}[[t]]$.

For an $\mathcal{O}_S$-module $\mathcal{F}$ we define its fiber at point $s \in S$ to be $\mathcal{F}_s/m_s\mathcal{F}_s$. In particular, if $\mathcal{F}$ is the sheaf of sections of a vector bundle F, then in this way one recovers the fibers of F. We will say that an $\mathcal{O}$-module $\mathcal{F}$ is *lisse* if it is the sheaf of sections of a finite-dimensional vector bundle. Note that every lisse sheaf is coherent, but converse is not true.

In general, for an open subset $U \subset S$ and a sheaf $\mathcal{F}$ on S, we denote by $\mathcal{F}(U)$ the vector space of sections of $\mathcal{F}$ over U. However, in the case when $U = C \setminus D$, where C is compact and D is a divisor, and $\mathcal{F}$—an $\mathcal{O}$-module over C, we will denote by $\mathcal{F}(C - D)$ the space of *meromorphic* sections of $\mathcal{F}$ over C which are regular outside of D. We hope it won't cause confusion.

We will use the following well known facts about complex curves. As before, all the curves are assumed to be compact and non-singular (unless specified otherwise), but not necesarily connected.

THEOREM 7.2.1 (Riemann-Roch). *Let C be a connected complex curve, and $p_1, \dots, p_n, q$—distinct points of C $(n \geq 0)$. Let us fix the principal parts of Laurent expansions $(f)_i \in \mathbb{C}((t_i))/\mathbb{C}[[t]]$ near p_i. Then there exists a function $f \in \mathcal{O}(C - \{p_1, \dots, p_n, q\})$ which has given principal parts of Laurent expansion at p_i and has a pole at q. Moreover, the order of pole at q can be bounded by a constant which only depends on the order of poles at p_i and the genus of the curve C.*

This theorem can be generalized to curve which may have ordinary double point singularities and may be disconnected. In this case, we have to allow poles at a collection of points $q_1, \dots, q_m$ such that on every component of C there is at least one of the points q_i.

THEOREM 7.2.2. *Let C be a complex curve (possibly disconnected and singular). Let $q \in C$ be a regular point, and t—a local parameter at q. Then the vector space*

$$\mathbb{C}((t))/\mathbb{C}[[t]] + \mathcal{O}(C - q),$$

is finite dimensional. Moreover, there exists $N \in \mathbb{Z}_+$ which only depends on the topology of C such that

$$\mathcal{O}(C - q) + \mathbb{C}[[t]] \supset t^{-N}\mathbb{C}[t^{-1}] + \mathbb{C}[[t]].$$

7.3. Conformal blocks: definition

In this section, we will define the vector spaces of coinvariants; later we will show that these vector spaces satisfy the axioms of a modular functor. The basic references for this section are [**TUY**], [**Be**] (with minor changes).

Fix a compact nonsingular complex curve C (not necessarily connected), a finite dimensional simple Lie algebra $\mathfrak{g}$, and a positive integer k.

Let $p_1, \dots, p_n$ be distinct points on C with local coordinates $t_1, \dots, t_n$ (recall that a local coordinate at a point p is a holomorphic function t in a neighborhood of p such that $t(p) = 0, (dt)_p \neq 0$). We will always assume that on every connected component of C there is at least one point. Let $V_1, \dots, V_n \in \mathcal{O}_k$ be some $\widehat{\mathfrak{g}}$-modules associated to these points.

We will use the notations

$$\vec{p} = (p_1, \dots, p_n),$$
$$V = V_1 \otimes \dots \otimes V_n.$$

In particular, if $V_i = L^k_{\lambda_i}$ are integrable modules, we will use the notation

$$\vec{\lambda} = (\lambda_1, \dots, \lambda_n), \quad L^k_{\vec{\lambda}} = L^k_{\lambda_1} \otimes \dots \otimes L^k_{\lambda_n}.$$

Consider the Lie algebra

$$\mathfrak{g}(C - \vec{p}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}(C - \vec{p}) \tag{7.3.1}$$

of $\mathfrak{g}$-valued functions on C which are regular outside the points $p_1, \dots, p_n$ and meromorphic at these points. We have Lie algebra homomorphisms

$$\gamma_i \colon \mathfrak{g}(C - \vec{p}) \to \mathfrak{g}((t))$$

given by Laurent expansion around the point p_i in the local coordinate t_i. This does not give a Lie algebra homomorphism $\mathfrak{g}(C - \vec{p}) \to \widehat{\mathfrak{g}}$ because of the central term in definition of $\widehat{\mathfrak{g}}$. However, by the Residue Theorem,

$$\vec{\gamma} = \gamma_1 \oplus \dots \oplus \gamma_n \colon \mathfrak{g}(C - \vec{p}) \to \mathfrak{g}((t)) \oplus \dots \oplus \mathfrak{g}((t))$$

can be lifted to a homomorphism

$$\vec{\gamma} \colon \mathfrak{g}(C - \vec{p}) \to U(\widehat{\mathfrak{g}})_k \otimes \dots \otimes U(\widehat{\mathfrak{g}})_k, \quad \vec{\gamma}(x) = \sum_{i=1}^{n} 1 \otimes \dots \otimes \gamma_i(x) \otimes \dots \otimes 1.$$

In particular, $\mathfrak{g}(C - \vec{p})$ acts on V.

DEFINITION 7.3.1. The space of *conformal blocks* is the vector space of coinvariants

$$\tau(C, \vec{p}, V) := V_{\mathfrak{g}(C - \vec{p})} = V / \mathfrak{g}(C - \vec{p}) V.$$

We will write $\tau(C, \vec{p}, \vec{t}, V)$ when we need to show the dependence on the choice of local parameters $\vec{t} = (t_1, \dots, t_n)$.

It is easy to see that the construction above also makes perfect sense if we allow t_i be *formal* local parameters at p_i, i.e., $t_i \in \widehat{\mathcal{O}}_{p_i}, (dt_i)_{p_i} \neq 0$. Note that once t_i is chosen, one has $\widehat{\mathcal{O}}_{p_i} = \mathbb{C}[[t_i]]$.

LEMMA 7.3.2 (Beauville [**Be**]). *Let $\vec{p}$, V be as above, and let $q \in C - \vec{p}$, $\lambda \in P^k_+$. As before, let V_λ be the corresponding finite-dimensional $\mathfrak{g}$-module, and let V^k_λ be the Weyl module over $\widehat{\mathfrak{g}}$. Then the inclusion $V_\lambda \hookrightarrow V^k_\lambda$ induces an isomorphism*

$$(V \otimes V_\lambda)_{\mathfrak{g}(C - \vec{p})} \xrightarrow{\sim} (V \otimes V^k_\lambda)_{\mathfrak{g}(C - \vec{p} - q)} = \tau(C, \vec{p} \cup q, V \otimes V^k_\lambda), \tag{7.3.2}$$

where $\mathfrak{g}(C - \vec{p})$ acts on V_λ via the evaluation map $a \otimes f \mapsto f(q)a$, $a \in \mathfrak{g}$, $f \in \mathcal{O}(C - \vec{p})$.

PROOF. Since the natural embedding $V \otimes V_\lambda \hookrightarrow V \otimes V^k_\lambda$ is clearly $\mathfrak{g}(C - \vec{p})$ equivariant, it induces a map from the left hand side of (7.3.2) to the right hand side.

By the Riemann–Roch formula, there exists a function z on C regular outside $\vec{p} \cup q$ and having a simple pole at the point q. Then

$$\mathcal{O}(C - \vec{p} - q) = \mathcal{O}(C - \vec{p}) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C} z^{-i},$$

therefore $\mathfrak{g}(C-\vec{p}-q) \simeq \mathfrak{g}(C-\vec{p}) \oplus \widehat{\mathfrak{g}}^-$.

By definition, V^k_λ is a free $U(\widehat{\mathfrak{g}}^-)$-module isomorphic to $U(\widehat{\mathfrak{g}}^-)V_\lambda$; hence, $V_\lambda \simeq (V^k_\lambda)_{\widehat{\mathfrak{g}}^-}$. Then (7.3.2) follows by tensoring with V and taking coinvariants with respect to $\mathfrak{g}(C-\vec{p})$. $\square$

LEMMA 7.3.3. *Let C be connected, and let V_i be quotients of Weyl modules: $V_i = V^k_{\lambda_i}/I_i$ (the ideals I_i may be zero, maximal, or anything in between). Assume also that at least one of V_i is integrable, i.e., equal to $L^k_{\lambda_i}$. Then the natural surjection $V = V_1 \otimes \cdots \otimes V_n \twoheadrightarrow L^k_{\lambda_1} \otimes \cdots \otimes L^k_{\lambda_n} = L^k_{\vec{\lambda}}$ gives rise to an isomorphism*

$$\tau(C,\vec{p},V) \xrightarrow{\sim} \tau(C,\vec{p},L^k_{\vec{\lambda}}). \tag{7.3.3}$$

PROOF. It suffices to prove that

$$(L^k_{\lambda_1} \otimes V_2 \otimes \cdots \otimes V_{n-1} \otimes V^k_{\lambda_n})_{\mathfrak{g}(C-\vec{p})} = (L^k_{\lambda_1} \otimes V_2 \otimes \cdots \otimes V_{n-1} \otimes L^k_{\lambda_n})_{\mathfrak{g}(C-\vec{p})}.$$

Let $Z = \{v \in V^k_{\lambda_n} \mid L_{\lambda_1} \otimes \cdots \otimes V_{n-1} \otimes v \subset \operatorname{Im} \mathfrak{g}(C-\vec{p})\}$. Obviously, this is a submodule in $V^k_{\lambda_n}$; our goal is to prove that $V^k_{\lambda_n}/Z$ is integrable. This is equivalent to the following statement: for every root α and $v \in V^k_{\lambda_n}$, one has $(e_\alpha[-1])^N v \in Z$ for $N \gg 0$ (in fact, it suffices to check this for $\alpha = \theta$). We leave it to the reader to check that if we choose $f \in \mathbb{C}((t))$ such that f has first order pole at 0, then the above condition is equivalent to $(e_\alpha f)^N v \in Z$ for $N \gg 0$ (in other words, the notion of an integrable module does not depend on the choice of local parameter).

Now let $f \in \mathcal{O}(C-p_1-p_n)$ be a function which has a first order pole at p_n. By the Riemann–Roch theorem, such a function exists if we allow it to have a pole of sufficiently high order at p_1. Since $L^k_{\lambda_1}$ is integrable, and f is regular at $p_2, \ldots, p_{n-1}$, we easily see that action of $e_\alpha f$ on $L^k_{\lambda_1} \otimes \cdots \otimes V_{n-1}$ is locally nilpotent. Therefore, for any $v_1 \in L^k_{\lambda_1}, \ldots, v_n \in V^k_{\lambda_n}$, one has $v_1 \otimes \cdots \otimes v_{n-1} \otimes (e_\alpha f)^N v_n \in \operatorname{Im} \mathfrak{g}(C-\vec{p})$. But this exactly means that $(e_\alpha f)^N v_n \in Z$ for $N \gg 0$. $\square$

This theorem can be rewritten in more invariant terms. For a module $V \in \mathcal{O}_k$, denote by V^{int} its maximal integrable quotient (it is easy to see that it is well-defined). Then the previous lemma immediately implies the following corollary.

COROLLARY 7.3.4. *Let $V_i \in \mathcal{O}^I_k NT$, and at least one of V_i is integrable. Then*

$$\tau(C,\vec{p},V_1 \otimes \cdots \otimes V_n) = \tau(C,\vec{p},V_1^{\text{int}} \otimes \cdots \otimes V_n^{\text{int}}).$$

COROLLARY 7.3.5. *Let $V = V_1 \cdots \otimes \mathtt{V_n}, \mathtt{V_i} \in \mathtt{O}^{\text{int}}_{\mathtt{k}}$. Then the embedding $\mathbb{C} = V_0 \hookrightarrow L^k_0$ induces an isomorphism*

$$\tau(C,\vec{p},V) \simeq \tau(C,\vec{p}\cup q, V \otimes L^k_0). \tag{7.3.4}$$

PROOF. This follows from Lemmas 7.3.2 and 7.3.3:

$$(V \otimes L^k_0)_{\mathfrak{g}(C-\vec{p}-q)} \simeq (V \otimes V^k_0)_{\mathfrak{g}(C-\vec{p}-q)} \simeq (V \otimes \mathbb{C})_{\mathfrak{g}(C-\vec{p})}.$$

$\square$

Having proved these results, we can prove now the following proposition.

PROPOSITION 7.3.6. *If $V = V_1 \cdots \otimes \mathtt{V_n}, \mathtt{V_i} \in \mathtt{O}^{\text{int}}_{\mathtt{k}}$, then the spaces of coinvariants $\tau(C,\vec{p},V)$ are finite dimensional.*

PROOF. We may assume that C is connected. Combining Lemma 7.3.2 and 7.3.3, we see that it suffices to prove the statement for $n = 1, V_1 = L^k_\lambda$. It follows from Theorem 7.2.2 that $\widehat{\mathfrak{g}}^+ + \mathfrak{g}(C-p) \supset \widehat{\mathfrak{g}}^+ + t^{-N}\widehat{\mathfrak{g}}^-$ for $N \gg 0$. Therefore, it suffices to prove that the vector space

$$W_N = L^k_\lambda / t^{-N}\widehat{\mathfrak{g}}^- V_\lambda$$

is finite-dimensional.

To prove this, note that one has a well-defined action of $\widehat{\mathfrak{g}}^{\leq 0} = \mathfrak{g}[t^{-1}]$ on W_N, which factors through the finite-dimensional quotient $\mathfrak{a} = \widehat{\mathfrak{g}}^{\leq 0}/t^{-N}\widehat{\mathfrak{g}}^{\leq 0}$. Obviously, $W_N = (U\mathfrak{a})v_{\lambda,k}$. On the other hand, $\mathfrak{a}$ is generated by e_α, f_α, $e_\alpha t^{-1}$, and all of these generators act nilpotently on W_N. Thus, all we need is to prove the following lemma.

LEMMA 7.3.7. *If $\mathfrak{a}$ is a finite-dimensional Lie algebra with generators $x_1, \ldots, x_n$, and W is a cyclic $\mathfrak{a}$-module such that the action of x_i in W is locally nilpotent, then W is finite-dimensional.*

To prove this lemma, we pass from the module W over $U\mathfrak{a}$ to the corresponding graded module $Gr\, W$ over $Gr(U\mathfrak{a}) = S(\mathfrak{a})$. Consider the variety $S = Supp(Gr\, W) \subset \mathfrak{a}^*$. Then it follows from the nilpotency condition that x_i, considered as a function on $\mathfrak{a}^*$, vanishes on S. By Gabber's integrability theorem [**Gab**], if x, y vanish on S, then $[x, y]$ also vanishes. Therefore, $S = \{0\}$. But every finitely generated module over the polynomial ring, which has a finite support, is finite-dimensional. This proves the lemma, and thus, the proposition. □

As an illustration, consider the simplest case $C = \mathbb{P}^1$.

PROPOSITION 7.3.8. *Let $C = \mathbb{P}^1$, $p_1, \ldots, p_n$—distinct points on C.*
(i) *Let $V^k_{\vec\lambda} = V^k_{\lambda_1} \otimes \ldots \otimes V^k_{\lambda_n}$, and $V_{\vec\lambda} = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$. Then the homomorphism*

$$(V_{\vec\lambda})_\mathfrak{g} \to \tau(C, \vec p, V^k_{\vec\lambda})$$

obtained by restricting the natural map $V^k_{\vec\lambda} \to V^k_{\vec\lambda}/\mathfrak{g}(C - \vec p)V^k_{\vec\lambda}$, is an isomorphism.

(ii) *Let z be a global coordinate on $\mathbb{P}^1$; assume that $z(p_i)$ is finite. Define the endomorphism $T\colon V_{\vec\lambda} \to V_{\vec\lambda}$ by*

$$T(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes z(p_i)\, e_\theta v_i \otimes \cdots \otimes v_n$$

Then one has an isomorphism

$$(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})_{\mathfrak{g} \oplus \mathbb{C}T^{k+1}} \simeq \tau(\mathbb{P}^1, \vec p, L^k_{\vec\lambda}).$$

PROOF. Part (i) is proved in the same way as Lemma 7.3.2, if we also note that for one point, $\mathfrak{g}(\mathbb{P}^1 - p) = \mathfrak{g} \oplus \widehat{\mathfrak{g}}^-$. As for part (ii), it can be deduced from the fact that $L^k_\lambda = V^k_\lambda / U\widehat{\mathfrak{g}}(e_\theta[-1])^{a+1} v_{\lambda,k}$. □

Let us relate this description with the one usually given in the physics literature. As before, let $C = \mathbb{P}^1$ with global coordinate z, and let the marked points be $0, z_1, \ldots, z_n, \infty$ with the local parameters $z, z - z_i, -1/z$ respectively. Let us assign to the points 0 and ∞ some $\mathcal{O}_k$-modules V_0, V_∞ respectively and assign to the points $z_1, \ldots, z_n$ Weyl modules $V^k_{\lambda_1}, \ldots, V^k_{\lambda_n}$. Then, by Lemma 7.3.2, we can replace in the definition of coinvariants $V^k_{\lambda_i}$ by V_{λ_i} and the algebra $\mathfrak{g}(\mathbb{P}^1 - \{0, z_i, \infty\})$ by $\mathfrak{g}(\mathbb{P}^1 - \{0,\infty\}) = \mathfrak{g}[z, z^{-1}]$. Thus

$$\begin{aligned} (7.3.5) \quad & \tau(\mathbb{P}^1, 0, z_1, \dots, z_n, \infty, V_0, \dots, V_\infty) \\ & = (V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \otimes V_\infty)/\big((x[n])_0 + \sum z_i^n x_i + (-1)^n (x[-n])_\infty\big) \end{aligned}$$

where $n \in \mathbb{Z}, x \in \mathfrak{g}$, and notation x_i means x acting on V_{λ_i}, etc. We can pass to the dual space τ^* which will be a subspace in

$$\mathrm{Hom}_{\mathbb{C}}(V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \otimes V_\infty, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}, \widehat{DV_\infty})$$

where $\widehat{W}$ is the completion of a $W \in \mathcal{O}_k$ with respect to the homogeneous grading. Rewriting the coinvariance condition, we get

$$(7.3.6) \quad \begin{aligned} \tau^* &= \{\Phi : V_0 \otimes V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \to \widehat{DV_\infty} \mid \Phi(x[n] + \sum z_i^n x_i) = x[n]\Phi\} \\ &= \mathrm{Hom}_{\mathfrak{g}[t,t^{-1}]}(V_0 \otimes V_{\lambda_1}(z_1) \otimes \dots V_{\lambda_n}(z_n), \widehat{DV_\infty}), \end{aligned}$$

where, as before, $V(z)$ is the evaluation representation.

For the case $\mathfrak{g} = \mathfrak{sl}_2, n = 1$ the dimensions of these spaces (which, as we will show below, play the role of multiplicity coefficients N^k_{ij} for the modular category $\mathcal{O}^{\mathrm{int}}_k$) were calculated in [**TK**]; their answer agrees with the formula for $U_q(\mathfrak{sl}_2), q = e^{\pi \mathrm{i}/(k+2)}$ given in (3.3.24)—as expected from Theorem 7.0.2.

REMARK 7.3.9. It is a natural question to generalize the definition of coinvariants, which can be viewed as Lie algebra homology in degree zero $H_0(\mathfrak{g}(C - \vec{p}), V)$ and consider all homology spaces $H_*(\mathfrak{g}(C-\vec{p}), V)$. To the best of out knowledge, this approach was first suggested by B. Feigin. One of the first results in this direction, proved in [**Tel**], is the vanishing theorem: if V_i are Weyl modules, then all higher homology vanish. In particular, this theorem allows one to calculate dimensions of the vector spaces of coinvariants $\tau(C, \vec{p}, L^k_{\vec{\lambda}})$, by writing for each of $L^k_{\lambda_i}$ a resolution consisting of Weyl modules, and then using the fact that for the Weyl modules, dimension of the space of coinvariants is known (see Lemma 7.3.8). This answer coincides with the dimension of the spaces of homomorphisms in the category of representations of quantum group at root of unity (see Proposition 3.3.23).

The meaning of the higher homology spaces ("higher conformal blocks") $H_i(\mathfrak{g}(C-\vec{p}), V)$ when V_i are integrable and the role they play in conformal field theory is still unclear.

7.4. Flat connection

In the previous section, we have defined and studied some properties of the vector spaces of coinvariants for a given curve C with marked points and chosen local parameters at these points. Now, let us study what happens with these spaces when we change the local parameters, or move the points. Let us assume that we have a smooth family of pointed curves $C_s, s \in S$ over a smooth base S. As mentioned above, it means that we have a smooth manifold C_S with a proper flat smooth morphism $\pi : C_S \to S$ such that each fiber $C_s = \pi^{-1}(s)$ is a complex curve; we also have n non-intersecting sections $p_i : S \to C_S$, and local parameters t_i, which are functions in a neighborhood of $p_i(S) \subset C_S$ such that $p_i(S)$ is the zero locus of t_i, and $dt_i \neq 0$ on $p_i(S)$. Such a data defines on each fiber a structure of a pointed complex curve, with a local parameter at each puncture; as before, we will assume taht on each connected component of C_s there is at least one marked point. Similarly to the construction of the previous section, it is convenient to allow t_i to be formal parameter, i.e. an element of the completed local ring $\widehat{\mathcal{O}}_{C_S,p_i(S)} \simeq \mathcal{O}_S[[t_i]]$.

We will denote by Θ_S the sheaf of vector fields on S. We will also denote by $\mathcal{O}(C_S - \vec{p}(S))$ the sheaf on S whose sections over $U \subset S$ are by definition meromorphic functions over $\pi^{-1}(U) \subset C_S$ which are regular outside of $p_i(S)$; when $S = \{point\}$, this coincides with the definition in the previous section. In a similar way, we define $\mathfrak{g}(C_S - \vec{p}(S)), \Theta(C_S - \vec{p}(S))$—all of them are sheaves on S.

Throughout this section, let us fix a family C_S as above, choose integrable $\widehat{\mathfrak{g}}$-modules $V_1, \dots, V_n \in \mathcal{O}_k^{\mathrm{int}}$, and let $V = V_1 \otimes V_n$. Then for every point $s \in S$ we can define the vector space of coinvariants

$$\tau_s = \tau(C_s, \vec{p}(s), V) = V/\mathfrak{g}(C_s - \vec{p}(s))V. \tag{7.4.1}$$

The main goal of this section is to prove the following theorem.

THEOREM 7.4.1. *Under the above assumptions, the vector spaces τ_s form a vector bundle τ_S over S which carries a natural projectively flat connection. The assignment $S \mapsto \tau_S$ is functorial in S: for every map $\psi\colon S' \to S$ and a family C_S over S as before, there is a canonical isomorphism $\tau_{S'} = \psi^*(\tau_S)$, where $C_{S'} := \psi^*(C_S)$.*

We remind that a connection is called projectively flat if $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is an operator of multiplication by a function for any two vector fields X, Y on S. The failure of the connection to be flat is, of course, related with the central term in the definition of $\widehat{\mathfrak{g}}$: for $k = 0$, the connection is flat (but of little interest, since the only integrable module of level 0 is $L_0^k = \mathbb{C}$). We will discuss this later.

The remaining part of this section is devoted to the construction of the flat connection and the proof of the theorem. For simplicity, we will assume that $n = 1$; the general case can be treated similarly. Our exposition follows [**BFM**] (somewhat simplified).

LEMMA 7.4.2. *The vector spaces τ_s form a $\mathcal{O}_S$-coherent sheaf over S, i.e., there exists a coherent sheaf τ_S such that $\tau_s = \tau_S/I_s\tau_S$, I_s being the ideal of functions vanishing at s.*

PROOF. Let $V_S = \mathcal{O}_S \otimes V$ (usual algebraic vector product, no completions); this is an $\mathcal{O}_S$-module, which carries an $\mathcal{O}_S$-linear action of the $\mathcal{O}_S$-module $\mathfrak{g}(C_S - p(S))$. Define the sheaf

$$\tau_S = V_S/\mathfrak{g}(C_S - p(S))V_S. \tag{7.4.2}$$

It is obvious that localizing τ_S at $s \in S$, we get the vector space of coinvariants τ_s. The coherency of τ_S can be proved in a way similar to the proof of finite-dimensionality of the spaces $\tau(C)$ in the previous section, using the following lemma.

LEMMA 7.4.3. *Let A be a finite-dimensional vector bundle of Lie algebras over S which is generated (as a Lie algebra) by sections $x_1, \dots, x_n$. Denote by $\mathcal{A}$ the sheaf of sections of A. Let $\mathcal{W}$ be an $\mathcal{O}_S$-module with an $\mathcal{O}_S$-linear action of $\mathcal{A}$. Assume that $\mathcal{W}$ is locally cyclic (i.e., locally there exists a section $w_0 \in \mathcal{W}$ such that $\mathcal{W} = \mathcal{A}w_0$) and action of x_i is locally nilpotent: for every section w, one has $x_i^N w = 0$ for $N \gg 0$. Then $\mathcal{W}$ is $\mathcal{O}_S$-coherent.*

To prove this lemma, it suffices to note that by Gabber's theorem, $Supp(\mathcal{W})$ is the zero section of the bundle A^*, and that every module over $\mathcal{O}_S[x_1, \ldots, x_m]$ whose support is given by $x_i = 0$, is $\mathcal{O}_S$-coherent. □

We will show that the sheaf τ_S has a natural structure of a twisted $\mathcal{D}_S$-module, i.e., a projective action of the sheaf Θ_S of vector fields on S which is compatible with the $\mathcal{O}_S$-module structure: $\xi(\phi\tau) = (\xi\phi)\tau + \phi(\xi\tau), \xi \in \Theta_S, \phi \in \mathcal{O}_S$. Since it is well known that every $\mathcal{O}$-coherent twisted $\mathcal{D}$-module is in fact a sheaf of sections of a vector bundle with a projectively flat connection, this will establish the theorem.

To construct an action of Θ_S on the sheaf of coinvariants, let us first consider the case when we have a fixed curve C with a marked point p, and S is the set of all possible choices of a formal local parameter t at p. This set has a natural structure of a projective limit of the smooth manifolds $S^{(N)} = \{N$-jets of local parameters at $p\}$. We have a tautological family of curves $C_S = C \times S$ over S, with the same marked point p and with the formal local parameter determined by $s \in S$.

This S is a torsor over the pro-Lie group (i.e., a projective limit of Lie groups) $K_0 = \operatorname{Aut} \mathbb{C}[[t]]$ of changes of local parameter. This group can be explicitly described as the group of power series of the form $a_1 t + a_2 t^2 + \ldots, a_1 \neq 0$, with the group operation being composition; it acts on the set of formal local parameters in an obvious way. The corresponding Lie algebra $\mathcal{T}_0 = \operatorname{Lie} K_0$ is given by $\mathcal{T}_0 = t\mathbb{C}[[t]]\partial_t$ (see [**TUY**, Section 1.4] for precise statements). Therefore, the tangent space to S at every point can be identified with $\mathcal{T}_0$. or, equivalently, $\mathcal{T}_0$ is the space of all K_0 left-invariant vector fields on S. Thus, to define an action of Θ_S on the bundle of coinvariants, one needs to define an action of $\mathcal{T}_0$.

Therefore, we see that the key step in this case would be to define an action of $\mathcal{T}_0 = t\mathbb{C}[[t]]\partial_t$ on V. In the general case, we will in fact need an action of a larger Lie algebra $\mathcal{T} = \mathbb{C}((t))\partial_t$, which is usually called the *Witt algebra*. It has a natural (topological) basis $L_n = -t^{n+1}\partial_t$, $n \in \mathbb{Z}$, with the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{7.4.3}$$

The subalgebra $\mathcal{T}_0$ is generated by L_n with $n \geq 0$. Similarly, we will also use the subalgebras $\mathcal{T}_1 = t^2\mathbb{C}[[t]]\partial_t$, $\mathcal{T}_{-1} = \mathbb{C}[[t]]\partial_t$ generated (as topological Lie algebras) by L_n with $n \geq 1$ (respectively, $n \geq -1$).

It is indeed possible to define a projective action of $\mathcal{T}$ on $\widehat{\mathfrak{g}}$-modules. This is known as the Sugawara construction. We formulate this result as a proposition, referring the reader to [**K1**] for details and the proof.

PROPOSITION 7.4.4. *One can define elements $L_n, n \in \mathbb{Z}$, in a certain completion of $U(\widehat{\mathfrak{g}})_k$ which have the following properties*:

(i) *In every module V from the category $\mathcal{O}_k$, the action of L_n is well-defined, and*

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c, \tag{7.4.4}$$

where

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee}. \tag{7.4.5}$$

(ii) *The operator L_n has degree n with respect to the homogeneous grading, and*

$$[L_n, a[m]] = -m\, a[m+n], \qquad a \in \mathfrak{g}. \tag{7.4.6}$$

(iii) *In the Weyl module* V_λ^k *(and thus, in* L_λ^k*), the operator* L_0 *acts by*

$$L_0 v = (\Delta_\lambda - \deg v)v, \qquad \Delta_\lambda = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(k + h^\vee)}. \tag{7.4.7}$$

Part (i) of this proposition can be reformulated as follows. Let

$$Vir = \mathbb{C}((t))\partial_t \oplus \mathbb{C}c \tag{7.4.8}$$

as before, this vector space has topological basis $c, L_n = -t^{n+1}\partial_t, n \in \mathbb{Z}$. We define the structure of Lie algebra on Vir by (7.4.4) (it can also be defined in a coordinate-free way, with the central term given as a residue of the $f'''g$). This algebra is called the *Virasoro algebra* and plays a central role in conformal field theory; by definition, it is a central extension of the Witt algebra $\mathbb{C}((t))\partial_t$. Thus, part (i) claims that every module $V \in \mathcal{O}_k$ is naturally a module over Vir with the central charge equal to $k \dim \mathfrak{g}/(k + h^\vee)$.

Note that when restricted to $\mathcal{T}_{-1} = \mathbb{C}[[t]]\partial_t$, the central term in (7.4.4) vanishes; thus, $\mathcal{T}_{-1}$ is a subalgebra in Vir and therefore acts on V. Hence, the same construction also defines an action $\mathcal{T}_0$ on V. Considering $\mathcal{T}_0$ as the Lie algebra of left-invariant vector fields on the set S of all choices of local parameter at p, one easily sees that this action can be uniquely extended to the action of the sheaf Θ_S of all vector fields on S on the sheaf $V_S = \mathcal{O}_S \otimes V$.

Let us now consider the general case, when not only the local parameter but also the the curve itself is allowed to vary.

First of all, let C be a complex curve, and t—a formal parameter at the point $p \in C$. Denote by $\Theta(C - p)$ the space of meromorphic vector fields on C which are holomorphic outside of p. Then we have a Lie algebra homomorphism $\gamma_p \colon \Theta(C - p) \to \mathcal{T}$ obtained by expanding a vector field in a neighborhood of p in power series in t. Similarly, if we have several marked points $p_1, \dots, p_n$, we can define a map

$$\gamma_{\vec{p}} = \bigoplus \gamma_{p_i} \colon \Theta(C - \vec{p}) \to \mathcal{T} \oplus \cdots \oplus \mathcal{T}. \tag{7.4.9}$$

On the other hand, Sugawara construction gives a projective action of the direct sum $\mathcal{T} \oplus \cdots \oplus \mathcal{T}$ on $V = V_1 \otimes \ldots \otimes V_n$; thus, we get a projective action of $\Theta(C - \vec{p})$ on V, which we will also denote by $\gamma_{\vec{p}}$.

LEMMA 7.4.5. (i) *The action of* $\Theta(C - \vec{p})$ *on* V*, given by* $\gamma_{\vec{p}}$*, is a true action, not a projective one.*

(ii) *The actions of* $\Theta(C - p)$ *and* $\mathfrak{g}(C - p)$ *on* V *agree as follows*:

$$[\gamma_{\vec{p}}(\xi), a \otimes f] = a \otimes \xi(f), \quad \xi \in \Theta(C - p), a \otimes f \in \mathfrak{g} \otimes \mathcal{O}_S.$$

(iii) *The induced action of* $\Theta(C - \vec{p})$ *on the space of coinvariants* $V_{\mathfrak{g}(C-\vec{p})}$ *is zero.*

PROOF. Part (i) follows from the fact that the central term in (7.4.4) can be written as a residue, and from the fact that the sum of residues of a meromorphic 1-form is equal to zero. The proof of part (ii) is immediate from (7.4.6). As for part (iii), the simplest way to prove it is to note that $\Theta(C - \vec{p})$ is a simple Lie algebra (see [**BFM**]), and therefore has no non-trivial finite-dimensional representations. Of course, this is a very artificial proof. A more natural proof can be obtained from the theory of chiral algebras. For readers familiar with this theory, we point out that the Sugawara construction in fact shows that the generating function $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a field in the vertex operator algebra (=chiral algebra on a formal punctured disk) generated by the Kac–Moody currents $a(z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$, $a \in \mathfrak{g}$ (see, e.g., [**K2**]); similarly, the Lie algebra $\Theta(C-p)$ is a subalgebra in the chiral algebra associated with the curve $C-p$. But since this chiral algebra is generated (in an appropriate sense) by the Kac–Moody currents, and these currents act on the space of coinvariants by zero, this whole chiral algebra acts by zero. Details can be found in [**Gai**]. $\square$

Part (iii) of the lemma may seem discouraging. Note, however, that what we are looking for is an action of Θ_S on the bundle of coinvariants, not an action of Θ_C, so we do not have a problem with the fact that $\Theta(C-p)$ acts by zero. In fact, it will be useful to us.

In order to define an action of Θ_S, we will first lift a vector field on S to a vector field on C_S, and then restrict to a formal neighborhood of p.

Let θ be a vector field on S. Let us lift it to a vector field $\tilde{\theta}$ on $C_S - p(S)$. Such a lifting is always possible, which follows from the fact that $\pi : C_S - p(S) \to S$ is affine, and therefore defines an exact functor on $\mathcal{O}$-coherent sheaves (this is where we need to allow poles at $p(S)$!).

Let us consider the vector field $\tilde{\theta}$ in a neighborhood of one of the sections $p_i(S)$ ("marked point"). Then the choice of local coordinate t_i allows us to define the notion of horizontal vector field: a vector field v in a punctured neighborhood of $p_i(S)$ is horizontal if $v(t) = 0$. Then we can define "vertical" component $\gamma_p(\tilde{\theta})$ by

$$\tilde{\theta} = \gamma_{p_i}(\tilde{\theta}) + \tilde{\theta}^{\text{horiz}}, \quad \tilde{\theta}^{\text{horiz}}(t) = 0.$$

Note that while one can easily define the notion of a vertical vector filed on C_S (v is vertical if its projection to S is zero), the notion of horizontal vector field, nad thus, of "vertical component" $\gamma_{p_i}(\tilde{\theta})$ depends on the choice of local parameter t_i. If we choose local coordinates x_i on S, so that $\theta = \sum f_i(x)\partial_{x_i}$, then (x_i, t) give a coordinate system in a neighborhood of $p_i(S)$, and we can write $\tilde{\theta} = g(x,t)\partial_t + \sum f_i(x)\partial_{x_i}$. Then $\gamma_{p_i}(\tilde{\theta}) = g(x,t)\partial_t$. The function $g(x,t)$ can have poles at $t_i = 0$, so it can be viewed as a local section of $\mathcal{O}_S((t_i))$, and thus $\gamma_{p_i}(\tilde{\theta}) \in \mathcal{O}_S \otimes \mathcal{T}$.

Repeating this for all points p_i, we define

$$\gamma_{\vec{p}}(\tilde{\theta}) = \sum \gamma_{p_i}(\tilde{\theta}) \in \mathcal{O}_S \otimes (\mathcal{T} \oplus \cdots \oplus \mathcal{T}) \tag{7.4.10}$$

(for $S = \{pt\}$, this coincides with the definition (7.4.9)).

Now, let us define the action of $\tilde{\theta}$ on $V_S = V \otimes \mathcal{O}_S$ by

$$\tilde{\theta}(fv) = (\theta(f))v + f\sum_i \gamma_{p_i}(\tilde{\theta})v,$$

where $\gamma_{p_i}(\tilde{\theta})$ acts on V_i by the Sugawara construction.

LEMMA 7.4.6. *The above defined action of $\tilde{\theta}$ on V_S has teh following properties:*

1. *It is compatible with the structure of $\mathcal{O}_S$-module: for $f \in \mathcal{O}_S, v \in V_S$, one has $\tilde{\theta}(fv) = (\theta(f))v + f\tilde{\theta}(v)$.*
2. *It is compatible with the action of $\mathfrak{g}(C_S - \vec{p}_S)$ on V_S: if $f \in \mathcal{O}_{C_S - \vec{p}(S)}, x \in \mathfrak{g}$, then $[\tilde{\theta}, fx] = (\tilde{\theta}(f))x$.*

PROOF. The first part immediately follows from the definition; the second one follows from Theorem 7.4.5(ii). $\square$

It immediately follows from part (ii) of this lemma that we have a well-defined action of $\tilde{\theta}$ on the bundle of coinvariants $\tau_S = V_S/\mathfrak{g}(C_S - \vec{p}_S)V_S$.

PROPOSITION 7.4.7. *The induced action of $\tilde{\theta}$ on the bundle of coinvariants depends only on θ and not on the choice of lifting $\tilde{\theta}$. It defines a projective action of the Lie algebra Θ_S on the bundle of coinvariants, which agrees with the structure of $\mathcal{O}_S$-module.*

PROOF. The only non-trivial statement is the independence of the choice of lifting. It follows from the fact that any two liftings differ by a vertical vector field. On the other hand, it follows from Theorem 7.4.5(iii) that vertical fields act by zero. □

This completes the proof of Theorem 7.4.1. □

More careful analysis also allows one to calculate explicitly the failure of the connection to be flat. Using the language of twisted $\mathcal{D}$-modules developed in Section 6.6 and the notion of determinant line bundle Q_S defined in Section 6.7, the result can be formulated as follows:

THEOREM 7.4.8. *Under the assumptions of Theorem* 7.4.1, *the sheaf τ_S carries a natural structure of a $\mathcal{D}_{Q^c}$-module, where c is the Virasoro central charge defined by* (7.4.5).

We do not give a proof of this theorem, referring the reader to [**BS**]. The proof is based on the fact that the central extension defining the Virasoro algebra can be defined using the action of the Lie algebra of vector fields on the space $\mathbb{C}((\vec{t})) = \oplus_i \mathbb{C}((t_i))$ and the "universal" cocycle defined by the the subspace $\mathbb{C}[[\vec{t}]] = \oplus_i \mathbb{C}[[t_i]] \subset \mathbb{C}((\vec{t}))$. This cocycle was first discovered by Tate [**Ta**] and rediscovered under different names by many authors (see [**BS**], [**ACK**]). On the other hand, it is well known that for a connected smooth curve C one has $\mathbb{C}((\vec{t}))/(\mathbb{C}[[\vec{t}]]+\mathcal{O}(C-\vec{p})) = H^1(C,\mathcal{O})$. This gives a relation between this cocycle and the determinant line bundle (recall that $Q_s = \det(H^1(C_s,\mathcal{O}))$). Details can be found in [**BS**] or [**BFM**].

EXAMPLE 7.4.9. Let us calculate this flat connection explicitly in the case when the curve C is fixed but the point p is allowed to move. Let u be a local coordinate on C, i.e. a biholomorphic map $u\colon C^0 \to U$, where C^0 is some open subset of C, and U an open subset of $\mathbb{C}$. We will denote by z a global coordinate on $\mathbb{C}$ and thus, on U. Let us define the following family of punctured curves over U: $C_U = C \times U$, $p(z) = u^{-1}(z)$, and the local parameter at p given by $t = u - z$ (considered as a function on $C \times U$). Note that both (z,u) and (z,t) can be considered as local coordinates on $C \times U$.

In this case, every vector field $f(z)\partial_z$ on U admits a canonical horizontal lifting to $C \times U$; in terms of the coordinate system (z,u) this lifting is given by $f(z)\partial_z \mapsto f(z)\partial_z + 0 \cdot \partial_u$. When we rewrite this in terms of (z,t), we get $f(z)(\partial_z - \partial_t)$. Therefore, the action of such a vector field on the bundle of coinvariants is given by $(f\partial_z)(\phi v) = f(\partial_z\phi)v + f\phi L_{-1}v$ (recall that $L_{-1} \in Vir$ corresponds to $-\partial_t$). In other words, the corresponding flat connection on U is induced from the connection on $V \otimes \mathcal{O}_S$ given by

$$\nabla = d + L_{-1}dz.$$

It is easy to see that for several points, we get

$$\nabla = d + \sum_i (L_{-1})_i dz_i, \tag{7.4.11}$$

where $(L_{-1})_i$ stands for L_{-1} acting in V_i.

Note that in this case every vector field on S can be lifted to a regular vector field on C_S. Therefore, we only need to use the Sugawara construction for the fields from $\mathbb{C}[[t]]\partial_t = \mathcal{T}_{-1}$. Since the central term in (7.4.4) vanishes when restricted to $\mathcal{T}_{-1}$, we get a true action, not a projective one.

Let us consider even more special case than in the previous example, namely when $C = \mathbb{P}^1$, with marked points $z_1, \dots, z_n \neq \infty$ and local parameters given by $t_i = z - z_i$. This defines a family of curves over $X_n = \mathbb{C}^n \setminus$diagonals. Assign to these points Weyl modules $V^k_{\lambda_1}, \dots, V^k_{\lambda_n}$. Then, by Proposition 7.3.8, the vector bundle of coinvariants $\tau(\mathbb{P}^1, z_1, \dots, z_n, V^k_{\lambda_1}, \dots, V^k_{\lambda_n})$ is a quotient of the trivial vector bundle with the fiber $(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})_{\mathfrak{g}}$ over X_n. Therefore, the construction above defines a flat connection in this quotient bundle. Passing to the dual vector bundle, we see get a flat connection in the vector subbundle

$$\big(\tau(\mathbb{P}^1, z_1, \dots, z_n; V^k_{\lambda_1}, \dots, V^k_{\lambda_n})\big)^* \subset \big(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}\big)^*_{\mathfrak{g}} = (V^*_{\lambda_1} \otimes \dots \otimes V^*_{\lambda_n})^{\mathfrak{g}}$$

Theorem 7.4.10 ([**KZ**]). *The flat connection described above coincides with the restriction of the KZ connection in $V^*_{\lambda_1} \otimes \dots \otimes V^*_{\lambda_n}$, defined by* (KZ$_n$).

A proof of this theorem can be found in the original paper [**KZ**] (only recommended for those familiar with the basics of conformal field theory). This proof is also repeated in a number of sources, for example, in [**EFK**], in a language more familiar to mathematicians. This theorem and comparison of the gluing isomorphisms, which we will do later, will be used to show that for $k \notin \mathbb{Q}$ the functor of coinvariants defined above for genus zero curves coincides with the modular functor defining Drinfeld's category—see Theorem 7.9.12. In particular, this modular functor can be defined in a way which doesn't refer to the affine Lie algebras at all. Note, however, that for $k \notin \mathbb{Q}$ this modular functor can not be extended to positive genus.

Example 7.4.11. Let $C, \vec{p}, \vec{t}$ be as before. Choose one of the points p_j and consider the family of curves $C \times \mathbb{C}^\times$ over $\mathbb{C}^\times$, with the the marked points $p_i(z) = p_i$ and local parameters $t_i(x, z) = t_i(x), x \in C, z \in \mathbb{C}$, except for $i = j$ when we set $t_i(x, z) = t_i(x)/z$. By the construction of this section, the corresponding vector bundle of coinvariants τ has a canonical flat connection. An easy calculation, similar to the one in Example 7.4.9, shows that this connection is induced from the connection

$$\nabla = d + (L_0)_j \frac{dz}{z}$$

in the trivial vector bundle with fiber $V_1 \otimes \dots \otimes V_n$. In particular, the monodromy of this connection around $z = 0$ is given by $e^{2\pi \mathrm{i} L_0}$, so if V_j is an irreducible module with highest weight λ, the monodromy operator is constant and equals $e^{2\pi \mathrm{i} \Delta_\lambda}$.

Note that if we pass from 1-jet of local parameter to tangent vector, we see that the tangent vector is given by $z\partial_{t_j}$, and thus, as z goes around the origin counterclocwise, so does the tangent vector. Recalling the relation between modular functor and tensor categories, we see that in the tensor category corresponding to the WZW modular functor, the universal twist is given by

$$\theta_{L^k_\lambda} = e^{2\pi i \Delta_\lambda} \operatorname{id}_{L^k_\lambda} \tag{7.4.12}$$

(compare with Remark 3.1.20), which agrees with the formulas for universal twist in Drinfeld's category (Theorem 2.2.7) and in the category of representations of a quantum group Exercise 2.2.6—which is another argument confirming equivalence of these categories.

In fact, this vector bundle on $\mathbb{C}^\times$ admits a canonical extension to a vector bundle on $\mathbb{P}^1$, and the connection has logarithmic singularities at $0, \infty$. Indeed, we can assume that $V_j = L^k_\lambda$. Denote $V = \otimes_{i\neq j} V_i$. The fiber of τ at point $z \in \mathbb{C}^\times$ is given by $\tau_z = W_z/\mathfrak{g}(C - \vec{p})W_z$, where $W_z = V \otimes L^k_\lambda$ does not depend on z. Note that the subspace $\mathfrak{g}(C-\vec{p})W_z$ depends on z, since the choice of local coordinate at p_j depends on z. Let us choose a different trivialization of the vector bundle $V \otimes L^k_\lambda$, namely, let us identify

$$\begin{aligned} V \otimes L^k_\lambda &\to (V \otimes L^k_\lambda)_z, \\ v \otimes v_j &\mapsto z^{\deg v_j} v \otimes v_j. \end{aligned}$$

In other words, in this trivialization constant sections are given by $z^{\deg v_j} v \otimes v_j$. Then one easily sees that in this trivialization, the subspace $\mathfrak{g}(C-\vec{p})W_z$ does not depend on z; thus, it also gives a trivialization of the vector bundle of coinvariants on $\mathbb{C}^\times$, and in this trivialization the flat connection is given by $\nabla = d + \Delta_\lambda dz/z$. Therefore, this gives an extension of our vector bundle with a flat connection to $\mathbb{P}^1$, and the connection has logarithmic singularities at $0, \infty$.

Note that for this definition of extension to $z = 0$, a function of the form $f(z)(v_1 \otimes \dots \otimes v_j \otimes \dots \otimes v_n)$ defines a section holomorphic at 0 iff $z^{-\deg v_j} f(z)$ is regular at $z = 0$ (we assume that v_j is homogeneous).

7.5. From local parameters to tangent vectors

In the previous section, we have studied properties of the vector spaces of coinvariants for a curve C with marked points and chosen local parameters at these points, or a family of such curves. In this section we will show that the vector space of coinvariants only depends on the 1-jet of local parameter: if t_i, t'_i are different choices of local parameter at p_i such that $d_{p_i} t_i = d_{p_i} t'_i$, then the vector spaces $\tau(C, \vec{p}, \vec{t}, \mathcal{L})$ and $\tau(C, \vec{p}, \vec{t'}, \mathcal{L})$ are canonically isomorphic, and similarly for families of curves.

Let us start with the case when we only have one curve C; as before, for simplicity we assume that it has only one marked point p. Let us fix a non-zero tangent vector $v \in T_pC$ and consider only such formal local parameters t at p that $\partial_v t = 1$; the set of formal local parameter form a pro-variety M. We want to show that for such local parameters t, the vector spaces $\tau(C, p, t, \mathcal{L})$ can be canonically identified. In order to do that, consider the family of curves $C_M = C \times M$ over M, with a marked point p (which does not depend on m) and the local parameter at $p \in C_m$ defined by $m \in M$. As discussed in the previous section, this defines a canonical flat connection on the bundle of coinvariants $\tau(C, p, t, \mathcal{L})$. We will show that this vector bundle with a flat connection is trivial. Indeed, it is easy to see that M is a torsor over the group

$$K_1 = \{k \in \operatorname{Aut}\mathbb{C}[[t]] \mid (k(t))'(0) = 1\}.$$

This group can be explicitly described as the group of all formal power series of the form $1 + \sum_2^\infty a_i t^i$, with the group operation being substitution of one series into another. The corresponding Lie algebra is $\operatorname{Lie} K = \mathcal{T}_1 = t^2\mathbb{C}[[t]]\partial_t$.

Now the triviality of the flat connection follows from the following two easy lemmas whose proofs are omitted.

LEMMA 7.5.1. *Let a manifold M be a torsor over a Lie group K, and E be a vector bundle with a flat connection over M. Then this flat connection is trivial iff the action of* $\operatorname{Lie} K$ *by vector fields on E can be lifted to an action of K on E.*

LEMMA 7.5.2. *The action of* $\operatorname{Lie} K_1 = \mathcal{T}_1$ *on an integrable module $\mathcal{L}$, defined by the Sugawara construction, can be integrated to an action of K_1 on $\mathcal{L}$.*

Combining these two lemmas, we get that in our case, the flat connection on the bundle of conformal blocks is trivial, and thus all the spaces $\tau(C, p, t, \mathcal{L})$ are canonically isomorphic. Therefore, we can define the space of coinvariants $\tau(C, p, v, \mathcal{L})$ as the space of global flat sections of the bundle $\tau(C, p, t, \mathcal{L})$ on M.

REMARK 7.5.3. Note that the action of $\mathcal{T}_0$ usually can not be integrated to the action of $\operatorname{Aut}\mathbb{C}[[t]]$. Indeed, in $\operatorname{Aut}\mathbb{C}[[t]]$ one has $e^{2\pi \mathrm{i} L_0} = 1$, but in a highest weight $\widehat{\mathfrak{g}}$ module with highest weight λ, one has

$$e^{2\pi \mathrm{i} L_0} =: \theta_\lambda = e^{2\pi \mathrm{i}\Delta_\lambda}$$

which is not equal to 1 unless $\Delta_\lambda \in \mathbb{Z}$. Therefore, we do need to specify a 1-jet of local parameter.

Now let us consider families of curves. Let $C_S, p(S)$ be a family of curves with a fixed 1-jet of local parameter t at $p(S)$. If we fix a formal local parameter t at $p(S)$ with given 1-jet, then, by the construction of the previous section, we get a vector bundle of coinvariants with a flat connection over S. Let us show that these vector bundles for different choices of t can be canonically identified.

Using the same idea as in the case $S = \{point\}$, consider the pro-variety $M = \{(s, t) \mid s \in S\}$; obviously, M is a principal K_1-bundle over S. The family C_S over S defines a family C_M over M and therefore defines a bundle of coinvariants τ_M with a flat connection over M. Our goal is to show that this flat connection is trivial along the fibers of the projection $M \to S$. A convenient framework for such proofs is provided by the formalism of Harish–Chandra pairs.

DEFINITION 7.5.4. A *Harish–Chandra pair* is a pair $(\mathfrak{g}, K)$, where $\mathfrak{g}$ is a Lie algebra, and K is a Lie group with the Lie algebra $\operatorname{Lie} K = \mathfrak{k} \subset \mathfrak{g}$. We also assume that we are given an action Ad of K on $\mathfrak{g}$ which agrees with both the standard Ad action of K on $\mathfrak{k}$ and ad action of $\mathfrak{k}$ on $\mathfrak{g}$.

As usual, we define a module V over a Harish–Chandra pair $(\mathfrak{g}, K)$ to be a vector space which has an action of both $\mathfrak{g}$ and K, and these actions agree on $\mathfrak{k}$.

These definitions can be suitably reformulated if we want to replace a Lie algebra $\mathfrak{g}$ by the sheaf of vector fields on a manifold M (or, more generally, by a Lie algebroid over M—see [**BFM**]). Let us assume that we have a manifold M with a free action of a Lie group K such that M is a principal K-bundle over a manifold S. We denote by $p : M \to S$ the projection. Denote by Θ_M the sheaf of vector fields on M. Then for every $U \subset M$, we have a natural embedding $\mathfrak{k} \subset \Theta_M(U)$, which is a Lie algebra homomorphism. We also have an adjoint action of K on Θ_X. Therefore, the pair (Θ_M, K) is a natural sheaf analogue of a Harish-Chandra pair.

DEFINITION 7.5.5. Let M, K, Θ_M be as above. A finite-dimensional (Θ_M, K)-*module* is a finite-dimensional vector bundle V with a flat connection over M with an action of K on V, which agrees an obvious sense with both the action of K on M and with the action of $\mathfrak{k} \subset \Theta_M$ by vector fields on V.

(A not necessarily finite-dimensional (Θ_M, K)-module can be defined in a similar way, replacing "vector bundle with a flat connection" by "$\mathcal{D}$-module.)

Our main reason in developing this technique is the following lemma.

LEMMA 7.5.6. *Any finite-dimensional (Θ_M, K)-module V defines a vector bundle with a flat connection V^K on $S = M/K$.*

PROOF. For every $s \in S$, define the vector space $V_s^K = (\Gamma(M_s, V))^K$, where $M_s = p^{-1}(s)$ is the fiber of the projection $p\colon M \to S$. It is easy to see that these vector spaces form a vector bundle over S of the same dimension as the original bundle V (it suffices to choose locally a section of the projection to show this). Note that any section ϕ of this bundle is killed by the vertical vector fields; thus, the quotient $\Theta_M/\Theta_M^v = \Theta_S$ acts on V^K. $\square$

Now we have all the prerequisites to prove the following theorem.

THEOREM 7.5.7. *Let C_S be a family of pointed curves over a smooth base S, and let $L_1^k, \dots, L_n^k$ be some integrable modules assigned to these points. Then we have a bundle of coinvariants τ_S over S which carries a natural projectively flat connection, and this bundle is functorial in S in the same sense as in Theorem* 7.4.1.

PROOF. Take $M = \{(s,t)\}, s \in S, t$–a local parameter at $p \in C_s$ with given differential. Obviously, M is a K^n–torsor over S, where $K = \mathrm{Aut}_1\, \mathbb{C}[[t]]$ and we have a tautological family C_M of curves over M with marked points and a local parameters at these points. By the construction of the previous section, this defines a vector bundle with a projectively flat connection over M. By Lemma 7.5.2, this connection is integrates to an action of K. Therefore, by Lemma 7.5.6, we have a flat connection on $S = M/K$. $\square$

COROLLARY 7.5.8. *For a fixed finite set A and a collection of modules $L_a^k \in \mathcal{O}_k^{\mathrm{int}}$, we have a vector bundle of coinvariants $\tau(\{L_a^k\})$ over the moduli stack $M_{*,A}$, which carries a natural projectively flat connection.*

As in Theorem 7.4.8, we can also explicitly describe the failure of the connection to be flat by saying that the sheaf of sections of the vector bundle $\tau(\{L_a^k\})$ is a $\mathcal{D}_{Q^c}$-module.

7.6. Families of curves over formal base

This section introduces some technical notions which will be used later for proving the gluing axiom for the WZW modular functor. Namely, we will generalize most of the results regarding the bundle of coinvariants to the case where the base is an infinitesimal neighborhood of a divisor D.

Throughout this section, we fix a non-singular variety S and a smooth divisor $D \subset S$. We also choose (locally) a function q on S such that the equation of D is $q = 0$, and $dq \neq 0$ on D. All our defintions and theorems will be local in S.

The main subject of this section is the study of the n-th infinitesimal neighborhood $D^{(n)}$ of D in S, where n is a fixed non-negative integer. As before, we will not really define $D^{(n)}$; instead, we will define the structure sheaf of $D^{(n)}$, $\mathcal{O}$-modules on $D^{(n)}$, family of curves over $D^{(n)}$, etc.

DEFINITION 7.6.1. The structure sheaf of $D^{(n)}$ is the sheaf of algebras $\mathcal{O}_D^{(n)}$ on D defined by $\mathcal{O}_D^{(n)} = \mathcal{O}_S/q^{n+1}\mathcal{O}_S$.

One also defines in an obvious way a notion of $\mathcal{O}_D^{(n)}$-module; it is called lisse if it is locally free module of finite rank. Every sheaf $\mathcal{F}$ over S defines a sheaf $\mathcal{F}^{(n)}$ over $D^{(n)}$ in an obvious way: $\mathcal{F}^{(n)} = \mathcal{F}_D/q^{n+1}\mathcal{F}_D$. It is easy to see that if $\mathcal{F}$ is $\mathcal{O}_S$-coherent, then $\mathcal{F}^{(n)}$ is finitely generated, and if $\mathcal{F}$ is lisse then so is $\mathcal{F}^{(n)}$. Unfortunately, the functor $\mathcal{F} \mapsto \mathcal{F}^{(n)}$ is not exact on $\mathcal{O}_S$-modules. However, we have the following result.

LEMMA 7.6.2. (i)*Let $\mathcal{F}$ be an $\mathcal{O}_S$-coherent sheaf such that its restriction to $S \setminus D$ is lisse and for every $n \geq 0$, $\mathcal{F}^{(n)}$ is lisse. Then $\mathcal{F}$ is lisse.*

(ii) *For every short exact sequence of quasicoherent $\mathcal{O}_S$-modules $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ such that $\mathcal{G}$ is $\mathcal{O}_S$-coherent, the sequence $0 \to \mathcal{E}^{(n)} \to \mathcal{F}^{(n)} \to \mathcal{G}^{(n)} \to 0$ is also exact.*

The proof of this lemma is left as an exercise to the reader.

EXAMPLE 7.6.3. Assume that $\dim S = 1$. Then $D =$ point, $\mathcal{O}_D^{(n)} = \mathbb{C}[q]/(q^{n+1})$, and $\mathcal{O}_D^{(n)}$ is just a module over this algebra.

We can also define vector fields and $\mathcal{D}$-modules for $D^{(n)}$. Note, however, that the only vector fields on S that can be restricted to $D^{(n)}$ are those tangent to D: the vector field ∂_q can not be restricted to $D^{(n)}$ as it does not preserve the relation $q^{n+1} = 0$. Thus, we can define an analogue of $\mathcal{D}_S^0$-module, but not of a $\mathcal{D}_S$-module (recall that $\mathcal{D}_S^0$ is generated by $\mathcal{O}_S$ and vector fields tangent to D, see (6.3.5)). Thus, we give the following definition:

$$\mathcal{D}^0_{D^{(n)}} = \mathcal{D}^0_S/q^{n+1}\mathcal{D}^0_S \tag{7.6.1}$$

For example, for $\dim S = 1$, $\mathcal{D}^0_{D^{(n)}}$ is generated by $\mathcal{O}_D^{(n)} = \mathbb{C}[q]/(q^{n+1})$ and $q\partial_q$.

Since a flat connection on S with logarithmic singularities at D is the same as a lisse sheaf on S with an action of $\mathcal{D}_S^0$, it is natural to give the following definition.

DEFINITION 7.6.4. A flat connection on $D^{(n)}$ with logarithmic singularities at D ($\log D$-connnection for short) is a lisse sheaf on $D^{(n)}$ with a structure of $\mathcal{D}^0_{D^{(n)}}$-module.

We have the following obvious lemma.

LEMMA 7.6.5. (i) *Every $\log D$ flat connection on S defines a $\log D$ flat connection $D^{(n)}$ by $\mathcal{F} \mapsto \mathcal{F}^{(n)}$*

(ii) *If the connection $\mathcal{F}$ is regular—i.e., has no poles at all—then $q\partial_q$ acts by zero in $\mathcal{F}^{(0)} = \mathcal{F}/q\mathcal{F}$.*

Now let us define families of curves over $D^{(n)}$ and the bundles of coinvariants.

DEFINITION 7.6.6. A family of curves over $D^{(n)}$ is the following collection of data:

– a family C_D of stable complex curves over D

– a sheaf of algebras $\mathcal{O}^{(n)}_{C_D}$ on C_D with a structure of a flat $\mathcal{O}^{(n)}_D$-module such that $\mathcal{O}^{(n)}_{C_D}/q\mathcal{O}^{(n)}_{C_D} = \mathcal{O}_{C_D}$.

The family is called *non-singular* if the family C_D is non-singular.

In a similar way, one can define a notion of families with marked points and local parameters at these points by adding to the data above a collection of points $p_i \in C_0$ and local parameters $t_i \in \mathcal{O}^{(n)}_{p_i}$ such that $t_i(p_i) = 0 \mod q, (dt_i)_{p_i} \neq 0 \mod q$. We can also define an analogue of the $\mathcal{O}_S$-sheaf $\mathcal{O}(C_S - \vec{p}(S))$. Namely, we define $\mathcal{O}^{(n)}_D$-module $\mathcal{O}^{(n)}(C - \vec{p})$ to be the space of global sections on C_D of the sheaf $\mathcal{O}^{(n)}_C[t_i^{-1}]$.

Obviously, every family of curves over S defines a family of curves over $D^{(n)}$: it suffices to take $\mathcal{O}^{(n)}_C = \mathcal{O}_{C_S}/q^{n+1}\mathcal{O}_{C_S}$; we will call this *restriction* of the family C_S to $D^{(n)}$). It turns out that if C_D is non-singular, then this statement can be reversed.

LEMMA 7.6.7. *Locally in S, every non-singular family of curves over $D^{(n)}$ can be obtained as a restriction of an analytic family of curves over a neighborhood of D in S.* ?!

Let us give an example of a singular family over $D^{(n)}$.

EXAMPLE 7.6.8. Let $\dim S = 1$, and let C_S be a family of curves over S such that C_s is smooth for $s \neq D$, and C_D is the curve with one double point a, so that in a neighbohood of a, C_S has local coordinates t_1, t_2 and the projection is given by $q = t_1t_2$; thus, C_0 is given by equation $t_1t_2 = 0$.

Let us describe the corresponding family of curves over $D^{(n)}$. In this case, the curve C_D is singular—it has double point a. To describe the sheaf $\mathcal{O}^{(n)}_C$, note that its stalk at a point $b \neq a$ is given by $\mathcal{O}^{(n)}_{C,b} \simeq \mathcal{O}_{C,b} \otimes \mathcal{O}^{(n)}_D$ (note: this doesn't define the sheaf yet, as we haven't defined the gluing maps—they depend on the map $\pi\colon C_S \to S$). However, the stalk at the double point is different:

$$\mathcal{O}^{(n)}_{C,a} = \mathcal{O}(t_1, t_2)/(t_1t_2)^{n+1}, \tag{7.6.2}$$

where $\mathcal{O}(t_1, t_2)$ is the ring of germs of analytic functions in t_1, t_2 near the origin $t_1 = t_2 = 0$.

To relate the stalk at the double point with the stalks at nearby points, let us describe $\mathcal{O}^{(n)}(U)$, where U is a punctured neighborhood of a in C_D. Since in a neighborhood of a, the curve C_D consists of two components given by equations $t_2 = 0$ and $t_1 = 0$, every small enough U can be presented as $U = U_1 \sqcup U_2$, where $U_1 = U \cap \{t_2 = 0\}, U_2 = U \cap \{t_1 = 0\}$. Thus, t_1 is a coordinate on U_1 and t_2 is a coordinate on U_2. From this it is easy to show that

$$\mathcal{O}^{(n)}(U_1) = \mathcal{O}(U_1) \otimes \mathcal{O}^{(n)}_D \simeq \mathcal{O}(U_1) \otimes (\mathbb{C}[t_2]/(t_2)^{n+1})$$

where the isomorphism is given by $f(t_1)q^k \mapsto f(t_1)t_1^kt_2^k$, and similarly for U_2. Thus:

$$\mathcal{O}^{(n)}(U) = \Big(\mathcal{O}(U_1) \otimes (\mathbb{C}[t_2]/(t_2)^{n+1})\Big) \oplus \Big(\mathcal{O}(U_2) \otimes (\mathbb{C}[t_1]/(t_1)^{n+1})\Big)$$

Now it is easy to see that for $f(t_1, t_2) \in \mathcal{O}^{(n)}_{C,a}$, its restriction to the punctured neighborhood of a is given by

$$t_1^k t_2^l \mapsto (t_1^k t_2^l) \oplus (t_1^k t_2^l) = (t_1)^{k-l} q^l \oplus (t_2)^{l-k} q^k \tag{7.6.3}$$

In particular, if $l > n$, then restriction of $t_1^k t_2^l$ to U_1 is zero, and if $k > n$, then restriction of $t_1^k t_2^l$ to U_2 is zero.

For every family C_S over S with marked points and modules $V_i \in \mathcal{O}_k^{\mathrm{int}}$ assigned to these points we have a sheaf of coinvariants $\tau(C_S)$ over S which gives rise to the sheaf $\tau^{(n)}$ over $D^{(n)}$; if C_S is a smooth family, then $\tau^{(n)}$ is lisse. It follows from Lemma 7.6.2(ii) that this module can be defined in terms of the n-th infinitesimal neighborhood of D, namely

$$\tau^{(n)} = V^{(n)} / \mathfrak{g}^{(n)}(C - \vec{p}) V^{(n)}, \tag{7.6.4}$$

where $\mathfrak{g}^{(n)}(C - \vec{p}) = \mathfrak{g} \otimes \mathcal{O}^{(n)}(C - \vec{p})$, and $V^{(n)} = V \otimes \mathcal{O}_D^{(n)}$.

Therefore, it is natural to take this formula as the definition of the sheaf of coinvariants for families over $D^{(n)}$.

PROPOSITION 7.6.9. *Let $C_{D^{(n)}}$ be a family of curves with marked points over $D^{(n)}$, with local parameters at these points, and integrable $\widehat{\mathfrak{g}}$-modules assigned to these points. Let $\tau^{(n)}$ be the $\mathcal{O}_D^{(n)}$-module defined by (7.6.4). Assume that C_D is nonsingular. Then $\tau^{(n)}$ is lisse and has a natural structure of a projective $\mathcal{D}^0_{D^{(n)}}$-module such that the action of $q\partial_q$ on $\tau^{(0)} = \tau^{(n)}/q\tau^{(n)}$ is zero.*

PROOF. By Lemma 7.6.7, such a family can be obtained as a restriction of some analytic family. Now existence of the flat connection and the fact that $\tau^{(n)}$ is lisse immediately follow from Theorem 7.4.1 and Lemma 7.6.7. To prove that $q\partial_q$ acts by zero on $\tau^{(0)}$, just note that for the analytic family, we have a well-defined action of ∂_q, and thus $q\partial_q = 0 \mod q$.

It is also important to note that the structure of $\mathcal{D}^0_{D^{(n)}}$-module can be defined completely in terms of $D^{(n)}$, without extending this to a family on S. Let $\Theta^{(n)}(C - \vec{p})$ be the space of global sections (on C_D) of the sheaf of derivations of $\mathcal{O}^{(n)}(C - \vec{p})$—this is the infinitesimal analogue of the algebra of vector fields. Then we can lift any vector field θ on S which is tangent to D—in particular, the vector field $q\partial_q$—to a "vector field" $\tilde{\theta} \in \Theta^{(n)}(C - \vec{p})$. The easiest way to prove this is to use Lemma 7.6.7.

As in the analytic case (see proof of Theorem 7.4.1), define the action of θ on the bundle of coinvariants by

$$\theta(fv) = (\theta(f))v + f \sum_i \gamma_{p_i}(\tilde{\theta})(v).$$

The same arguments as in Theorem 7.4.1 show that this is indeed defines the structure of a projective $\mathcal{D}^0_{D^{(n)}}$-module on the sheaf of coinvariants. □

7.7. Coinvariants for singular curves

In this section, we give a description of the vector space $\tau(C, \vec{p}, V)$ for a singular curve C. This description will be used in the next section to prove that the bundle of conformal blocks satisfies the gluing axiom and in particular has regular singularities on the boundary of the moduli space.

Let $C, \vec{p}, \vec{t}$ be stable singular curve with marked points and local parameters at these points. Choose modules $V_1, \dots, V_n \in \mathcal{O}_k^{\text{int}}$ assigned to these points. We define the space of coinvariants $\tau(C, \vec{p}, V)$ (or, for brevity, $\tau(C, V)$) by the same formula as for non-singular curves (see Definition 7.3.1). For simplicity, let us only consider the case when C has only one double point; general case is completely parallel.

Denote by $C^\vee$ the *normalization* of C, i.e. the non-singular curve such that C is obtained by identifying points $a', a'' \in C$. Let us choose the local coordinates t', t'' near a', a''.

Theorem 7.7.1. *The map*

$$V \to \bigoplus_\lambda V \otimes (L_\lambda^k \otimes DL_\lambda^k)$$

$$v_1 \otimes \dots \otimes v_n \mapsto \bigoplus v_1 \otimes \dots \otimes v_n \otimes 1_\lambda,$$

where DL_λ^k is defined as in Section 7.1, and $1_\lambda \in V_\lambda \otimes V_\lambda^* \subset L_\lambda^k \otimes DL_\lambda^k$ is the canonical $\mathfrak{g}$-invariant vector, induces an isomorphism of the spaces of coinvariants

$$\tau(C, V) \simeq \bigoplus_{\lambda \in P_+^k} \tau(C^\vee, V \otimes L_\lambda^k \otimes DL_\lambda^k)$$

with the modules $L_\lambda^k, DL_\lambda^k$ assigned to the points a', a'' respectively.

Proof. The basic observation is that $\mathcal{O}(C - \vec{p}) = \{f \in \mathcal{O}(C^\vee - \vec{p}) \mid f(a') = f(a'')\}$. Therefore,

$$\mathfrak{g}(C - \vec{p}) = \{f \in \tilde{\Gamma} \mid (\gamma_{a'} \oplus \gamma_{a''})f \in \left(\widehat{\mathfrak{g}}^+ \oplus \widehat{\mathfrak{g}}^+ \oplus \Delta(\mathfrak{g})\right) \subset \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}\} \tag{7.7.1}$$

where $\Delta(\mathfrak{g}) = \{x \oplus x\}, x \in \mathfrak{g}$, $\tilde{\Gamma} = \mathfrak{g}(C^\vee - \vec{p} - a' - a'')$.

Next, let us define the $U\widehat{\mathfrak{g}}_k \otimes U\widehat{\mathfrak{g}}_k$-module U as follows:

$$U = \operatorname{Ind}_{\tilde{U}}^{U\widehat{\mathfrak{g}}_k \otimes U\widehat{\mathfrak{g}}_k} \mathbb{C}\mathbf{1}$$

where $\tilde{U} \subset U_k\widehat{\mathfrak{g}} \otimes U_k\widehat{\mathfrak{g}}$ is the subalgebra generated by $\widehat{\mathfrak{g}}^+ \otimes 1, 1 \otimes \widehat{\mathfrak{g}}^+, x \otimes 1 + 1 \otimes x, x \in \mathfrak{g}$, which acts trivially on $\mathbb{C}$:

$$(\widehat{\mathfrak{g}}^+ \otimes 1)\mathbf{1} = (1 \otimes \widehat{\mathfrak{g}}^+)\mathbf{1} = (a \otimes 1 + 1 \otimes a)\mathbf{1} = 0. \tag{7.7.2}$$

(By Poincare-Birkhoff-Witt theorem, U is isomorphic to $U\mathfrak{g} \otimes (U\widehat{\mathfrak{g}}^-)^{\otimes 2}$ as a graded vector space.)

Since U is a $(U(\widehat{\mathfrak{g}})_k)^{\otimes 2}$-module, we can define the space of coinvariants $\tau(C^\vee, \vec{p} \cup a' \cup a'', V \otimes U)$.

Lemma 7.7.2. *The map $v \mapsto v \otimes \mathbf{1}$ is an isomorphism $\tau(C, V) \xrightarrow{\sim} \tau(C^\vee, V \otimes U)$.*

The proof of this lemma is more or less standard: one has to check that this map is well-defined, which follows from (7.7.2); injectivity follows from the fact that U is free over $U\widehat{\mathfrak{g}}^- \otimes U\widehat{\mathfrak{g}}^-$. Proof of surjectivity is is only slightly more difficult: it suffices to prove that for every $v \in V, u \in u$ one can find $v' \in V$ such that $v \otimes u \equiv v' \otimes \mathbf{1} \mod \operatorname{Im} \tilde{\Gamma}$. It follows from the fact that for every $a \oplus b \in \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}, u \in u$ there exists a function $f \in \Gamma$ such that $(\gamma_{q'} \oplus \gamma_{q''})(f)u = au$, and therefore $v \otimes (a \oplus b)u \equiv -(\gamma_{\vec{p}} f)v \otimes u$.

Lemma 7.7.3. *Maximal integrable quotient of U is equal to $\bigoplus_{\lambda \in P_+^k} L_\lambda^k \otimes DL_\lambda^k$.*

Indeed, let us define the homomorphism of $(U(\widehat{\mathfrak{g}})_k)^{\otimes 2}$ modules $\pi : U \to \bigoplus L^k_\lambda \otimes DL^k_\lambda$ by $1 \mapsto \bigoplus 1_\lambda$ (since U is the induced module, this uniquely defines π). It is an easy exercise to show that $\bigoplus 1_\lambda$ is a cyclic vector in $\bigoplus L^k_\lambda \otimes DL^k_\lambda$ (with respect to the action of $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$), and therefore, the above map is surjective; thus, $\bigoplus L^k_\lambda \otimes DL^k_\lambda$ is indeed an integrable quotient of U. On the other hand, every integrable $(U(\widehat{\mathfrak{g}})_k)^{\otimes 2}$-module is of the form $\bigoplus_{\lambda,\mu \in P^k_+} N_{\lambda\mu} L^k_\lambda \boxtimes L^k_\mu$. Since U is generated by a vector 1 which is $\Delta(\mathfrak{g})$ invariant, it easily follows that any integrable quotient of U must have $N_{\lambda,\mu} \le \delta_{\lambda,\mu^*}$. Details are left to the reader.

These two lemmas, combined with Lemma 7.3.3, give the proof of the theorem. $\square$

7.8. Bundle of coinvariants for a singular family

In this section, we continue the study of coinvariants for singular curves. This time, we will consider a family of pointed curves C_S over a smooth base S such that C_s is stable and non-singular for $S \setminus D$, and C_s is a stable singular curve with one double point for $s \in D$, where D is a smooth divisor in S (without loss of generality we may assume that D is connected). As before, we assume that we have some integrable modules $V_1, \dots, V_n$ assigned to the marked points $p_1, \dots, p_n$. Then, by the construction of the previous sections, this data defines a vector bundle of coinvariants $\tau = \tau(C_S, \vec{p}, V)$ over $S \setminus D$.

Let us extend τ to the whole of S as an $\mathcal{O}$-module. Define the sheaf τ on S in the obvious way, as in Lemma 7.4.2. The restriction of this sheaf to $S \setminus D$ is lisse, and its fiber at a point $s \in D$ is the vector space $\tau(C_s, \vec{p}, V)$ which was discussed in the previous section. The same arguments as before show that τ_S is $\mathcal{O}_S$-coherent sheaf. The goal of this section is to prove the following theorem, which is the key step in proving the gluing axiom.

THEOREM 7.8.1. *Under the assumptions above, the sheaf τ_S is lisse.*

The remaining part of this section is devoted to the proof of this theorem. Note that by Theorem 7.4.1, the restriction of τ to $S \setminus D$ is lisse, so the only problem is analyzing the behavior of τ at D.

PROOF. The proof consists of several steps. The main idea is to use the results of the previous section, relating coinvariants for the singular fibers $C_s, s \in D$ with the coinvariants for nonsingular curve $C^\vee_s$ obtained by normalization of C_s, and extend it to an isomorphism of sheaves of coinvariants in some neighborhood of D. Unfortunately, it is impossible to do this directly: we can not extend $C^\vee$ to a family of nonsingular curves $C^\vee_S$ over S with a natural map $C^\vee_S \to C_S$. However, this becomes possible if instead of constructing a family over S we restrict ourselves to an infinitesimal neighborhood of D, as defined in Section 7.6, which is sufficient for our purposes. For simplicity, we will assume that S is a disk in the complex plane with coordinate q and $D = \{0\}$. The general case can be treated quite similarly; however, it is not even necessary to do that due to Lemma 6.3.13. We will choose coordinates t_1, t_2 in the neighborhood of the double point $a \in C_S$ such that $t_1 t_2 = q$ (this is always possible).

By Lemma 7.6.2, it suffices to prove that for every $n \ge 0$, the module $\tau^{(n)}$ over $\mathcal{O}^{(n)}_D$ defined by (7.6.4) for our family of curves is free of finite rank.

In order to prove that $\tau^{(n)}$ is free over $\mathcal{O}^{(n)}_D$, let us construct another family $C^\vee$ of curves over $D^{(n)}$. Namely, take $C^\vee_0$ to be the normalization of C_0; this is a nonsingular curve with the same marked points as C_0, plus two more marked points which we denote a', a''. The choice of local coordinates t_1, t_2 on C_S defines local coordinates t_1, t_2 in the neighborhood of $a' \in C^\vee$ (respectively, a'').

Now, let us define the sheaf $\mathcal{O}^{(n)}_{C^\vee}$ as follows. Let $U = C_0^\vee \backslash \{a', a''\} = C_0 \backslash \{a\}$. By definition, let $\mathcal{O}^{(n)}_{C^\vee}|_U = \mathcal{O}^{(n)}_C|_U$. To extend it to the points a', a'', define the stalks $\mathcal{O}^{(n)}_{a'} = \mathcal{O}(t_1) \otimes \mathcal{O}^{(n)}_D$, where $\mathcal{O}(t_1)$ is the ring of germs of analytic functions in t_1 in a neighborhood of $t_1 = 0$, and similarly for a''. Obviously, each $f \in \mathcal{O}^{(n)}_{a'}$ also defines a section of $\mathcal{O}^{(n)}_{C^\vee}|_U$ on some punctured neighborhood of a' by $t_1 \mapsto t_1, q \mapsto t_1 t_2$, and thus we can glue the sheaf $\mathcal{O}^{(n)}_{C^\vee}$ from its restriction to U and stalks at a', a''. This defines on $C^\vee$ a structure of a family of curves over $D^{(n)}$; this family is non-singular.

Now let us assign the modules $L^k_\lambda, DL^k_\lambda$ to the points a', a'' and take direct sum over all $\lambda \in P^k_+$. By Proposition 7.6.9, this defines a lisse module $\tau^{\vee(n)}$ over $\mathcal{O}^{(n)}_D$.

PROPOSITION 7.8.2. *The map*

$$\begin{aligned} \phi\colon V^{(n)} &\to V^{(n)} \\ v &\mapsto \sum_{\lambda,i} q^{-\deg e_{\lambda,i}} v \otimes e_{\lambda,i} \otimes e^*_{\lambda,i}, \end{aligned} \tag{7.8.1}$$

where $e_{\lambda,i}$ is a homogeneous basis in L^k_λ, and $e^*_{\lambda,i}$ is the dual basis in DL^k_λ, induces an isomorphism of $\mathcal{O}^{(n)}_D$-modules $\tau^{(n)} \to \tau^{\vee(n)}$.

PROOF. First of all, we have to check that this map descends to the bundle of coinvariants. To do this, note that it is immediate from the definition that we have an embedding $A : \mathcal{O}^{(n)}(C - p) \hookrightarrow \mathcal{O}^{(n)}(C^\vee - p - a' - a'')$. Near the double point this map is given by

$$\begin{aligned} \mathcal{O}^{(n)}(C-p) &\to \Big(\mathbb{C}((t_1))[[q]] \oplus \mathbb{C}((t_2))[[q]]\Big)/(q^{n+1}) \\ t_1^k t_2^l &\mapsto t_1^{k-l} q^l \oplus t_2^{l-k} q^k \end{aligned}$$

(compare with (7.6.3)). We leave it to the reader to check that in fact the image of this embedding is analytic functions.

It is also easy to show by explicit calculation that the vector

$$w_\lambda = \sum_i q^{-\deg e_{\lambda,i}} e_{\lambda,i} \otimes e^*_{\lambda,i} \in (L^k_\lambda \otimes DL^k_\lambda)^{(n)} \tag{7.8.2}$$

is invariant under the image of the embedding

$$\mathfrak{g}[[t_1, t_2]]/(t_1 t_2)^{n+1} \to \Big(\mathfrak{g}((t_1))[q] \oplus \mathfrak{g}((t_2))[q]\Big)/q^{n+1}.$$

Indeed, it suffices to show this for $xt_1^n t_2^m, x \in \mathfrak{g}$. In this case, it follows from the following sequence of identities:

$$\begin{aligned}
(x[n-m]q^m \otimes 1 &+ 1 \otimes x[m-n]q^n)w_\lambda \\
&= (x[n-m]q^m \otimes 1 + 1 \otimes x[m-n]q^n)(q^{-d} \otimes 1)\sum_i e_{\lambda,i} \otimes e^*_{\lambda,i} \\
&= (q^{-d} \otimes 1)(x[n-m]q^n \otimes 1 + 1 \otimes x[m-n]q^n)\sum_i e_{\lambda,i} \otimes e^*_{\lambda,i} \\
&= (q^{-d+n} \otimes 1)(x[n-m] \otimes 1 + 1 \otimes x[n-m])1 \\
&= 0,
\end{aligned}$$

where $1 = \sum_i e_{\lambda,i} \otimes e^*_{\lambda,i}$ is considered as a vector in a certain completion of $L^k_\lambda \otimes (L^k_\lambda)^*$. Note that in the last line we replaced DL^k_λ by $(L^k_\lambda)^*$, which resulted in
?! replacing $x[m-n]$ by $x[n-m]$—see (7.1.4). We leave it to the reader to check that the fact that 1 does not lie in $L^k_\lambda \otimes (L^k_\lambda)^*$ but only in some completion does not cause any problems.

Therefore, if $f \in \mathfrak{g}^{(n)}(C_S - \vec{p}), v \in V$, then $\phi(f(v)) = A(f)\phi(v)$ and thus the map ϕ descends to the space of coinvariants; we will denote the corresponding map also by ϕ.

Now the proof of proposition is easy. Indeed, we have a morphism of $\mathcal{O}^{(n)}_D$-modules $\phi : \tau^{(n)} \to \tau^{\vee(n)}$. By Theorem 7.7.1, ϕ induces an isomorphism on the fibers at zero $\tau^{(n)}/q\tau^{(n)} \xrightarrow{\sim} \tau^{\vee(n)}/q\tau^{\vee(n)}$. Since $\tau^{\vee(n)}$ is free over $\mathcal{O}^{(n)}_D$, this immediately implies that ϕ is surjective. To prove that ϕ is injective, choose a basis $v_1, \dots, v_k$ in $\tau^{(n)}/q\tau^{(n)}$. Since $\tau^{\vee(n)}$ is free, this implies that $v_1, \dots, v_k$ are linearly independent over $\mathcal{O}^{(n)}_D$. On the other hand, it follows from the definition that the module $K = \tau^{(n)}/\langle v_1, \dots, v_k\rangle$ satisfies $qK = K$; since $q^{n+1} = 0$, this implies $K = 0$. Thus, $\tau^{(n)}$ is freely generated by $v_1, \dots, v_k$. Therefore, ϕ is an isomorphism, which completes the proof of the proposition. □

Since by Proposition 7.6.9 the sheaf $\tau^{\vee(n)}$ is lisse, this proposition implies that the same holds for $\tau^{(n)}$ and thus completes the proof of Theorem 7.8.1. □

7.9. Proof of the gluing axiom

In this section we give a proof of the gluing axiom for the WZW modular functor. Recall that this axiom describes the behaviour of the bundle of coinvariants in a neighborhood of the boundary of the moduli space; in particular, it claims that the connection has first regular singularities at the boundary, and describes the specialization of this connection.

Recall that the boundary of the moduli space consists of the stable curves with ordinary double points (see Section 6.2) and that it suffices to check the regularity condition for an open part of the boundary. Thus, we need to prove regularity and calculate specialization of the connection in τ_S, where $S, C_S, D, \dots$ are same as in the beginning of the previous section. By the construction of the previous sections, τ_S carries a natural projectively flat connection over $S \setminus D$. Also, we have shown in the previous section that τ_S is lisse, i.e., is a sheaf of sections of a vector bundle on S.

THEOREM 7.9.1. *Under the assumptions above, the connection in τ_S has logarithmic singularities at D.*

PROOF. As before, choose a local coordinate q in a neighborhood of D such that $q = 0$ is the equation of D. Recall (see (6.3.5)) that $\mathcal{D}_S^0 \subset \mathcal{D}_S$ be the subsheaf generated (as sheaf of algebras) by $\mathcal{O}_S$ and vector fields which are tangent to D.

PROPOSITION 7.9.2. *The sheaf τ has a natural structure of a $\mathcal{D}_S^0$-module.*

This proposition is a generalization of Theorem 7.4.1, and is proved in the same way. The only change is that instead of claiming that any vector field on S can be lifted to a vector field on $C_S - \vec{p}(S)$, we use the following lemma.

LEMMA 7.9.3. *Let θ be vector field on S which is tangent to D. Then locally in S, such a field can be lifted to a vector field on C_S which has poles at the marked points.*

EXAMPLE 7.9.4. Let S be a neighborhood of zero in $\mathbb{C}$, with coordinate q, $D = \{0\}$. As before, introduce coordinates t_1, t_2 near the double point in C_S such that $q = t_1 t_2$. Then in the neighborhood of the double point, the lifting of the vector field $q\partial_q$ must be of the form $\alpha t_1 \partial_{t_1} + \beta t_2 \partial_{t_2}$ for some α, β satisfying $\alpha + \beta = 1$.

This proposition, along with the fact that τ_S is lisse, immediately implies the statement of the theorem. $\square$

EXAMPLE 7.9.5. Let S be a neighborhood of zero in $\mathbb{C}$, with coordinate q. Define the family $C_S \subset \mathbb{C}P^2 \times S$ by the equation

$$uv = qw^2, \qquad (u : v : w) \in \mathbb{C}P^2, \quad q \in S$$

with the marked points $p_1(q) = (1 : 0 : 0), p_2(q) = (0 : 1 : 0)$, and local parameters at these points $t_1 = w/u, t_2 = w/v$. The same argument as in Example 6.2.4 shows that for $q \neq 0$, the curve C_q is isomorphic to a sphere $\mathbb{P}^1$, with marked points $p_1 = 0, p_2 = \infty$ and local parameters $z, 1/z$ respectively. For $q = 0$, the fiber C_0 consists of two components, each of them isomorphic to a sphere $\mathbb{P}^1$, with coordinates $z' = u/w, z'' = v/w$ respectively, which have one common point $z' = z'' = 0$. The marked points p_1 and p_2 are the points ∞', ∞'' —infinite points of the first and the second spheres respectively, with local coordinates $t_1 = 1/z', t_2 = 1/z''$ respectively.

It is easy to see that any vector field of the form

$$\tilde{v} = \alpha u \partial_u + \beta v \partial_v + q\partial_q, \quad \alpha + \beta = 1$$

defines a vector field on C_S which is a lifting of the vector field $q\partial_q$ on S. Rewriting $\tilde{v}$ in terms of coordinates t_1, q, we get $\tilde{v} = -\alpha t_1 \partial_{t_1} + q\partial_q$, and thus $\gamma_{p_1}(\tilde{v}) = \alpha L_0$. Similarly, expansion near p_2 gives $\gamma_{p_2}(\tilde{v}) = \beta L_0$. Therefore, the action of $q\partial_q$ on coinvariants is given by $\alpha(L_0)_{p_1} + \beta(L_0)_{p_2}$.

This statement also has an infinitesimal analogue. Recall the notation $\tau^{(n)} = \tau_S / q^{n+1}\tau_S$ (see the previous section). This is a lisse $\mathcal{O}_D^{(n)}$-module. It immediately follows from Proposition 7.9.2 that $\tau^{(n)}$ has a natural action of the sheaf of algebras $\mathcal{D}^0_{D^{(n)}} = \mathcal{D}_S^0 / q^{n+1}\mathcal{D}_S^0$.

Similar result also holds for the sheaf $\tau^{\vee(n)}$ described in the previous section: it follows from Proposition 7.6.9 that $\tau^{\vee(n)}$ has a natural structure of a projective $\mathcal{D}^0_{D^{(n)}}$-module. Let us twist this action, defining a new action of $q\partial_q$ by adding to the old action the constant Δ_λ, defined by (7.4.7) (cf. Example 7.4.11). We will denote this new action by $\nabla^\vee$.

Note that a lifting of the vector field $q\partial_q$ to $C^\vee_{D^{(n)}}$ can be explicilty described as follows: lift $q\partial_q$ to a derivation $\tilde{v}$ of $\mathcal{O}^{(n)}(C-\vec{p})$; as was discussed in Example 7.9.4, this lifting in a neighborhood of the double point has the form $\alpha t_1\partial_{t_1}+\beta t_2\partial_{t_2}, \alpha+\beta=1$. Define $v^\vee$ by $v^\vee=\tilde{v}$ on $C^\vee\setminus\{a',a''\}=C_0\setminus\{a\}$, and $v^\vee=\alpha t_1\partial_{t_1}+q\partial_q$ at a'; similarly, let $\tilde{v}=\beta t_2\partial_{t_2}+q\partial_q$ at a''. It is easy to check that this defines an element of $\Theta^{(n)}(C^\vee-\vec{p})$.

EXAMPLE 7.9.6. Under the assumptions of Example 7.9.5, the lifting of the vector field $q\partial_q$ is given by $v^\vee=\alpha z'\partial_{z'}+q\partial_q$ on the first component, and by $v^\vee=\beta z''\partial_{z''}+q\partial_q$ on the second one. Therefore, its action on the bundle of coinvariants is given by

$$\nabla^\vee_{q\partial_q}=q\partial_q+\alpha\big((L_0)_{p_1}-(L_0)_{a'}\big)+\beta\big((L_0)_{p_2}-(L_0)_{a''}\big)+\Delta_\lambda. \tag{7.9.1}$$

PROPOSITION 7.9.7. *The isomorphism* $\phi\colon \tau^{(n)}\to\tau^{\vee(n)}$, *defined by* (7.8.1), *is an isomorphism of* $\mathcal{D}^0_{D^{(n)}}$*-modules.*

PROOF. It suffices to check that ϕ commutes with the action of the vector field $q\partial_q$. To prove this, it suffices to check that

$$\nabla^\vee_{q\partial_q}(v\otimes w_\lambda)=(\nabla_{q\partial_q}v)\otimes w_\lambda$$

where w_λ was defined in (7.8.2). But this is immediate from the definition of $\nabla^\vee$:

$$\begin{aligned}\nabla^\vee_{q\partial_q}(v\otimes w_\lambda)-(\nabla_{q\partial_q}v)\otimes w_\lambda&=v\otimes(q\partial_q-\alpha(L_0)_{a'}-\beta(L_0)_{a''}+\Delta_\lambda)w_\lambda\\&=v\otimes(-d+\Delta_\lambda-\alpha(L_0)_{a'}-\beta(L_0)_{a''})w_\lambda\\&=0.\end{aligned}$$

□

Now let us calculate the specialization of the connection in τ_S. Let us recall the definition of the specialization functor, slightly modifying it for our needs. As in Chapter 6, assume that (F,∇) is flat connection with first order poles at D. As before, we denote by $\mathcal{F}$ the sheaf of sections of F, and $\mathcal{F}^{(0)}=\mathcal{F}/q\mathcal{F}$. $\mathcal{F}^{(0)}$ is a sheaf on D which has a natural action of the sheaf of algebras $\mathcal{D}^{(0)}_D=\mathcal{D}^0_S/q\mathcal{D}^0_S$. It turns out that the specialization Sp_DF can be defined using only $\mathcal{F}^{(0)}$ as follows.

LEMMA 7.9.8. *Let* $(G,\tilde{\nabla})$ *be a vector bundle on the normal bundle* ND *with a monodromic* $\log D$ *flat connection, and let* i *be a homeomorphism identifying a neighborhood of* D *in* S *with a neighborhood of* D *in* ND, *as in* (6.2.8). *Then an isomorphism of vector bundles with connections* $Sp_DF\to G$ *is the same as an isomorphism of* $\mathcal{D}^{(0)}_S$*-modules*

$$\mathcal{F}^{(0)}\to i_*\mathcal{G}^{(0)}. \tag{7.9.2}$$

As before, we leave the proof of this lemma to the reader.

Now we need to calculate the specialization of the vector bundle of coinvariants τ_S. To do so, recall first that by Lemma 6.2.5, the normal bundle to D is $ND=\{(d,v)\}, d\in D, v\in T^{(1)}_aC_d\otimes T^{(2)}_aC_d$, where C_d is the curve with one double point a, and $T^{(1)},T^{(2)}$ are the tangent spaces to the two components of C_d at a. Choice of coordinate q on S and coordinates t_1,t_2 on C_S such that $t_1t_2=q$ gives an identification of a neighborhood of D in S with a neighborhood of D in ND by

$$i\colon(d,q)\mapsto(d,q\partial_{t_1}\otimes\partial_{t_2}),$$

or, passing from vectors to covectors,

$$i\colon (d,q) \mapsto (d, \frac{dt_1 \otimes dt_2}{q}). \tag{7.9.3}$$

Now, let us define a family of pointed curves over ND by $C_{d,q} = C_d^\vee$ with the parameters at a', a'' given by $t_1/q, t_2$. This defines a bundle of coinvariants $\tilde{\tau}$ on a neighborhood of D in S.

THEOREM 7.9.9. *The map*

$$\begin{aligned} \mathcal{O}_S \otimes V &\to \mathcal{O}_{ND} \otimes \sum_\lambda V \otimes L_\lambda^k \otimes DL_\lambda^k \\ f(s)v &\mapsto \sum_\lambda f(i(s))v \otimes w_\lambda \end{aligned} \tag{7.9.4}$$

where $1_\lambda \in V_\lambda \otimes V_\lambda^* \subset L_\lambda^k \otimes DL_\lambda^k$ is the canonical $\mathfrak{g}$-invariant vector, gives rise to an isomorphism of $\mathcal{D}_S^{(0)}$-modules $\tau_S^{(0)} \to \tilde{\tau}^{(0)}$.

PROOF. We will use as an intermediate step the sheaf $\tau^{\vee(0)}$ introduced in the previous section. By Proposition 7.9.7, the isomorphism $\phi\colon \tau^{(0)} \to \tau^{\vee(0)}$, defined by (7.8.1) is an isomorphism of $\mathcal{D}_D^{(0)}$-modules. On the other hand, let us show that the map $V \otimes L_\lambda^k \otimes DL_\lambda^k \to V \otimes L_\lambda^k \otimes DL_\lambda^k$, given by

$$v \otimes v' \otimes v'' \mapsto q^{\deg v'} v \otimes v' \otimes v''$$

gives rise to an isomorphism of $\tau^{\vee(0)}$ and $\tilde{\tau}^{(0)}$ as $\mathcal{D}_D^0$-modules. Indeed, let us compare the action of the vector field $q\partial_q$ on both spaces. For $\tilde{\tau}^{(0)}$ it is given by $-(L_0)_{a'}$, and for $\tau^{\vee(0)}$, it is given by

$$\gamma_{a'}(v^\vee) + \gamma_{a'}(v^\vee) + \sum \gamma_{p_i}(v^\vee) + \Delta_\lambda.$$

It follows from Proposition 7.6.9 that the only non-zero term in this sum is Δ_λ, and therefore, (7.9.4) is indeed an isomorphism of modules.

Combining the isomorphisms $\tau^{(0)} \to \tau^{\vee(0)} \to \tilde{\tau}^{(0)}$, we get the statement of the theorem. □

Now we can prove the main result of this chapter.

THEOREM 7.9.10. *The sheaves of coinvariants $\tau(C, \vec{p}, V_i)$, $V_i \in \mathcal{O}_k^{\mathrm{int}}$, form a modular functor with additive central charge c.*

PROOF. According to Definition 6.4.1, we need to define the gluing isomorphism and the vacuum propagation isomorphism for the spaces of coinvariants. Vacuum propagation isomorphism is given by Corollary 7.3.5; the gluing isomorphism is obtained by combining Lemma 7.9.8 and Theorem 7.9.9. Checking all the compatibility conditions for these isomorphisms is trivial. □

For technical reasons, it is more convenient to pass to the dual sheaf

$$\tau^*(C, \vec{p}, V_i) = \big(\tau(C, \vec{p}, DV_i)\big)^*.$$

Obviously, the previous theorem immediately implies that the sheaves $\tau^*(C, \vec{p}, V_i)$ also form a modular functor with the additive central charge c. This functor will be called *Wess-Zumino-Witten modular functor*.

As a corollary, we have proved the theorem formulated in the introduction to this chapter.

COROLLARY 7.9.11. *The category $\mathcal{O}_k^{\mathrm{int}}$ has a structure of a modular tensor category, with $\mathbf{1} = L_0^k, \theta_V = e^{2\pi \mathrm{i} L_0}$, and the tensor product $\dot\otimes$ defined by*

$$\mathrm{Hom}_{\mathcal{O}_k^{\mathrm{int}}}(\mathbf{1}, V_1 \dot\otimes \dots \dot\otimes V_n) = \big(\tau(C, DV_1 \otimes \dots \otimes DV_n)\big)^*$$

where C is the "standard" n-punctured sphere, as in (6.4.3).

As a matter of fact, we have not yet proved the rigidity (recall that modular functor only defines weak rigidity); however, it can be shown that this category is
?! indeed rigid.

A weaker version of this result is the following:

THEOREM 7.9.12. *Let $k \notin \mathbb{Q}$. Then the vector spaces of coinvariants $\tau(C, \vec{p}, V_{\vec\lambda}^k)$ define a genus zero modular functor. The corresponding ribbon category is the Drinfeld's category.*

PROOF. The proof is obtained by noticing that we have used integrability of L_λ^k only in two places: when checking finite-dimensionality of the spaces of coinvariants, and in the proof of Theorem 7.7.1, identifying the coinvariants for a singular curve C and its normalization $C^\vee$. On the other hand, if we restrict ourselves to genus zero curves, then the vector spaces of coinvariants are finite-dimensional by Proposition 7.3.8. It is also easy to show that the proof of Theorem 7.7.1 remains valid for $k \notin \mathbb{Q}$ if we replace $\bigoplus L_\lambda^k \otimes DL_\lambda^k$ by (infinite) sum $\bigoplus_{\lambda \in P_+} V_\lambda^k \otimes DV_\lambda^k$.

The fact that the corresponding category is exactly the Drinfeld's category follows from comparison of this modular functor with the modular functor defining Drinfeld category (see Proposition 6.5.4). Indeed, Proposition 7.3.8 shows that the corresponding vector spaces of conformal blocks can be identified, Theorem 7.4.10 shows that this identification preserves the flat connections, and Theorem 7.9.9 shows that the gluing map for these two modular functors also coincides. $\square$

REMARK 7.9.13. One can note that we have most of the arguments above were quite general and didn't use much information about the coinvarints. Most of the time we were only using the action of the Virasoro algebra on integrable modules, given by the Sugawara construction. The only places were we actually used the definition of coinvariants and properties of integrable modules were the proof of finite-dimensionality of the vector spaces of coinvariants and the proof of Theorem 7.7.1, identifying the coinvariants for a singular curve C and its normalization $C^\vee$. Thus, if we could repeat these two steps in other setups—for example, replacing the category $\mathcal{O}_k^{\mathrm{int}}$ by a suitable category of Virasoro modules—we would again get a modular functor. Indeed, it is rather easy to modify these arguments to define the modular functor related to the so-called minimal models of Conformal Field Theory, in which the modules L_λ^k are replaced by irreducible unitary modules over Vir with a suitable central charge. If we try to pursue this idea as far as we can and see what is the most general situation in which we can apply the same proof, we will arrive at the notion of Rational Conformal Field Theory (or, to be more precise, the holomorphic (chiral) half of RCFT). The number of references on this subject is tremendous; some of the more suitable for mathematical audience are [**Hua**], influential but unpublished manuscript [**BFM**], and [**Gai**]. For more physical exposition and extra references, see [**FMS**].

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Index

Index of Notation